

Melting crystal model

to emergence of integrable hierarchy
technical tools: free fermion, quantum, vertex operators

1. Undeformed model

$$Z = \sum_{\lambda \in P} s_{\lambda} (q-p)^2 Q^{|\lambda|}$$

(partition function)

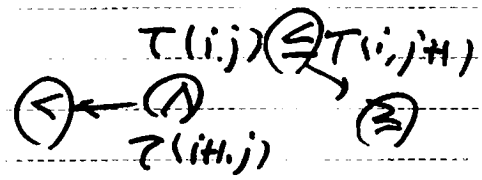
P : set of all partitions
 $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$: partition
 $|\lambda| = \sum \lambda_i$ $\longleftrightarrow$ Young diagram

$S_{\lambda}(X) = S_{\lambda}(x_1, x_2, \dots)$ Schur function
 symmetric function

$$= \sum_{T: \text{S.S. tableau of shape } \lambda} X^T$$

$$T = (T(i,j))_{(i,j) \in \lambda}$$

$$X^T = \prod_{(i,j) \in \lambda} X_{T(i,j)}$$

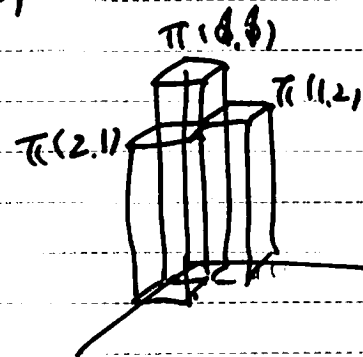


$$q-p = (q^{1/2}, q^{3/2}, \dots, q^{-(1/2)}, \dots) \quad (|q| < 1)$$

(or formal)

$$Q=1: Z = \sum_{\pi \in PP} q^{|\pi|}$$

$\pi = (\pi_{ij})_{i,j=0}^{\infty}$ plane partition



$$\pi_{ij} \geq \pi_{i+1,j} \geq \pi_{i,j+1}$$

3D Young diagram

By Cauchy identity

$$(x \rightarrow q-p, y \rightarrow Qq-p)$$

$$\sum_{\lambda} S_{\lambda}(x_1, x_2, \dots) S_{\lambda}(y_1, y_2, \dots) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1}$$

$$Z = \prod_{i,j=1}^{\infty} (1 - Qq^{i+j})^{-1} = \prod_{n=1}^{\infty} (1 - Qq^n)^{-n}$$

2]

2. Deformation model

"time"

deformation parameters $t = (t_1, t_2, \dots)$ continuous $s \in \mathbb{Z}$ discrete

potentials $\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} q^k (\lambda_i + s - i + 1) - q^k (-i + 1)$

(formal expression)

$$= \sum_{i=1}^{\infty} (q^k (\lambda_i + s - i + 1) - q^k (s - i + 1))$$

finite sum
(resummation)polynomial in q

motivation / origin

q-analogue of Casimir invariants

SDUV SUSY gauge theory
contribution of Wilson loops

$$Z(s, t) = \sum_{\lambda \in p} S_{\lambda}(q^{-p})^2 Q^{|\lambda| + s(s+1)/2} e^{\Phi(\lambda, s, t)}$$

$$\Phi(\lambda, s, t) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s)$$

Fact (K.T. V. Narasimha 2007) $\left\{ \begin{array}{l} \text{CMP 285 (2009), 445-468} \\ \text{Adv. Stud. Pure Math. 59 (2010), 201-223} \end{array} \right.$

$$Z(s, t) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}\right) q^{-s(s+1)(2s+1)/6} \tau(s, L(t))$$

$$L(t) = (-t_1, t_2, -t_3, \dots, t_1, k, t_k, \dots)$$

$$\tau(s, t) = \langle s | \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g | s \rangle = \langle s | g \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) | s \rangle$$

— a KP (or 1D Toda)
tau function3. Tau functions $\langle s |, |s \rangle$: ground states in charge- s sector
of fermionic Fock space

$$\psi_n, \psi_n^* \ (n \in \mathbb{Z}) \quad \psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m+n,0}, \quad \psi_m \psi_n + \psi_n \psi_m = 0,$$

$$\langle s | = \langle -\infty | \dots \psi_{s-2}^* \psi_{s-1}^* \psi_0^*$$

$$|s \rangle = \psi_{-s} \psi_{-s+1} \dots |-\infty \rangle$$

slightly unusual
convention

$$J_k = \sum_{n \in \mathbb{Z}} \psi_n \psi_{k-n}^*$$

$$[J_m, J_n] = m \delta_{m+n,0}$$

$$\mathcal{H} = \bigoplus_{s \in \mathbb{Z}} \mathcal{H}_s$$

normal ordering $(\psi_m \psi_n^* : \psi_m \psi_n^* - \langle 0 | \psi_m \psi_n^* | 0 \rangle)$

$$g = q^{W_0/2} \Gamma_{-}(q^{-p}) \Gamma_{+}(q^{-p}) Q^{L_0} \Gamma_{-}(q^{-p}) \Gamma_{+}(q^{-p}) q^{W_0/2} \in GL(\infty)$$

where $L_0 = \sum_{n \in \mathbb{Z}} n : \psi_n \psi_n^{\dagger}$, $W_0 = \sum_{n \in \mathbb{Z}} n^2 : \psi_n \psi_n^{\dagger}$

$$\Gamma_{\pm}(x) = \prod_{i=1}^{\infty} \Gamma_{\pm}(x_i), \quad \Gamma_{\pm}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right)$$

(vertex operators)
QORV

KP tau function

$$\tau_{KP}(\#) = \underbrace{\langle 0 | \exp\left(\sum_{k=1}^{\infty} t_k J_k\right)}_{\gamma_{+}(\#)} g \underbrace{| 0 \rangle}_{| W \rangle}, \quad g \in GL(\infty)$$

$| W \rangle \in \hat{Gr}$
 $\uparrow$
 $W \subset V = \mathbb{C}((z^{-1}))$
($W \in Gr$)

2D Toda tau function

$$\tau_{Toda}(s, \#, \tilde{\#}) = \langle s | \underbrace{\exp\left(\sum_{k=1}^{\infty} t_k J_k\right)}_{\gamma_{+}(\#)} g \underbrace{\exp\left(\sum_{k=1}^{\infty} \tilde{t}_k J_{-k}\right)}_{\gamma_{-}(\tilde{\#})} | s \rangle$$

