KP and Toda tau functions in Bethe ansatz: a review

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1. Introduction

Determinant formulas in algebraic Bethe ansatz:
• Partition function of 6 vertex model under domain wall boundary condition (DWBC) (Korepin; Izergin)
• Scalar product of Bethe states in finite spin 1/2 XXZ chain (Korepin; Slavnov; Kitanine, Maillet & Terras)
• Scalar products of states in models "at $q = 0$" (Bogolioubov et al.)

Foda, Wheeler and Zuparic observed that these quantities are special tau functions of KP and Toda hierarchies:
• "Domain wall partition functions and KP", arXiv:0901.2251
• "XXZ scalar products and KP", arXiv:0903.2611
• "Phase model expectation values and the 2-Toda hierarchy", arXiv:0906.3358

This talk is a review of their results.

2. Tau functions

Tau functions of KP hierarchy

Tau functions $\tau[\boldsymbol{t}]$ of the KP hierarchy depend on an infinite series $\boldsymbol{t} = (t_1, t_2, \ldots)$ of time variables. A general formula of tau functions (M. & Y. Sato, 1981) reads

$$\tau[\boldsymbol{t}] = \sum_{\lambda:\text{partitions}} c_\lambda s_\lambda[\boldsymbol{t}]$$

• c_λ's are Plücker coordinates of a point of an infinite dimensional Grassmann manifold $\mathrm{Gr}(\frac{\infty}{2}, \infty)$.
• $s_\lambda[\boldsymbol{t}]$'s, $\lambda = (\lambda_1, \lambda_2, \ldots)$, are Schur functions redefined by the Jacobi-Trudy formula

$$s_\lambda[\boldsymbol{t}] = \det(h_{\lambda_i - i + j}[\boldsymbol{t}])_{i,j=1}^{\infty}, \qquad \sum_{n=0}^{\infty} h_n[\boldsymbol{t}] z^n = \exp(\sum_{n=1}^{\infty} t_n z^n)$$

as functions of the universal "power sum" variables $t_n = p_n/n$.

Schur functions

When t_n's are parametrized by N variables $\boldsymbol{x} = (x_1, \ldots, x_N)$ as

$$t_n = \frac{p_n}{n} = \frac{1}{n} \sum_{j=1}^{N} x_j^n,$$

• $s_\lambda[\boldsymbol{t}]$ vanishes for all partitions of length $> N$.
• $s_\lambda[\boldsymbol{t}]$'s for partitions of length $\leq N$ become the Schur functions

$$s_\lambda(\boldsymbol{x}) = \frac{\det(x_j^{\lambda_i - i + N})_{i,j=1}^{N}}{\det(x_j^{-i+N})_{i,j=1}^{N}}.$$

• $h_n[\boldsymbol{t}]$'s become the complete symmetric functions

$$h_n(\boldsymbol{x}) = \sum_{N \geq i_1 \geq \cdots \geq i_n \geq 1} x_{i_1} \cdots x_{i_n}.$$

Tau functions given by finite determinant

Plücker coordinates c_λ of points in $\mathrm{Gr}(N,\infty) \subset \mathrm{Gr}(\frac{\infty}{2},\infty)$ are given by finite-dimensional determinants

$$c_\lambda = \det(f_{i,l_j})_{i,j=1}^N, \quad l_i = \lambda_i - i + N.$$

The $N \times \infty$ matrix $(f_{i,l})_{i=1,\dots,N,\, l=0,1,\dots}$ represent a point of the Grassmann manifold $\mathrm{Gr}(N,\infty)$. By the Cauchy-Binet formulas, the tau function $\tau[\boldsymbol{t}] = \tau(\boldsymbol{x})$ can be expressed as

$$\tau(\boldsymbol{x}) = \sum_{\infty > l_1 > \cdots > l_N \geq 0} \frac{\det(f_{i,l_j}) \det(x_i^{l_j})}{\Delta(\boldsymbol{x})} = \frac{\det(f_i(x_j))_{i,j=1}^N}{\Delta(\boldsymbol{x})},$$

where $\Delta(\boldsymbol{x})$ is the Vandermonde determinant of $x_1, \dots, x_N$ and $f_i(x)$'s are the power series $f_i(x) = \sum_{l=0}^{\infty} f_{i,l} x^l$.

