Combinatorial properties of toric topological string partition functions

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Plan

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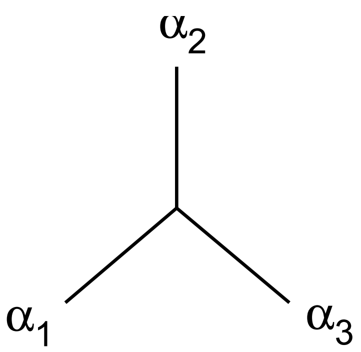
Main reference M. Mariño, *Chern-Simons Theory, Matrix Models, And Topological Strings* (Oxford UP).

Topological vertex

Combinatorial definition

$$C_{\alpha_1\alpha_2\alpha_3} = q^{\kappa(\alpha_3)/2} s_{\alpha_2}(q^\rho) \sum_\mu s_{\alpha_1/\mu}(q^{\,{}^t\alpha_2+\rho}) s_{\,{}^t\alpha_3/\mu}(q^{\alpha_2+\rho})$$

$\alpha_1, \alpha_2, \alpha_3$ are partitions (or Young diagrams),



$$\kappa(\lambda) = \sum_{i\geq 1} \lambda_i(\lambda_i - 2i + 1) = 2 \sum_{(i,j)\in\lambda} (j - i),$$

$$\rho = \left(-\frac{1}{2}, -\frac{3}{2}, \cdots, -i + \frac{1}{2}, \cdots\right),$$

$$q^{\lambda+\rho} = \left(q^{\lambda_1-1/2}, q^{\lambda_2-3/2}, \cdots, q^{\lambda_i-i+1/2}, \cdots\right).$$

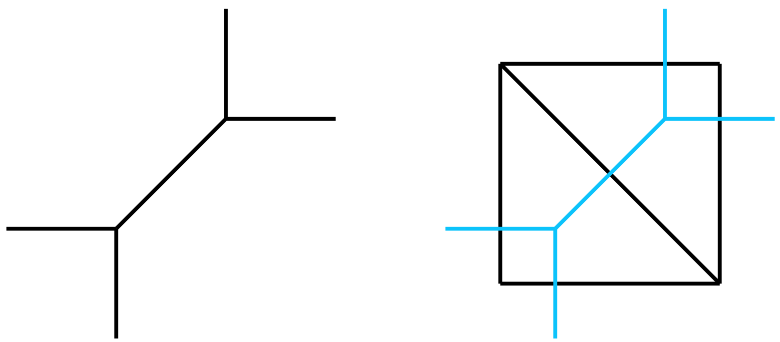
Cyclic symmetry $C_{\alpha_1\alpha_2\alpha_3} = C_{\alpha_2\alpha_3\alpha_1} = C_{\alpha_3\alpha_1\alpha_2}$ (non-trivial!)

Hook formula $C_{\emptyset\lambda\emptyset} = s_\lambda(q^\rho) = \dfrac{q^{\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda}(q^{h(i,j)/2} - q^{-h(i,j)/2})}.$

Web diagrams

Example   Web diagram for resolved conifold

$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$

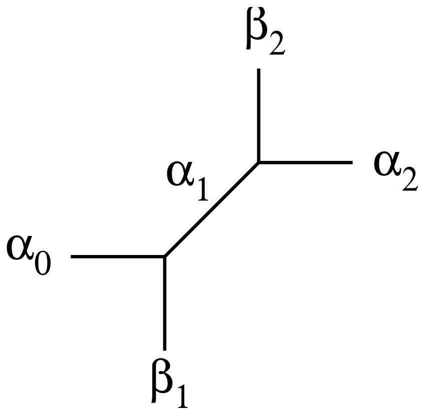


web diagram (left) and toric diagram (right)

The topological vertex itself corresponds to the simplest Calabi-Yau 3-fold $X = \mathbf{C}^3$. General toric (more precisely, locally toric) Calabi-Yau 3-folds are obtained by "gluing" several $\mathbf{C}^3$'s. The gluing data are encoded in the toric diagram.

Partition function

The partition function of (open) topological strings on a toric Calabi-Yau 3-fold X is obtained by "gluing" topological vertices on the web diagram. Internal lines carry Kähler parameters Q_n. α_n's on the internal lines are summed over all partitions.