(switched for 2-KP interpretation)

1D reduction:

$$J_k g = g J_{-k} \quad (k \geq 1, 2, \dots)$$

$$\Rightarrow \tau_{Toda}(s, \#, \tilde{\#}) = \tau_{Toda}(s, \# - \tilde{\#}, 0) \quad - \text{symmetrized KP tau function}$$

(rem) $| W \rangle = |\lambda\rangle = \psi_{-\lambda_1 - s} \psi_{-\lambda_2 - s + 1} \dots \psi_{-\lambda_i - s + i - 1} \dots | -\infty \rangle$

$$\tau(\#) = \langle 0 | \underbrace{\exp\left(\sum_{k=1}^{\infty} t_k J_k\right)}_{\gamma_{+}(\#)} | \lambda \rangle = S_2(\#)$$

$(t_k = \frac{1}{k} \sum_{i=1}^{\infty} x_i)$

4. How to derive $\tau(s, \#)$ from $Z(s, \#)$

(1) Start from fermionic expression of $Z(s, \#)$

cf. Okounkov, Reshetkin, Vafa, crystal model of topological strings

$$Z(s, \#) = \langle s | \Gamma_{+}(q^{-p}) Q^{L_0} H(\#) \Gamma_{-}(q^{-p}) | s \rangle,$$

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[Key] Shift symmetry in quantum torus algebra

$$V_m^{(k)} = q^{-km/2} \sum_{n \in \mathbb{Z}} q^{kn} : \psi_{m-n} \psi_n^k : \quad (k, m \in \mathbb{Z})$$

(— specialization of Okounkov & Pandharipande's $\sum_m$ quantum)

$$\downarrow 8/2q \quad H_k = V_0^{(k)}, \quad J_k = V_k^{(1)}$$

• "Shift symmetries"

$$\begin{aligned} (1) \quad \Gamma_-(q^P) / \Gamma_-(q^{-P}) (V_m^{(k)} - \frac{q^k}{1-q^k} \delta_{m,0}) \\ = (-1)^k (V_{m+k}^{(k)} - \frac{q^k}{1-q^k} \delta_{m+k,0}) \Gamma_-(q^P) / \Gamma_-(q^{-P}) \quad \text{for } k > 0, m \in \mathbb{Z} \end{aligned}$$

$$(2) \quad q^{w_0/2} V_m^{(k)} q^{-w_0/2} = V_m^{(k-m)} \quad \text{for } k \in \mathbb{Z}, m \in \mathbb{Z}$$

5. Modified model

$$Z' = \sum_{\lambda} s_{\lambda}(q^{-P}) s_{\lambda_2}(q^{-P}) Q^{|\lambda|} = \prod_{n=1}^{\infty} (1 + Q q^n)^n$$

~ conjugate

$$Z'(s, t, \tilde{t}) = \sum_{\lambda} s_{\lambda}(q^{-P}) s_{\lambda_2}(q^{-P}) Q^{|\lambda| + s|\lambda| + 1/2} e^{\Phi(s, t, \tilde{t})}$$

$$\Phi(s, t, \tilde{t}) = \sum_{k=1}^{\infty} \Phi_k(t, \tilde{t}) + \sum_{k=1}^{\infty} \tilde{t}_k \Phi_{-k}(t, s)$$

Foot (K.T., 2012-13) (J Phys A 46 (2013), 245202)

$$(i) \quad Z'(s, t, \tilde{t}) = \exp \left(\sum_{k=1}^{\infty} \frac{\tilde{t}_k t_k - \tilde{t}_k}{1-q^k} \right) Z'(s, t, \tilde{t}),$$

$$Z'(s, t, \tilde{t}) = \langle s | \exp \left(\sum_{k=1}^{\infty} \tilde{t}_k J_k \right) q' \exp \left(\sum_{k=1}^{\infty} \tilde{t}_k J_{-k} \right) | s \rangle$$

$\gamma'_+(t) \quad \gamma'_-(\tilde{t})$ (2D Toda tau function)

(ii) τ is the tau function of the Ablowitz-Ladik (or Relativistic Toda) hierarchy embedded in the 2D Toda hierarchy.

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$$g' = q^{W_0/2} \Gamma_-(q^{-p}) \Gamma_+(q^{-p}) \otimes L_0 \Gamma_-(q^{-p}) \Gamma_+(q^{-p}) q^{-W_0/2}$$

$$\Gamma_{\pm}'(x) = \prod_{i=1}^{\infty} \Gamma_{\pm}'(x_i), \quad \Gamma_{\pm}'(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k}\right)$$

(Bryan & Young)

$$\langle 0 | \Gamma_{+}'(x) | \lambda \rangle = \langle \lambda | \Gamma_{-}'(x) | 0 \rangle = \tilde{s}_{\lambda}(\mathbb{D})$$

• $z' \rightarrow \tau'$: another set of shift symmetry (for $V_m^{(-k)}$'s)

• Relation to AL (RT) hierarchy

~~Ex~~ ~~Ex~~ fermion bilinears $\leftrightarrow$ $\mathbb{R} \times \mathbb{R}$ matrices
(exponential of)

(b) Lax formalism of 2D Toda hierarchy
by $\mathbb{R} \times \mathbb{R}$ matrices

(c) Characterization of AL/RT hierarchy

$$\begin{aligned} L &= BC^{-1}, & B &= \Lambda - b, & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ \tilde{L} &= -CB^{-1}, & C &= 1 + c\Lambda^{-1} & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \end{aligned}$$

(Brini, Carlet, Rossi, Physica D 241 (2012), 2156-67)

Remarks ○ Geometric backgrounds

Brini, Local Gromov-Witten theory of $(\mathbb{C}P^1)$ and integrable hierarchies, CMP 313 (2012), 571-605
— conjecture on relevance of AL(RT)

Brini, Carlet, Rossi, loc. cit.