Tau functions of 2-KP hierarchy

Tau functions $\tau[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ of the two-component KP (2-KP) hierarchy are functions of two series $\boldsymbol{t} = (t_1, t_2, \ldots)$, $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$ of time variables, and can be expressed as

$$\tau[\boldsymbol{t}, \bar{\boldsymbol{t}}] = \sum_{\lambda,\mu} c_{\lambda\mu} s_\lambda[\boldsymbol{t}] s_\mu[\bar{\boldsymbol{t}}],$$

where $c_{\lambda\mu}$'s are Plücker coordinates of a point of $\mathrm{Gr}(\infty, 2\infty)$.

Tau functions $\tau[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ for points on the submanifold $\mathrm{Gr}(M + N, 2\infty)$ are given by finite determinants. If the time variables are parametrized as $t_n = \frac{1}{n} \sum_{j=1}^{M} x_j^n$, $\bar{t}_n = \frac{1}{n} \sum_{k=1}^{N} y_k^n$, the determinant formula reads

$$\tau(\boldsymbol{x}, \boldsymbol{y}) = \frac{\det(f_i(x_j) \mid g_i(y_k))_{i=1,\ldots,M+N;j=1,\ldots,M,\,k=1,\ldots,N}}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})}$$

Tau functions of Toda hierarchy

Tau functions $\tau[s,\boldsymbol{t},\overline{\boldsymbol{t}}]$ of the Toda hierarchy are functions of a discrete variable $s \in \mathbf{Z}$ and two series $\boldsymbol{t} = (t_1, t_2, \ldots)$ and $\overline{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$ of time variables, and can be expressed as

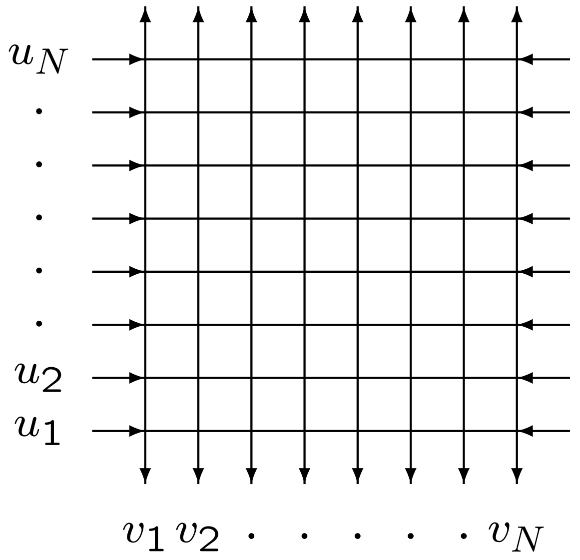
$$\tau[s,\boldsymbol{t},\overline{\boldsymbol{t}}] = \sum_{\lambda,\mu} c_{s\lambda\mu} s_\lambda[\boldsymbol{t}] s_\mu[-\overline{\boldsymbol{t}}] = \sum_{\lambda,\mu} c_{s\lambda\mu} (-1)^{|\mu|} s_\lambda[\boldsymbol{t}] s_{{}^{\mathrm{t}}\mu}[\overline{\boldsymbol{t}}],$$

where $c_{s\lambda\mu}$'s are Plücker coordinates of an infinite dimensional flag manifold.

For each value of s, $\tau[s,\boldsymbol{t},\overline{\boldsymbol{t}}]$ is a tau function of the 2-KP hierarchy. $c_{s\lambda\mu}(-1)^{|\mu|}$'s are identified with Plücker coordinates of the Grassmann manifold $\mathrm{Gr}((\frac{\infty}{2}+s)+(\frac{\infty}{2}-s), 2\infty)$.