Example Resolved conifold with Kähler parameter Q

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = \sum_{\alpha_1} C_{\alpha_1\beta_1\alpha_0}(-Q)^{|\alpha_1|} C_{{}^t\alpha_1\beta_2\alpha_2}$$

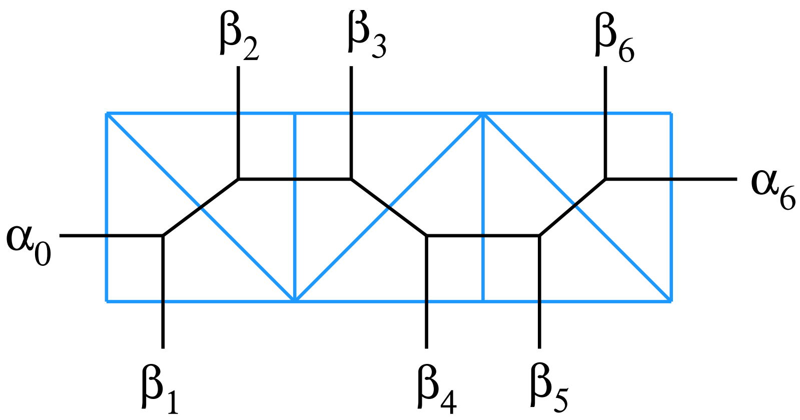
General case The weight of the n-th internal line is $(-Q_n)^{|\alpha_n|}$ times the "framing factor" determined by an integer r_n:

$$Z = \sum_{\cdots,\alpha_n,\cdots} \cdots C_{\alpha_n\mu_n\nu_n}(-Q_n)^{|\alpha_n|}(-1)^{r_n|\alpha_n|} q^{-r_n\kappa(\alpha_n)/2} C_{{}^t\alpha_n\mu'_n\nu'_n} \cdots$$

Generalized conifolds

— this is the case where the toric diagram consists of neighboring triangles of height 1 in a "strip" (a rectangle of height 1). The associated Calabi-Yau 3-folds are called "generalized conifolds".

The web diagram is acyclic, and has N vertical and 2 non-vertical external legs, where N is the number of triangles. Partitions assigned to these legs are denoted by $\beta_1, \cdots, \beta_N$ and α_0, α_N, respectively.



Partition function for generalized conifolds

(i) [Iqbal and Kashani-Poor] If $\alpha_0 = \alpha_N = \emptyset$, the partition function $Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N}$ can be calculated in a factorized form as

$$Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N} = s_{\beta_1}(q^\rho)\cdots s_{\beta_N}(q^\rho)\cdot(\text{product of factors}).$$

The last factors are of the form $\prod_{i,j=1}^{\infty}(1-Qq^{\lambda_i+\mu_j-i-j+1})^{\pm 1}$. Technical tools are Cauchy identities.

(ii) [Eguchi and Kanno for a special case; Nagao, Sulkowski for the general case] The partition functions has a fermionic formula

$$Z^{\alpha_0\alpha_N}_{\beta_1\cdots\beta_N} = q^{\epsilon_0\kappa(\alpha_0)/2}q^{\epsilon_N\kappa(\alpha_N)/2}s_{\beta_1}(q^\rho)\cdots s_{\beta_N}(q^\rho)\langle\,{}^{\mathrm{t}}\alpha_0|\mathcal{O}|\alpha_N\rangle$$

where $\epsilon_0, \epsilon_N \in \{0,1\}$ depend on the position of α_0, α_N in the web diagram. $\mathcal{O}$ is a product of vertex operators and propagators. This result is reviewed below.