— more evidence from Frobenius structures

generalization to orbifold model

Relation to topological vertex

証明
要旨

$\Gamma_-(q^{-p}) \Gamma_+(q^{-p})$
 $\Gamma_-(q^{-p}) \Gamma_+^*(q^p)$

building blocks of topological vertex

$(q^{-p} \rightarrow q^{-2-p})$

$\mathcal{Z}(0, t)$
 $\mathcal{Z}'(0, t, \tilde{t})$

generating functions of open string amplitudes

$(\tilde{t}_h = \frac{1}{k} \mathcal{Z}(\chi_h), \tilde{\chi}_h = \frac{1}{k} \mathcal{Z}(\tilde{\chi}_h^{-1}))$

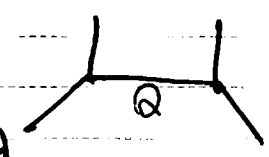
$\langle 0 | \exp(\sum_{h=1}^{\infty} \tilde{t}_h \chi_h) q^{W_0/2} \Gamma_-(q^{-p}) \Gamma_+(q^{-p}) Q^{L_0} \Gamma_-(q^{-p}) \Gamma_+(q^{-p}) q^{W_0/2} | 0 \rangle$

$q^{K/2} V_m^{(k)} q^{-L(A)}$
 $= q^m V_m^{(k-m)}$

$W_0 = \sum n^2 : \psi_{-n} \psi_n^* :$
 $K = \sum (n - \chi_0)^2 : \psi_{-n} \psi_n^* :$

almost the same
(K can be used in place of W_0)

$\langle 0 | \Gamma_+(\chi) q^{K/2} \Gamma_-(q^{-p}) \Gamma_+(q^{-p}) Q^{L_0} \Gamma_-(q^{-p}) \Gamma_+(q^{-p}) | 0 \rangle$
 $= \sum_{\lambda} S_{\lambda}(\chi) q^{K/2} \langle \lambda | \Gamma_+ \Gamma_- 0^{L_0} \Gamma_- \Gamma_+ | 0 \rangle$



Moreover, if χ is specialized to $-Qq^{-p}$, this happens to give the partition function of another diagram

$\chi = -Qq^{-p}$
 $e^{\Phi(\lambda, t)}, t = \frac{(-Q)^k q^{k/2}}{k(1-q^h)} \dots e^{H(t)}$

external potential ... geometry change

has a special meaning ($\rightarrow$ lecture 3)

$q^{K/2} V_m^{(k)} q^{-L(A)} = q^{-km/2} \sum_n q^{kn} q^{(n-m-k)/2 - (n-k)/2} : \psi_{-n} \psi_n^* :$
 $= q^m q^{-(k-m)m/2} \sum_n q^{(k-m)n} : \psi_{-n} \psi_n^* : = q^m V_m^{(k-m)}$

tan functions in topological vertex formalism

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(Nakatsu, T.
J. Phys. A 49
(2016) 023201
paper from arXiv
projection
of branes
where some
action is kept

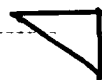
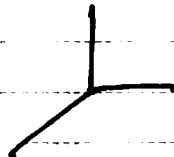
1. Topological vertex

local toric $\subset Y$ three fold $\longleftrightarrow$ web diagram

$$X = \mathbb{C}^{N+3} // U(1)^N = \bigwedge_{A=1}^N \mu_A^*(t_A) / U(1)^N \quad \downarrow \text{dual}$$

Symplectic quotient toric diagram

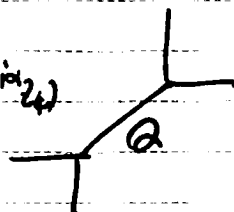
Examples: (1) $X = \mathbb{C}^3$



$$(1) X = \text{resolved conifold} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1) \rightarrow \mathbb{C}P^1$$

$$|z_1|^2 + |z_4|^2 = |z_2|^2 + |z_3|^2 = t$$

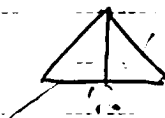
$$(z_1, z_2, z_3, z_4) \rightarrow (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3, e^{i\theta_4} z_4)$$



θ : Kähler parameter



$$(2) X \simeq \mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{C}P^1$$



vertex $\longleftrightarrow \nabla \longleftrightarrow \mathbb{C}P^3$

edge $\longleftrightarrow$ edge $\longleftrightarrow \mathbb{S}^2$

(re) topological vertex formalism of A-model

(AKMV
ADKMV)

web diagram

+ partition
external lines (legs)

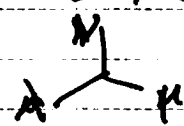
unnormalized
string amplitude
(partition function)

Z

(stack of D-branes)

(a) vertex

vertex weight $C_{\lambda\mu\nu} = q^{K(\mu)/2} s_{\lambda\nu}(q^p)$



$$\times \sum_{\eta \in \mathcal{P}} s_{\lambda\mu\eta}(q^{-v-p}) s_{\eta\nu\eta}(q^{-v-p})$$

famous expression

$$C_{\lambda\mu\nu} = s_{\lambda\nu}(q^p)$$

$$\times \langle \lambda | \Gamma_{-}(q^{-v-p}) \Gamma_{+}(q^{-v-p}) q^{K(\mu)/2} | \mu \rangle$$

cf. yesterday's talk

$$K(\mu) = \sum_{i \geq 1} \mu_i (\mu_i - 2i + 1) = 2 \sum_{(j) \in \mu} (j - i)$$

$s_{\lambda\mu}(x)$: skew Schur function

$$q^{-v-p} = (q^{-v_1+p_1}, q^{-v_2+p_2}, \dots, q^{-v_i+p_i}, \dots)$$

$$\langle \lambda | \Gamma_{-}(x) | \mu \rangle$$

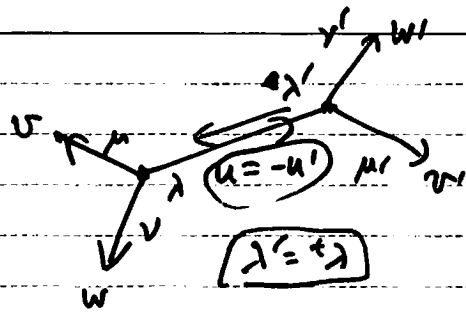
$$\langle \mu | \Gamma_{+}(x) | \lambda \rangle$$

(b) edge (internal lines)

edge weight

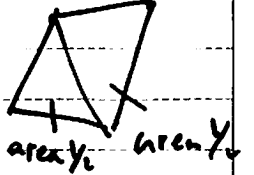
$$(-Q)^{|\lambda|} (-1)^{n(u)} q^{-n(u)/2}$$

framing factor



$$u+v+w=0$$

$$u'+v'+w'=0$$



$$n = v' \wedge v = w' \wedge w \quad (\det \begin{pmatrix} v \\ w \end{pmatrix} = \det \begin{pmatrix} w' \\ v' \end{pmatrix})$$