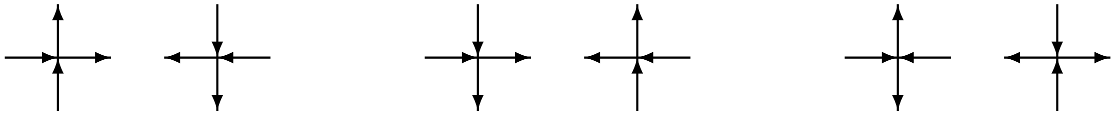
3. 6-vertex model with DWBC

6VM on $N \times N$ lattice with DWBC



• Inhomogeneity parameters $\boldsymbol{u} = (u_1, \ldots, u_N)$ and $\boldsymbol{v} = (v_1, \ldots, v_N)$ are assigned to the rows and columns.

- Vertex weights $w_{i,j}$



$a(u_i - v_j)$ $b(u_i - v_j)$ $c(u_i - v_j)$

$$a(u) = \sinh(u + \gamma), \quad b(u) = \sinh u, \quad c(u) = \sinh \gamma$$

- Partition function

$$Z_N = Z_N(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\text{configuration}} \prod_{(i,j)} w_{i,j}$$

Izergin-Korepin determinant formula

$$
\begin{aligned}
Z_N =& \frac{\prod_{i,j=1}^{N} \sinh(u_i - v_j + \gamma) \sinh(u_i - v_j)}{\prod_{1 \le i < j \le N} \sinh(u_i - u_j) \sinh(v_j - v_i)} \\
& \times \det \left(\frac{\sinh \gamma}{\sinh(u_i - v_j + \gamma) \sinh(u_i - v_j)} \right)_{i,j=1}^{N} \\
=& \frac{\sinh^N \eta}{\prod_{1 \le i < j \le N} \sinh(u_i - u_j) \sinh(v_j - v_i)} \\
& \times \det \left(\frac{\prod_{k=1}^{N} \sinh(u_i - v_k + \gamma) \sinh(u_i - v_k)}{\sinh(u_i - v_j + \gamma) \sinh(u_i - v_j)} \right)_{i,j=1}^{N} \\
=& \frac{\sinh^N \gamma}{\prod_{1 \le i < j \le N} \sinh(u_i - u_j) \sinh(v_j - v_i)} \\
& \times \det \left(\frac{\prod_{k=1}^{N} \sinh(u_k - v_j + \gamma) \sinh(u_k - v_j)}{\sinh(u_i - v_j + \gamma) \sinh(u_i - v_j)} \right)_{i,j=1}^{N}
\end{aligned}
$$

Changing to rational variables

Define $x_i := e^{2u_i}$, $y_i := e^{2v_i}$, $q := e^{-\gamma}$. Then up to a simple factor C_N (an exponential function of u_i's and v_i's), the partition function becomes a rational function of x_i's and y_i's:

$$
\begin{aligned}
Z_N =& C_N \frac{\prod_{i,j=1}^{N}(x_i q^{-1} - y_j q)(x_i - y_j)}{\prod_{1\le i<j\le N}(x_i - x_j)(y_j - y_i)} \det\left(\frac{q^{-1}-q}{(x_i q^{-1} - y_j q)(x_i - y_j)}\right)_{i,j=1}^{N} \\
=& C_N \frac{(q^{-1}-q)^N}{\prod_{1\le i<j\le N}(x_i - x_j)(y_j - y_i)} \det\left(\frac{\prod_{k=1}^{N}(x_i q^{-1} - y_k q)(x_i - y_k)}{(x_i q^{-1} - y_j q)(x_i - y_j)}\right) \\
=& C_N \frac{(q^{-1}-q)^N}{\prod_{1\le i<j\le N}(x_i - x_j)(y_j - y_i)} \det\left(\frac{\prod_{k=1}^{N}(x_k q^{-1} - y_j q)(x_k - y_j)}{(x_i q^{-1} - y_j q)(x_i - y_j)}\right)
\end{aligned}
$$

The second and third lines show that Z_N is a KP tau function with respect to both $\boldsymbol{x} = (x_1, \ldots, x_N)$ and $\boldsymbol{y} = (y_1, \ldots, y_N)$.