Tools for fermionic formula (Okounkov and Reshetikin)

Fermions and vertex operators

$$\Gamma_{\pm}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} a_{\pm k}\right), \quad a_k = \sum_n :\psi_n \psi^*_{n+k}: \ (k \in \mathbf{Z}),$$

$$\Gamma'_{\pm}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^k}{k} a_{\pm k}\right) = \Gamma_{\pm}(-z)^{-1},$$

$$\Gamma_{\pm}(q^{\lambda+\rho}) = \prod_{i=1}^{\infty} \Gamma_{\pm}(q^{\lambda_i - i + 1/2}), \quad \Gamma'_{\pm}(q^{\lambda+\rho}) = \prod_{i=1}^{\infty} \Gamma'_{\pm}(q^{\lambda_i - i + 1/2})$$

Matrix elements

$$\langle \lambda | \Gamma_-(q^{\alpha+\rho}) | \mu \rangle = \langle \mu | \Gamma_+(q^{\alpha+\rho}) | \lambda \rangle = s_{\lambda/\mu}(q^{\alpha+\rho}),$$

$$\langle \lambda | \Gamma'_-(q^{\alpha+\rho}) | \mu \rangle = \langle \mu | \Gamma'_+(q^{\alpha+\rho}) | \lambda \rangle = s_{{}^t\lambda/{}^t\mu}(q^{\alpha+\rho})$$

Commutation relations

$$\Gamma_+(q^{\lambda+\rho})\Gamma_-(q^{\mu+\rho}) = \prod_{i,j=1}^{\infty}(1-q^{\lambda_i+\mu_j-i-j+1})^{-1}\cdot\Gamma_-(q^{\mu+\rho})\Gamma_+(q^{\lambda+\rho}),$$

$$\Gamma'_+(q^{\lambda+\rho})\Gamma'_-(q^{\mu+\rho}) = \prod_{i,j=1}^{\infty}(1-q^{\lambda_i+\mu_j-i-j+1})^{-1}\cdot\Gamma'_-(q^{\mu+\rho})\Gamma'_+(q^{\lambda+\rho}),$$

$$\Gamma_+(q^{\lambda+\rho})\Gamma'_-(q^{\mu+\rho}) = \prod_{i,j=1}^{\infty}(1+q^{\lambda_i+\mu_j-i-j+1})\cdot\Gamma'_-(q^{\mu+\rho})\Gamma_+(q^{\lambda+\rho}),$$

$$\Gamma'_+(q^{\lambda+\rho})\Gamma_-(q^{\mu+\rho}) = \prod_{i,j=1}^{\infty}(1+q^{\lambda_i+\mu_j-i-j+1})\cdot\Gamma_-(q^{\mu+\rho})\Gamma'_+(q^{\lambda+\rho}),$$

$$\Gamma_\pm(q^{\lambda+\rho})Q^{L_0} = Q^{L_0}\Gamma_\pm(Q^{\pm1}q^{\lambda+\rho}),$$

$$\Gamma'_\pm(q^{\lambda+\rho})Q^{L_0} = Q^{L_0}\Gamma'_\pm(Q^{\pm1}q^{\lambda+\rho}),$$

$$L_0 = \sum_n n{:}\psi_n\psi_n^*{:}$$

Simplest examples

(1) Topological vertex ($X = \mathbf{C}^3$)

The definition of the topological vertex can be cast into two different fermionic expressions

$$C_{\alpha_0\beta\alpha_2} = q^{\kappa(\alpha_2)} s_\beta(q^\rho)\langle {}^{\mathrm{t}}\alpha_0|\Gamma'_-(q^{{}^{\mathrm{t}}\beta+\rho})\Gamma'_+(q^{\beta+\rho})|\alpha_2\rangle$$

and

$$C_{\alpha_0\beta\alpha_2} = q^{\kappa(\alpha_2)} s_\beta(q^\rho)\langle {}^{\mathrm{t}}\alpha_2|\Gamma_-(q^{\beta+\rho})\Gamma_+(q^{{}^{\mathrm{t}}\beta+\rho})|\alpha_0\rangle.$$

They correspond to two way of viewing the vertex as a web diagram on a strip. For $\alpha_0 = \alpha_2 = \emptyset$, the fermionic expression reduces to

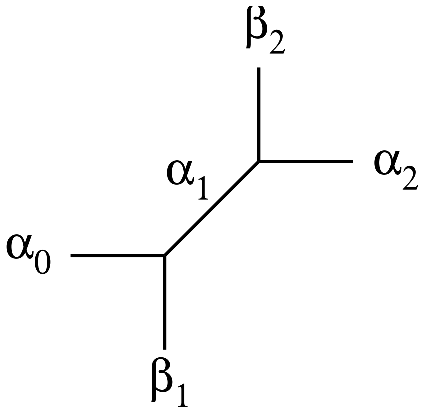
$$C_{\emptyset\beta\emptyset} = s_\beta(q^\rho).$$