(c) summation over all partitions on internal lines

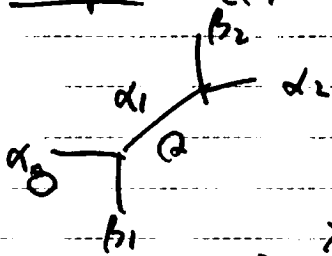
Cyclic symmetry
 $C_{\mu\nu} = C_{\nu\mu}$
 $= C_{\mu\mu}$

Examples

(1) resolved vertex $\simeq \mathcal{O}(1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$

$$\sum_{\beta_1 \beta_2} \alpha_0 \alpha_2 = \sum_{\alpha_1 \in \mathcal{P}} C_{\alpha_1, \alpha_0 \beta_1} (-Q)^{|\alpha_1|} C_{\alpha_1, \alpha_2 \beta_2}$$

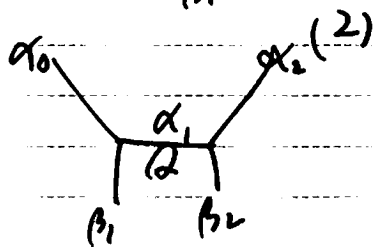
(framing factor is trivial)



$X \simeq \mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$

$$\sum_{\beta_1 \beta_2} \alpha_0 \alpha_2 = \sum_{\alpha_1 \in \mathcal{P}} C_{\alpha_1, \alpha_0 \beta_1} (-Q)^{|\alpha_1|} (-1)^{|\alpha_1|} q^{-K(\alpha_1)/2}$$

framing factor

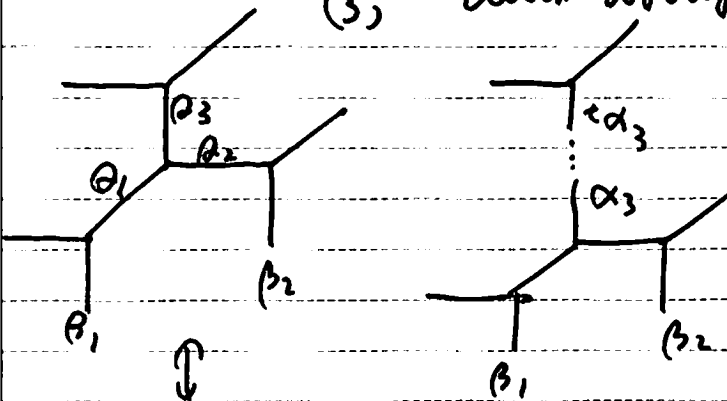


(3) closed topological vertex (spineless)

$$\sum_{\beta_1 \beta_2} \alpha_3 = \sum_{\alpha_3 \in \mathcal{P}} \sum_{\beta_1 \beta_2 \beta_3} (-Q_3)^{|\alpha_3|}$$

$\leftarrow \sum_{\beta_1 \beta_2 \beta_3} \alpha_3$ (partial sum) $\times C_{\alpha_3, \beta_1 \beta_2}$

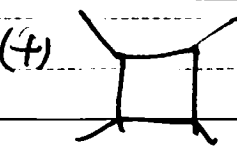
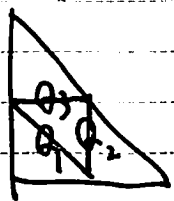
(framing factor is trivial)



$$C_{\alpha_3, \beta_1 \beta_2} = S_{\alpha_3}(q^{-p}) \dots C^3$$

(1) (2) — strip geometry

(3) (4) — not strip



local $\mathbb{CP}^1 \times \mathbb{CP}^1$

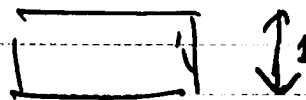
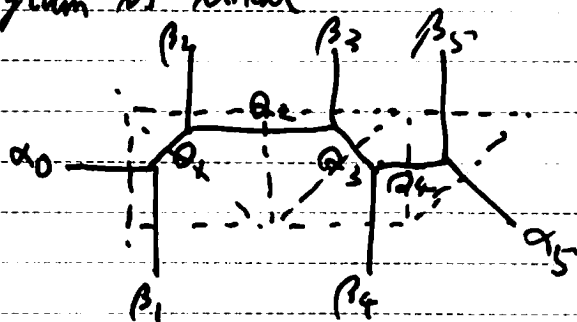


2. Strip geometry

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Case where toric diagram is a triangulation of a strip

web diagram is "linear"



vertical legs $\beta_1, \dots, \beta_N$
non-vertical legs α_0, α_N
internal lines $\alpha_1, \dots, \alpha_{N-1}$
Kähler param. $\alpha_1, \dots, \alpha_{N-1}$

$$\sum \alpha_0 \alpha_N = \sum_{\alpha_1, \dots, \alpha_{N-1} \in \mathcal{P}} \dots$$

$$\textcircled{1} \quad Z_{\beta_1, \dots, \beta_N}^{\alpha_0, \alpha_N} = S_{\beta_1}(q^{-p}) \dots S_{\beta_N}(q^{-p}) \prod_{1 \leq m < n \leq N} (1 - \alpha_{mn} q^{-\beta_1^{(m)} - \beta_j^{(n)} + i j})^{\sigma_{mn}}$$

(Igel &

Kurathin)

$$\alpha_{mn} = \alpha_m \alpha_{m+1} \dots \alpha_{n-1}$$

$$\beta_i^{(n)} = \begin{cases} \beta_n & \text{if } \sigma_n = +1 \\ +\beta_n & \text{if } \sigma_n = -1 \end{cases}$$

$$\sigma_n = \begin{cases} +1 \\ -1 \end{cases}$$

$$\beta_n = \begin{cases} \beta_n \\ -\beta_n \end{cases}$$

$$\textcircled{2} \quad Z_{\beta_1, \beta_N}^{\alpha_0, \alpha_N} = \begin{cases} 1 & (\sigma_1 = +1) \\ q^{K(\alpha_0)/2} & (\sigma_1 = -1) \end{cases} \times \begin{cases} q^{K(\alpha_N)/2} & (\sigma_N = +1) \\ 1 & (\sigma_N = -1) \end{cases}$$