Z_N as KP tau function

• The essential part of the second expression of Z_N is $\dfrac{\det(f_j(x_i))_{i,j=1}^N}{\Delta(\boldsymbol{x})}$, where f_j's are polynomials of the form

$$f_j(x) = \frac{\prod_{k=1}^N (xq^{-1} - y_k q)(x - y_k)}{(xq^{-1} - y_j q)(x - y_j)}.$$

As a function of $t_n = \dfrac{1}{n}\sum_{i=1}^N x_i^n$, this quantity is a polynomial tau function of the KP hierarchy.

• The same interpretation holds for the essential part $\dfrac{\det(g_i(y_j))_{i,j=1}^N}{\Delta(\boldsymbol{y})}$ of the third expression, where g_i's are polynomials of the form

$$g_i(y) = \frac{\prod_{k=1}^N (x_i q^{-1} - yq)(x_i - y)}{(x_i q^{-1} - yq)(x_i - y)}.$$

• It is an open question whether the first expression of Z_N implies that Z_N is a tau function of the 2-KP hierarchy with respect to $t_n = \frac{1}{n}\sum_{i=1}^{N} x_i^n$ and $\bar{t}_n = \frac{1}{n}\sum_{i=1}^{N} y_i^n$. Presumably, this is not the case for generic values of q. The case where $q = e^{\pi i/3}$ is exceptional. According to a result of Soichi Okada, Z_N with $q = e^{\pi i/3}$ coincides, up to a simple factor, with a single Schur function of the $2N$ variables $(\boldsymbol{x}, \boldsymbol{y})$,

$$Z_N = (\text{simple factor}) s_\lambda(\boldsymbol{x}, \boldsymbol{y}),$$

where λ is the "double staircase" partition

$$\lambda = (N-1, N-1, N-2, N-2, \ldots, 1, 1).$$

On the other hand, for any partition λ, $s_\lambda(\boldsymbol{x}, \boldsymbol{y}) \prod_{i,j=1}^{N} (x_i - y_j)$ turns out to be a 2-KP tau function. Thus Z_N with $q = e^{\pi i/3}$ is substantially a 2-KP tau function.

4. Scalar product of Bethe states in finite spin 1/2 XXZ chain

Spin 1/2 XXZ chain of length N

- R-matrix

$$R(u-v) = \begin{pmatrix} a(u-v) & 0 & 0 & 0 \\ 0 & b(u-v) & c(u-v) & 0 \\ 0 & c(u-v) & b(u-v) & 0 \\ 0 & 0 & 0 & a(u-v) \end{pmatrix}$$

- L-operators

$$L(u-\xi_l) = (L_{\alpha\beta}(u-\xi_l))_{\alpha,\beta=1,2} \quad (l = 1, 2, \ldots, N),$$

$$L_{11}(u) = a(u)\frac{1+\sigma^3}{2} + b(u)\frac{1-\sigma^3}{2}, \quad L_{12}(u) = c(u)\sigma^-,$$

$$L_{21}(u) = c(u)\sigma^+, \quad L_{22}(u) = b(u)\frac{1+\sigma^3}{2} + a(u)\frac{1-\sigma^3}{2}$$

The matrix elements of $L(u-\xi_l)$ are understood to act on the space $\mathbf{C}^2$ of spin states. Let $L^{(l)}(u-\xi_l)$ denote the L-matrix whose matrix elements act on the l-th component of the full quantum space $\bigotimes_{l=1}^{N} \mathbf{C}^2$.