(2) Resolved conifold ($X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$)

The fermionic expression of the partition function reads

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = q^{\kappa(\alpha_0)/2} q^{\kappa(\alpha_2)/2} s_{\beta_1}(q^\rho) s_{\beta_2}(q^\rho)$$

$$\times \langle {}^t\alpha_0 | \Gamma_-(q^{\beta_1+\rho}) \Gamma_+(q^{{}^t\beta_1+\rho}) (-Q)^{L_0} \Gamma'_-(q^{{}^t\beta_2+\rho}) \Gamma'_+(q^{\beta_2+\rho}) | \alpha_2 \rangle$$

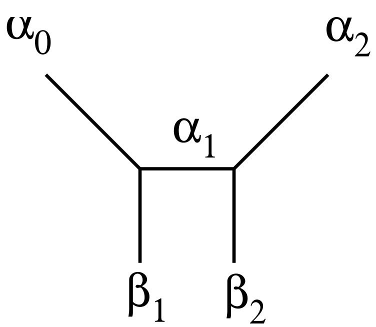


$\Gamma_-(\cdots)\Gamma_+(\cdots)$ and $\Gamma'_-(\cdots)\Gamma'_+(\cdots)$ correspond to the two vertices, and $(-Q)^{L_0}$ is the propagator along the internal line connecting the two vertices.

For $\alpha_0 = \alpha_2 = \emptyset$, the partition function can be calculated in a factorized form as

$$Z^{\emptyset\emptyset}_{\beta_1\beta_2} = s_{\beta_1}(q^\rho) s_{\beta_2}(q^\rho) \prod_{i,j=1}^{\infty} (1 - Q q^{{}^t\beta_{1,i} + {}^t\beta_{2,j} - i - j + 1}).$$

(3) Another fundamental example ($X = \mathcal{O} \oplus \mathcal{O}(-2) \to \mathbf{CP}^1$)



In this web diagram, a nontrivial framing factor shows up in the gluing of the two vertices:

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = \sum_{\alpha_1} C_{\alpha_1\beta_1\alpha_0}(-Q)^{|\alpha_1|}(-1)^{|\alpha_1|}q^{\kappa(\alpha_1)/2}C_{\alpha_2\beta_2\,{}^{\mathrm{t}}\alpha_1}$$

The fermionic expression of the partition function reads

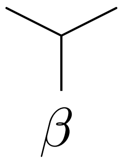
$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = s_{\beta_1}(q^\rho)s_{\beta_2}(q^\rho)q^{\kappa(\alpha_2)/2}$$
$$\times \langle {}^{\mathrm{t}}\alpha_0|\Gamma_-(q^{\beta_1+\rho})\Gamma_+(q^{{}^{\mathrm{t}}\beta_1})Q^{L_0}\Gamma_-(q^{\beta_2+\rho})\Gamma_+(q^{{}^{\mathrm{t}}\beta_2})|\alpha_2\rangle$$

and for $\alpha_0 = \alpha_2 = \emptyset$,

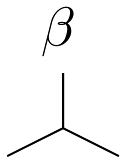
$$Z^{\emptyset\emptyset}_{\beta_1\beta_2} = s_{\beta_1}(q^\rho)s_{\beta_2}(q^\rho)\prod_{i.j=1}^{\infty}(1 - Qq^{{}^{\mathrm{t}}\beta_{1,i}+\beta_{2,j}-i-j+1})^{-1}.$$

General rules (Sulkowski)