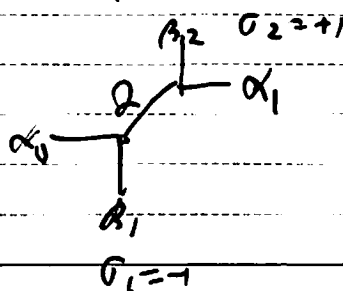
$$\times \langle \alpha_0 | \Gamma_{-}^{\sigma_1}(q^{-\beta_1^{(1)}} - p) \Gamma_{+}^{\sigma_1}(q^{-\beta_1^{(1)}} - p) (\sigma_1 \alpha_1 \sigma_2)^{L_0} \dots$$

$$\dots (\sigma_{N-1} \alpha_{N-1} \sigma_N)^{L_0} \Gamma_{-}^{\sigma_N}(q^{-\beta_N^{(N)}} - p) \Gamma_{+}^{\sigma_N}(q^{-\beta_N^{(N)}} - p) | \alpha_N \rangle$$

(Frenkel-Kurathin; Nagao; Subbarashi)

$$\Gamma_{\pm}^{\sigma}(x) = \begin{cases} \Gamma_{\pm}(x) & (\sigma = +1) \\ \Gamma'_{\pm}(x) & (\sigma = -1) \end{cases}$$

Ex. $N=1$ ($\sigma_1 = +1$) $Z_{\beta_1}^{\alpha_0, \alpha_1} = C_{\alpha_0, \alpha_1, \beta_1} = q^{K(\alpha_0)/2} S_{\beta_1}(q^{-p}) \langle \alpha_0 | \Gamma_{-}^{\sigma_1}(q^{-\beta_1^{(1)}} - p) \Gamma_{+}^{\sigma_1}(q^{-\beta_1^{(1)}} - p) | \alpha_1 \rangle$



$$Z_{\beta_1, \beta_2}^{\alpha_0, \alpha_1} = S_{\beta_1}(q^{-p}) S_{\beta_2}(q^{-p}) \prod_{i,j=1}^{\infty} (1 - \alpha_{ij} q^{-\beta_1^{(i)} - \beta_2^{(j)} + i j})^{\sigma_{ij}}$$

3 Tan Functions

ՀԱՆՐԱՊԵՏԱԿԱՆ ԿՈՄԻՏԵ ԲՆԾ 3 824/19

$$(1) \sum_{\alpha_0, \alpha_N} S_{\alpha_0}(x) S_{\alpha_N}(y) \sum_{\beta_1, \beta_N} \alpha_N$$

$$= \langle 0 | \Gamma_+(x) \left\{ \begin{matrix} 1 & (\sigma_1 = +1) \\ q^{K/2} & (\sigma_1 = -1) \end{matrix} \right\} \Gamma_-^{\sigma_1}(q^{-\beta_1/2} p) \Gamma_+^{\sigma_1}(q^{-\beta_1/2} p) (\sigma_1 \sigma_2) \dots$$

→ Transfer function of 2-comp. KP

$$(t_k = \frac{1}{k} \sum_{i=1}^k x_i, \tilde{t}_k = -\frac{1}{k} \sum_{i=1}^k y_i)$$

$$(2) \int_{\beta_n} S_t(x) Z^{\phi\phi} \dots \phi \beta_n \phi \dots / Z^{0\dots\phi}$$

$$= \sum_{\beta_n} S_{\beta_n}(x) S_{\beta_n}(q^{-p}) \frac{\prod_{m < n} (1 - q_{nm} q^{-\beta_j^{(n)} + i+j-1})^{-\sigma_n \sigma_m}}{\prod_{j=1}^{\infty} (1 - q_{nn} q^{-\beta_j^{(n)} + i+j-1})^{-\sigma_n \sigma_n}} \times \prod_{m > n} \prod_{j=1}^{\infty} (1 - q_{nm} q^{-\beta_j^{(n)} + i+j-1})^{-\sigma_n \sigma_m} \prod_{j=1}^{\infty} (1 - q_{nn} q^{-\beta_j^{(n)} + i+j-1})^{-\sigma_n \sigma_n}$$

Fact: This is a ran function of KPhierarchy

(non-trivial)

with a formative expression

$$Z_n(x) = \langle 0 | h[-(q^{-p})] | 0 \rangle,$$

$$h = \exp\left(\sum_n \log h_n! \psi_n \psi_n^*\right)$$

Show a few examples, then a full on save

Example

$$Z_B = S_{t_B}(q-p)$$

$$\tau(x) = \sum_{\beta} S_{\alpha\beta}(x) S_{\beta\alpha}(y^{-1})$$

(without the factors)

$$\prod_{m \leq n} () \times \prod_{m \geq n} (\dots)$$

$$f_{\text{ensemble}} \leftarrow f_{\text{model}} = \prod_{j=1}^{\infty} (1 - q^{i-k_2} x_j)^{-1}$$

$$= \langle 0 | \Gamma_c(x) \Gamma_c(x) | 0 \rangle = \exp \left(\sum_{k=1}^{\infty} \frac{g^{k/2} x_{k1}}{k(1-q^k)} \right)$$

$$t_k = \frac{1}{k} \sum_{i=1}^k x_i^h$$

Guayana del Sur

exponential (捷) tan function

(almost trivial)

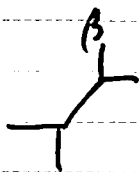
• (can further)

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Example resolved conifold ($N=2, \sigma_1=-1, \sigma_2=1$)

$n=2$ ($n=1$ is effectively the same)



~~vertex~~

$$Z_\beta = S_{t_\beta}(q^{-p}) \prod_{i,j=1}^{\infty} (1 - Q q^{-\beta i + (i+j)-1}) \quad Z_\beta / Z_\phi$$

$$Z(\lambda) = \sum_{\beta} S_{t_\beta}(\lambda) S_{t_\beta}(q^{-p}) \frac{\prod_{i,j=1}^{\infty} (1 - Q q^{-\beta i + (i+j)-1})}{\prod_{i,j=1}^{\infty} (1 - Q q^{i+j-1})} \quad \text{"(normalized amplitudes)"} \quad \text{Nakata.T.}$$

[Formula] $\rightarrow$

$$\frac{\prod_{i,j=1}^{\infty} (1 - Q q^{-\lambda i + (i+j)-1})}{\prod_{i,j=1}^{\infty} (1 - Q q^{i+j-1})} = \langle t_\lambda | \exp \left(\sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} V_0^{(k)} \right) | t_\lambda \rangle$$