- Local intertwining relations

The Yang-Baxter equations for $R(u-v)$ imply the intertwining relations

$$R(u-v)L^{(k)}(u-\xi_k)L^{(l)}(v-\xi_l) = L^{(l)}(v-\xi_l)L^{(k)}(u-\xi_k)R(u-v)$$

- T-operator

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L^{(1)}(u-\xi_1)\cdots L^{(N)}(u-\xi_N),$$

$$R(u-v)T^{(1)}(u)T^{(2)}(v) = T^{(1)}(v)T^{(2)}(u)R(u-v).$$

Algebraic Bethe ansatz

- Pseudo vacuum

$$|0\rangle = \bigotimes_{l=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \bigotimes_{l=1}^{N} \mathbf{C}^2$$

$A(u)$ and $D(u)$ act on $|0\rangle$ and its dual $\langle 0|$ as

$$A(u)|0\rangle = \alpha(u)|0\rangle, \quad D(u)|0\rangle = \delta(u)|0\rangle,$$
$$\langle 0|A(u) = \alpha(u)\langle 0|, \quad \langle 0|D(u) = \delta(u)\langle 0|,$$

where

$$\alpha(u) = \prod_{l=1}^{N} \sinh(u - \xi_l + \gamma), \quad \delta(u) = \prod_{l=1}^{N} \sinh(u - \xi_l).$$

• Bethe states

Define $r(u) := \dfrac{\alpha(u)}{\delta(u)}$. If $v_1, \ldots, v_n$ satisfy the Bethe equations

$$r(v_i) \prod_{j=1, j\neq i}^{n} \frac{\sinh(v_i - v_j - \gamma)}{\sinh(v_i - v_j + \gamma)} = 1 \quad (i = 1, \ldots, n),$$

$\prod_{i=1}^{n} B(v_i)|0\rangle$ becomes an eigenvector of $t(u) = A(u) + D(u)$:

$$t(u) \prod_{i=1}^{n} B(v_i)|0\rangle = \left(\alpha(u) \prod_{i=1}^{n} f(v_i - u) + \delta(u) \prod_{i=1}^{n} f(u - v_i)\right) \prod_{i=1}^{n} B(v_i)|0\rangle,$$

where

$$f(u) = \frac{a(u)}{b(u)} = \frac{\sinh(u + \gamma)}{\sinh u}.$$

• The operators $A(u), B(u), C(u), D(u)$ are related to a row-to-row transfer matrix of the 6VM with open boundaries. For N free (namely, not required to satisfy the Bethe equations) variables, one can thus deduce that

$$\prod_{i=1}^{N} B(u_i)|0\rangle = Z_N(u_1, \ldots, u_N, \xi_1, \ldots, \xi_N)|\bar{0}\rangle,$$

where $|\bar{0}\rangle = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This is a key to connect the 6VM with DWBC and the algebraic Bethe ansatz of the finite XXZ chain.

Scalar product of Bethe and non-Bethe states

• Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ be free variables and $\boldsymbol{v} = (v_1, \ldots, v_n)$ satisfy the Bethe equations. The associated scalar product

$$S_n(\boldsymbol{u}, \boldsymbol{v}) = \langle 0| \prod_{i=1}^{n} C(u_i) \prod_{i=1}^{n} B(v_i) |0\rangle$$

of Bethe and non-Bethe states is known to have a determinant formula (Slavnov; Kitanine, Maillet & Terras):

$$S_n(\boldsymbol{u}, \boldsymbol{v}) = \frac{\prod_{i=1}^{n} \delta(u_i)\delta(v_i) \prod_{i,j=1}^{n} \sinh(u_i - v_j + \gamma)}{\prod_{1 \le i < j \le n} \sinh(u_i - u_j) \sinh(v_j - v_i)} \det(H_{ij})_{i,j=1}^{n},$$

where

$$H_{ij} = \frac{\sinh \gamma}{\sinh(u_i - v_j + \gamma) \sinh(u_i - v_j)} \left(1 - r(u_i) \prod_{k \neq i} \frac{\sinh(u_i - v_k - \gamma)}{\sinh(u_i - v_k + \gamma)} \right).$$