Vertex operators

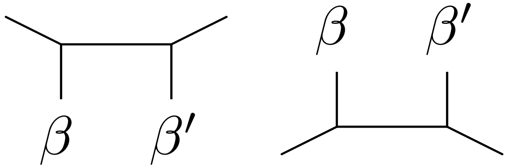


$$\longrightarrow \quad \Gamma_-(q^{\beta+\rho})\Gamma_+(q^{{}^t\beta+\rho})$$

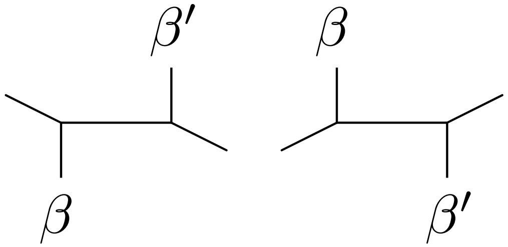


$$\longrightarrow \quad \Gamma'_-(q^{{}^t\beta+\rho})\Gamma'_+(q^{\beta+\rho})$$

Propagators



$$\longrightarrow \quad Q^{L_0}$$



$$\longrightarrow \quad (-Q)^{L_0}$$

Method for calculating partition functions for $\alpha_0 = \alpha_N = \emptyset$

$$Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N} = s_{\beta_1}(q^\rho)\cdots s_{\beta_N}(q^\rho)$$
$$\times \langle 0|G_-^{(1)}G_+^{(1)}(\sigma_1\sigma_2 Q_1)^{L_0}G_-^{(2)}G_+^{(2)}(\sigma_2\sigma_3 Q_2)^{L_0}\cdots G_-^{(N)}G_+^{(N)}|0\rangle$$

where $\sigma_n = \pm$ depends on the direction ($+$ for "down" and $-$ for "up") of the n-th leg, and

$$G_-^{(n)} = \begin{cases}\Gamma_-(q^{\beta_n+\rho}), \\ \Gamma'_-(q^{{}^t\beta_n+\rho}),\end{cases} \qquad G_+^{(n)} = \begin{cases}\Gamma_+(q^{{}^t\beta_n+\rho}) & \text{if } \sigma_n = +, \\ \Gamma'_+(q^{\beta_n+\rho}) & \text{if } \sigma_n = -.\end{cases}$$

The vacuum expectation values can be calculated by moving $G_-^{(n)}$'s to left, $G_+^{(n)}$'s to right, and noting the vacuum conditions

$$\langle 0|G_-^{(n)} = \langle 0|, \quad G_+^{(n)}|0\rangle = |0\rangle.$$

Recovering Iqbal and Kashani-Poor's result

As $G_+^{(m)}$ and $G_-^{(n)}$ interchange, a factor of the form

$$\prod_{i,j=1}^{\infty} (1 - Q_m Q_{m+1} \cdots Q_{n-1} q^{\,{}^{\mathrm{t}}\beta_i^{(m)} + \beta_j^{(n)} - i - j + 1})^{-\sigma_m \sigma_n}$$

shows up, where

$$\beta^{(n)} = \begin{cases} \beta_n & \text{if } \sigma_n = +, \\ {}^{\mathrm{t}}\beta_n & \text{if } \sigma_n = -. \end{cases}$$

$Z^{\emptyset\emptyset}_{\beta_1 \cdots \beta_N}$ is given by a product of these factors:

$$Z^{\emptyset\emptyset}_{\beta_1 \cdots \beta_N} = s_{\beta_1}(q^\rho) \cdots s_{\beta_N}(q^\rho)$$

$$\times \prod_{m<n} \prod_{i,j=1}^{\infty} (1 - Q_m Q_{m+1} \cdots Q_{n-1} q^{\,{}^{\mathrm{t}}\beta_i^{(m)} + \beta_j^{(n)} - i - j + 1})^{-\sigma_m \sigma_n}.$$

This explains the result of Iqbal and Kashani-Poor.

Quantum mirror curve

On "B-model" side of topological strings (Aganagic et al 1993) —

• Complex curve (mirror curve) $A(x,y)=0$ and its quantization $\hat{A}(\hat{x},\hat{y})\Psi(x)=0$: The mirror curve shows up as a reduced from of the 3D mirror manifold of X.

• Integrable hierarchies (KP, Toda, multi-component KP, ...): (Closed string) partition functions are expected to be tau functions of integrable hierarchies.

• Recent "remodeling" program (Bourchard, Mariño et al. 2007) based on the topological recursion relation in random matrix theory (Eynard and Orantin): The mirror curves are identified with spectral curves in random matrix theory.