Proof:

$e^{H(\lambda)}$ with $t_k = \frac{Q^k}{k(1-q^k)}$
in Lecture 1

$$\prod_{i,j=1}^{\infty} (1 - Q q^{-\lambda i + (i+j)-1}) = \exp \left(\sum_{k=1}^{\infty} \log \prod_{i,j=1}^{\infty} (1 - Q q^{-\lambda i + (i+j)-1}) \right)$$

$$= \exp \left(- \sum_{k=1}^{\infty} \sum_{i,j=1}^{\infty} \frac{(Q q^{-\lambda i + (i+j)-1})^k}{k} \right) = \exp \left(- \sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} \frac{k(\lambda i + (i+j)-1)}{q^{k(i+j-1)}} \right)$$

$$\frac{\prod_{i,j=1}^{\infty} (1 - Q q^{-\lambda i + (i+j)-1})}{\prod_{i,j=1}^{\infty} (1 - Q q^{i+j-1})} = \exp \left(- \sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} \left(\sum_{i,j=1}^{\infty} q^{i+j-1} - \sum_{i,j=1}^{\infty} q^{i+j-1} \right) \right)$$

(somewhat tricky)

$$= \exp \left(- \sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} V_0^{(k)} \right) = \langle t_\lambda | V_0^{(k)} | t_\lambda \rangle$$

$$= \langle t_\lambda | \exp \left(\sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} V_0^{(k)} \right) | t_\lambda \rangle \quad \square$$

Remark another formula

$$\frac{\prod_{i,j=1}^{\infty} (1 - Q q^{-\lambda i + (i+j)-1})}{\prod_{i,j=1}^{\infty} (1 - Q q^{i+j-1})} = \langle t_\lambda | \exp \left(- \sum_{k=1}^{\infty} \frac{Q^k q^k}{k(1-q^k)} V_0^{(k)} \right) | t_\lambda \rangle$$

(More straightforward)

$$\tau(x) = \sum_{\beta} \langle 0 | \Gamma_{\pm}(x) |^{\pm} \lambda \rangle \langle {}^{\pm} \lambda | \exp \left(\sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} V_0^{(k)} \right) |^{\pm} \lambda \rangle \langle {}^{\pm} \lambda | \Gamma_{\mp}(q^{-1}) | 0 \rangle$$

$$= \langle 0 | \Gamma_{\pm}(x) | h \Gamma_{\mp}(q^{-1}) | 0 \rangle,$$

where $h = \exp \left(\sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} V_0^{(k)} \right)$ ($\rightarrow$ key to KS operators Section 3)

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General strip geometry $22^2 \mp 13$??

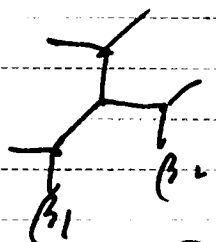
$$\tau(x) = \langle 0 | \Gamma_{\pm}(x) | h \Gamma_{\mp}(q^{-1}) | 0 \rangle, \text{ where,}$$

If $\sigma_n = +1$,

$$h = \exp \left(- \sum_{m < n} \sigma_m \sum_{k=1}^{\infty} \frac{Q_{mn}^k}{k(1-q^k)} V_0^{(k)} + \sum_{m > n} \sigma_m \sum_{k=1}^{\infty} \frac{Q_{nm}^k q^k}{k(1-q^k)} V_0^{(-k)} \right).$$

If $\sigma_n = -1$,

$$h = \exp \left(- \sum_{m < n} \sigma_m \sum_{k=1}^{\infty} \frac{Q_{mn}^k q^k}{k(1-q^k)} V_0^{(-k)} + \sum_{m > n} \sigma_m \sum_{k=1}^{\infty} \frac{Q_{nm}^k}{k(1-q^k)} V_0^{(k)} \right).$$

Remarks. $\tau_0(x)$ and $\tau_{n+1}(x)$ are distinct. (much similar to $\mathbb{CP}^3$) $22^2 \mp 13$??5. Closed topological vertex

$$\begin{cases} \tau_1(x) = \sum_{\beta_1} S_{\beta_1}(x) Z_{\beta_1, \beta} / Z_{\beta, 0} \\ \tau_2(x) = \sum_{\beta_2} S_{\beta_2}(x) Z_{\beta, \beta_2} / Z_{\beta, \beta} \end{cases}$$

Fact

(NT 2015)

$$Z_{\beta_1 \beta_2} = q^{K(\beta_2)/2} \prod_{i,j=1}^{\infty} (1 - Q_i Q_j^{-\beta_{1i} - \beta_{2j} + i + j})^{-1}$$

τ Nakata, K. T
J Phys A 49
(2016) 025201

$$\times \langle {}^{\pm} \beta_1 | \Gamma_{-}(q^{-1}) \Gamma_{+}(q^{-1}) (-Q_1)^{L_0} \Gamma_{-}'(q^{-1}) \Gamma_{+}'(q^{-1})$$

$$\times (-Q_3)^{L_0} \Gamma_{-}(q^{-1}) \Gamma_{+}(q^{-1}) (-Q_2)^{L_0} \Gamma_{-}'(q^{-1}) \Gamma_{+}'(q^{-1}) | {}^{\pm} \beta_2 \rangle$$

 $\tau_1(x), \tau_2(x)$ are KP tau functions

Fermionic expressions are more complicated.

~~$$\Gamma_1(x) = \langle 0 | \Gamma_1(x) | 0 \rangle, \quad \Gamma_2(x) = \langle 0 | \Gamma_2(x) | 0 \rangle$$~~

$$\begin{aligned} \Gamma_1(x) &= \langle 0 | \Gamma_1(x) g_1 | 0 \rangle, \\ \Gamma_2(x) &= \langle 0 | \Gamma_2(x) g_2 | 0 \rangle, \quad \Gamma = \langle 0 | g_2 \Gamma_2(x) | 0 \rangle \end{aligned}$$

$$g_1 = \exp\left(-\sum_{k=1}^{\infty} \frac{(Q_1 Q_2)^k}{k(1-q^k)} V_0^{(k)}\right) \Gamma_-(q^{-P}) \Gamma_-'(-Q_1 q^{-P}) \Gamma_-(Q_1 Q_3 q^{-P})$$