- Equivalent expression:

$$S_n(\boldsymbol{u},\boldsymbol{v}) = \frac{\sinh^N\gamma \prod_{i=1}^n \delta(v_i)}{\prod_{1\le i<j\le n}\sinh(u_i-u_j)\sinh(v_j-v_i)}\det(K_{ij})_{i,j=1}^n,$$

where

$$K_{ij} = \frac{\delta(u_i)\prod_{k\neq j}\sinh(u_i-v_k+\gamma) - \alpha(u_i)\prod_{k\neq j}\sinh(u_i-v_k-\gamma)}{\sinh(u_i-v_j)}.$$

Scalar product as KP tau function

These determinant formulas of the scalar product can be cast into a rational form with the new variables $x_i = e^{2u_i}$, $y_i = e^{2v_i}$ and $q = e^{-\gamma}$. The main part of $S_n(\boldsymbol{u}, \boldsymbol{v})$ thereby turns out to be of the form $\dfrac{\det(f_i(x_j))_{i,j=1}^n}{\Delta(x_1, \dots, x_n)}$, hence can be identified with a polynomial tau function of the KP hierarchy with time variables $t_n = \dfrac{1}{n} \sum_{i=1}^n x_i^n$.

5. Scalar product of (non-Bethe) states in models at $q = 0$

• Some solvable models (the phase model, the 4-vertex model, TASEP, etc.) has an R-matrix of the form

$$R(u-v) = \begin{pmatrix} f(u-v) & 0 & 0 & 0 \\ 0 & 1 & g(u-v) & 0 \\ 0 & g(u-v) & 0 & 0 \\ 0 & 0 & 0 & f(u-v) \end{pmatrix},$$

where

$$f(u-v) = \frac{u^2}{u^2 - v^2}, \quad g(u-v) = \frac{uv}{u^2 - v^2}.$$

This is achieved by a “crystal” limit, as $q \to 0$, of the R-matrix of the 6VM and the XXZ chain.

• The scalar product $S_n(\boldsymbol{u},\boldsymbol{v}) = \langle 0| \prod_{i=1}^{n} C(u_i) \prod_{i=1}^{n} B(v_i)|0\rangle$ with both $\boldsymbol{u} = (u_1,\ldots,u_n)$ and $\boldsymbol{v} = (v_1,\ldots,v_n)$ being free variables is known to have a determinant formula (Bogoliubov et al.),

$$S_n(\boldsymbol{u},\boldsymbol{v}) = C_n \prod_{1\le i<j\le n} \frac{u_i u_j}{u_i^2 - u_j^2} \frac{v_j v_i}{v_j^2 - v_i^2} \det(K_{ij})_{i,j=1}^{n},$$

where C_n is a simple factor, and K_{ij}'s are given by

$$K_{ij} = \frac{\delta(u_i)\alpha(v_j)(u_i/v_j)^{n-1} - \alpha(u_i)\delta(v_j)(v_j/u_i)^{n-1}}{(u_i^2 - v_j^2)/u_i v_j}.$$

By changing variables to $x_i = u_i^2$ and $y_i = v_i^2$, this function, too, can be interpreted as a KP tau function with respect to $t_n = \frac{1}{n}\sum_{i=1}^{n} x_i^n$ and $\bar{t}_n = \frac{1}{n}\sum_{i=1}^{n} y_i^n$ separately.

• In particular, for the phase model (equivalently, the 4-vertex model), the scalar product gives a generating function for counting "boxed plane partitions". Actually, this generating function coincides, up to a simple factor, with a single Schur function $s_{(N^n)}(u_1^2, \ldots, u_N^2, v_1^2, \ldots, v_N^2)$ of a rectangular Young diagram, and can be decomposed into a sum of products of two Schur functions as

$$S(\boldsymbol{u}, \boldsymbol{v}) = (\text{simple factor}) \sum_{\lambda \subset (N^n)} s_\lambda(u_1^2, \ldots, u_n^2) s_\lambda(v_1^{-2}, \ldots, v_n^{-2}).$$

This case has been studied in more detail by Zuparic, who pointed out a link with the Toda hierarchy.