The quantum mirror curves in special cases ($\mathbf{C}^3$, resolved conifold, etc.) are also derived as an analogue of A-polynomials of knot invariants (Gukov and Sulkowski 2011).

Quantum mirror curves for generalized conifolds

Closing external legs with auxiliary Schur functions

Choose one of the external β-legs, introduce a (possibly infinite) set of auxiliary variables $\boldsymbol{x} = (x_1, x_2, \cdots)$ and consider the generating function

$$Z_n(\boldsymbol{x}) = \sum_{\beta_n} Z^{\emptyset\emptyset}_{\emptyset\cdots\emptyset\beta_n\emptyset\cdots\emptyset} s_{\beta_n}(\boldsymbol{x}).$$

Remark   This is a tau function of the KP hierarchy with respect to the time variables $t_k = \sum_{i\geq 1} x_i^k/k$, $k = 1, 2, \cdots$. In the same sense,

$$Z(\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(N)}) = \sum_{\beta_1,\cdots,\beta_N} Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N} s_{\beta_1}(\boldsymbol{x}^{(1)}) \cdots s_{\beta_N}(\boldsymbol{x}^{(N)})$$

will be a tau function of the N-component KP hierarchy (thought there is no proof).

Specialization of variables

Specialize $\boldsymbol{x}$ to $\boldsymbol{x} = (x, 0, 0, \cdots)$. Since

$$s_\beta(x, 0, 0, \cdots) = \begin{cases} x^k & \text{if } \beta = (k),\ k = 0, 1, 2, \cdots, \\ 0 & \text{otherwise,} \end{cases}$$

the foregoing generating function $Z_n(\boldsymbol{x})$ reduces to

$$Z_n(x) = \sum_{k=0}^{\infty} Z^{\emptyset\emptyset}_{\emptyset\cdots\emptyset(k)\emptyset\cdots\emptyset} x^k.$$

Normalize this function by its value at $x = 0$ as

$$\Psi_n(x) = \frac{Z_n(x)}{Z_n(0)}.$$

Remark $\Psi_n(x)$ amounts to a (dual) Baker-Akhierzer function, with all time variables t_k, $k = 1, 2, \ldots$, specialized to $t_k = 0$, in the theory of integrable hierarchies.

q-difference equation

There are Laurent polynomials $B_n(y)$ and $C_n(y)$ such that $\Psi_n(x)$ can be expressed as

$$\Psi_n(x) = \sum_{k=0}^{\infty} \frac{C_n(1)C_n(q)\cdots C_n(q^{k-1})}{B_n(1)B_n(q)\cdots B_n(q^{k-1})} \frac{q^{k(k-1)/4}}{[1]\cdots[k]} x^k,$$

where $[k] = q^{k/2} - q^{-k/2}$. In particular, $\Psi_n(x)$ satisfies the q-difference equation

$$B_n(q^{-1}q^{x\partial/\partial x})[x\partial/\partial x]\Psi_n(x) = xC_n(q^{x\partial/\partial x})\Psi_n(q^{1/2}x)$$

or, by substituting $x \to q^{1/2}x$,

$$B_n(q^{-1}q^{x\partial/\partial x})(q^{x\partial/\partial x} - 1)\Psi_n(x) = q^{1/2}xC_n(q^{x\partial/\partial x})\Psi_n(qx).$$

Remark $\Psi_n(x)$ is substantially a basic hypergeometric series.

Quantum mirror curve

The q-difference equation represents a quantum curve defined by the operator ($\hat{x} = x$, $\hat{y} = q^{x\partial/\partial x}$)

$$\hat{A}_n(\hat{x}, \hat{y}) = B_n(q^{-1}\hat{y})(\hat{y} - 1) - q^{1/2}\hat{x}C_n(\hat{y})\hat{y}.$$

In the "classical limit" as $q \to 1$, this operator turns into the Laurent polynomial

$$A_n(x, y) = B_n(y)(y - 1) - xC_n(y)y$$

that defines the mirror curve.

Example For the resolved conifold,

$$\hat{A}_1 = \hat{A}_2 = \hat{y} - 1 - q^{1/2}\hat{x}(1 - Q\hat{y})\hat{y}.$$

This differs from a usual one by a "framing transformation".