$$\times \Gamma_-'(-Q_1 Q_2 Q_3 q^{-P})$$

$$g_2 = q^{\frac{K/2}{2}} \exp\left(\sum_{k=1}^{\infty} \frac{(Q_1 Q_2 Q_3)^k}{k(1-q^k)} V_0^{(k)}\right) \Gamma_-'(q^{-P}) \Gamma_-(-Q_2 q^{-P}) \Gamma_-'(Q_2 Q_3 q^{-P})$$

$$\times \Gamma_-(-Q_1 Q_2 Q_3 q^{-P})$$

[Derivative] $\prod_{i=1}^n (1 - Q_i Q_j q^{-P_i P_j})^{-1}$

$$= \langle \beta_1 | \exp\left(-\sum_{k=1}^{\infty} \frac{(Q_1 Q_2)^k}{k(1-q^k)} V_0^{(k)}\right) | \beta_1 \rangle,$$

$$\langle \beta_1 | \Gamma_-(q^{-P}) \Gamma_+(q^{-P}) (-Q_1)^0 \Gamma_-'(q^{-P}) \Gamma_+'(q^{-P}) (-Q_2)^0 \dots | 0 \rangle$$

└ move to the right ─┘

$$\Gamma_+(x) \Gamma_-(y) = \prod_{i,j=1}^n (1 - x_i y_j)^{-1} \Gamma_-(y) \Gamma_+(x), \text{ etc.}$$

$$\Gamma_+^0(x) | 0 \rangle = 0, \quad Q_1^0 \Gamma_-^0(x) = \Gamma_-^0(Q_1 x) Q_1^0, \dots$$

$$= \text{const.} \langle \beta_1 | \Gamma_-(q^{-P}) \Gamma_-'(-Q_1 q^{-P}) \Gamma_-(Q_1 Q_3 q^{-P}) \Gamma_-'(-Q_1 Q_2 Q_3 q^{-P}) | 0 \rangle$$

└ indep of β_1 ─┘

Remark Hypergeometric Toda tau function (Okounkov & Schenker)

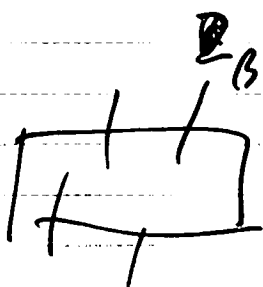
$$\tau(s, h, \vec{\mu}) = \langle s | \delta_+(h) h \gamma_-(\vec{\mu}) | s \rangle,$$

$$h = \exp\left(\sum_n \log h_n : \psi_{-n} \psi_n^* : \right),$$

[6]

(8/31)

Per. rotation groups now come in top. order formalism



$$\Psi(\eta) = \sum_{k=0}^{\infty} \eta^k Z_{(1k)} / Z_{\emptyset}$$

$\Downarrow ?$

$$\hat{H}(x, a^0) \Psi = 0$$

$H(1,4) \approx 20$ - min. curve

→ general strip geometry ✓

Q.7. arxiv:1301.4545

→ closed topological vertex ✓

Q.7. Nakahara J. Phys. A 49 (2016)

025201

入学年度	回生	学部	組	学科	コース	学号	氏名	学生証番号 (10桁)	ふりがな	氏名
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※ 氏名等の記入欄は、ペン又はボールペンで記入すること。

Kac-Schwartz operators and quantum curves

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1. Kac-Schwartz operators

Ref {Alexandrov, CMP 338 (2015), 195-244
Alexandrov et al
arXiv: 1512.07026

point of Jaco Grassmannian $\longleftrightarrow W \subset \mathbb{C}((x))$ ($x = z^{-1}$)
big cell of

$\exists$ admissible basis $\{\Phi_j(x)\}_{j=0}^{\infty}$

$$W = \text{Span}(\Phi_j(x))_{j=0}^{\infty}, \Phi_j(x) = \sum_{i=-j}^{\infty} \phi_{ij} x^i = \phi_{-j} x^j + \dots$$

Case 0.
(Kac, Schwarz)

(Rem ϕ_{ij} can be normalized to 1)

Suppose $\exists G: V \rightarrow V$, invertible, $\Phi_j(x) = G x^j$
(generating operator)

Case 1.1
(Alexandrov et al)

$$G = G(x, D) \quad D = x \frac{d}{dx}$$

differential operator of infinite order

cf. KS operator for Witten-Kontsevich
(P, q) 2D gravity
 $A = \frac{d}{dx} + x^2 + \dots$
 $B = x^2$

Define $A = G \cdot (-D) \cdot G^{-1}, \quad B = G \cdot x^{-1} \cdot G^{-1}$

then $\begin{cases} A \Phi_j(x) = j \Phi_j(x), & B \Phi_j(x) = \Phi_{j+1}(x) \\ [A, B] = B \end{cases}$

AWCW, BWCW

$A \Phi_0(x) = 0 \rightarrow$ quantum curve (for Hurwitz numbers)

Example: generating function of double Hurwitz numbers

$$|W\rangle = q^{\frac{1}{2}} e^{\beta K/2} \gamma_{-}(\tilde{\tau}) |0\rangle$$

$$L_0 \leftrightarrow D, \quad K \leftrightarrow (D - \gamma_2)^2, \quad J_k \leftrightarrow x^k$$

$$\begin{aligned} L_0 &\leftrightarrow \Delta - \log(\gamma_2) \in \mathbb{D} \\ K &\leftrightarrow (\Delta - \gamma_2)^2 \\ J_k &\leftrightarrow \Lambda^{-k} \end{aligned}$$

$$\Rightarrow G = e^{\beta D - x_2^2/2} \exp\left(\sum_{k=1}^{\infty} \tilde{\tau}_k x^k\right)$$

$$\begin{cases} A = -D + \sum_{k=1}^{\infty} \tilde{\tau}_k k e^{\beta k(k-1)x_2/2} x^k e^{k\beta D} \\ B = D^{-1} e^{-\beta D} x^{-1} \end{cases}$$

differential operators

$A \Psi(x) = 0$... cf. Mal'nev - Shubrik - Spitz
Comm. Math. Phys. 7 (2013) 1-19
for spin-1, q-Hurwitz

Case 2
topological
string on
local curve
CY3

$G = G(x, q^D)$, q -difference operators

Defin: $A = G \cdot q^{-D} \cdot G^{-1}$, $B = G \cdot x^{-1} \cdot G^{-1}$

Then $\begin{cases} A \Phi_j(z) = q^j \Phi_j(x), & B \Phi_j(z) = \Phi_{j+1}(z) \\ AB = qBA \end{cases}$

Examples

- $\mathbb{CP}^3$
- resolved conifold
- general strip geometry
- closed topological vertex

} below

$$A \Phi_0(z) = \Phi_0(x)$$

quantum mirror

$\Phi_0(x)$: generating function of
open string amplitudes

quantum mirror curves
derived

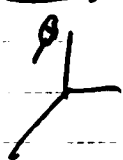
2. Examples

K.T.

arXiv:1301.4548

T. Nakatsu, K.T. — JHEP04 (2016), 025201

(1) $\mathbb{CP}^3$



open string amplitude

$$Z_\beta = S_{\beta}(q^{-D}), \quad Z_\emptyset = 1$$

$$\tau(x) = \sum_{\beta \in P} S_{\beta}(x) Z_{\beta} = \langle 0 | \Gamma_+(x) \Gamma_-(q^{-D}) | 0 \rangle$$

$$|W\rangle = \Gamma_-(q^D) |0\rangle$$

$$G \leftrightarrow g = \Gamma_-(q^{-D}) = \prod_{k=1}^{\infty} \exp \left(\frac{q^k}{k} J_k \right)$$

$$= \exp \left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{-k} \right)$$

$$G = \prod_{i=1}^{\infty} \exp \left(\sum_{h=1}^{\infty} \frac{q^{h(i-v_2)}}{h} x^h \right)$$

$$= \prod_{i=1}^{\infty} (1 - q^{i-v_2} x)^{-1} \quad \text{— quantum dilog}$$

$$A = G \cdot q^{-D} \cdot G^{-1} = \prod_{i=1}^{\infty} (1 - q^{i-v_2} x)^{-1} \cdot q^{-D} \cdot \prod_{i=1}^{\infty} (1 - q^{i-v_2} x)$$

$$= q^{-D} \prod_{i=1}^{\infty} (1 - q^{i+v_2} x)^{-1} \cdot \prod_{i=1}^{\infty} (1 - q^{i-v_2} x)$$

$$= q^{-D} (1 - q^{v_2} x)$$

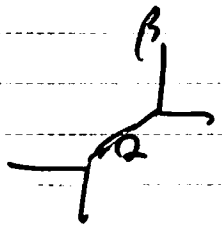
$$B = x^{-1} \quad \text{— trivial}$$

$$A \Phi_0(z) = \Phi_0(x) \leftrightarrow (1 - q^{v_2} x) \Phi_0(x) = q^0 \Phi_0(x) \leftrightarrow (1 - q^{v_2} x) \Phi_0(x) = \Phi_0(x)$$

classical limit: $1 - x - y = 0$



(2) resolvent isn't fold



$$Z_\beta = s_{+\beta}(q^{-\rho}) \prod_{i,j=1}^{\infty} (1 - Q q^{-\beta(i+j-1)})$$

$$Z(x) = \sum_{\beta} s_{+\beta}(Q) \underline{Z_\beta} / \underline{Z_\phi} \quad \text{normalized amplitudes}$$

$$= \langle 0 | \Gamma_+(x) | h, \Gamma_-(q^{-\rho}) | 0 \rangle$$

$$h = \exp \left(\sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} V_0^{(k)} \right)$$

$V_0^{(k)} \leftrightarrow q^{kD} \leftrightarrow -q^{hD}$
 $\sum q^{kD} \dots$

$$\dots \rightarrow g = h \Gamma_-(q^{-\rho})$$

$V_0^{(k)} \leftrightarrow q^{kD}$

$$G = \exp \left(\sum_{k=1}^{\infty} \frac{Q^k}{k(1-q^k)} q^{kD} \right) \exp \left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} x^k \right)$$

$$= \prod_{i=1}^{\infty} (1 - Q q^{i-1+D})^{-1} \cdot \prod_{i=1}^{\infty} (1 - q^{i-1/2} x)^{-1}$$

$$\Phi_j(x) = \prod_{i=1}^{\infty} (1 - Q q^{i-1+D})^{-1} \left(x^{-j} \prod_{i=1}^{\infty} (1 - q^{i-1/2} x)^{-1} \right)$$

$\sum_{k=0}^{\infty} \frac{q^{k/2}}{(q; q)_k} x^{k-j}$

$\leftarrow \begin{matrix} \text{expansion} \\ \text{of } q\text{-digamma} \end{matrix}$

$$= \sum_{k=0}^{\infty} \frac{q^{k/2}}{(q; q)_k} \prod_{i=1}^{\infty} (1 - Q q^{i-1+D})^{-1} x^{k-j}$$

$\nearrow$ q -hypergeometric (basic hypergeometric)

$$= \underbrace{\prod_{i=1}^{\infty} (1 - q^{i-j+D})^{-1}}_{\text{constant}} \cdot \sum_{k=0}^{\infty} \frac{(Q q^{-j}; q)_k q^{k/2}}{(q; q)_k} x^{k-j}$$

$$(a; q)_k = \begin{cases} (1-a)(1-qa) \dots (1-q^{k-1}a) & (k > 0) \\ 1 & (k = 0) \end{cases}$$

$$\left(\rightarrow (q; q)_k = (1-q)(1-q^2) \dots (1-q^k) \right) \quad \cdot \text{Zhuang (2012, 2015)}$$

$$\Phi_0(x) = \text{const} \sum_{k=0}^{\infty} \frac{(Q; q)_k q^{k/2}}{(q; q)_k} x^k$$

Hyun & Yi
JHEP 0611:075 (2006)
Kashani-Pon
JHEP 0704:044 (2007)

$\Psi(x)$
generating function of partition numbers for $\beta = (16)$

20

CP's

$$\begin{aligned}
 \textcircled{2} \quad A &= \prod_{i=1}^{\infty} (1 - q^{i-1+\nu})^{-1} \cdot q^{-\nu} (1 - q^{\nu/2} x) \cdot \prod_{i=1}^{\infty} (1 - 2q^{i-1+\nu}) \\
 &= q^{-\nu} - q^{-\nu} q^{\nu/2} x \prod_{i=1}^{\infty} (1 - 2q^{i-1+\nu})^{-1} \cdot q^{\nu/2} \cdot \prod_{i=1}^{\infty} (1 - 2q^{i-1+\nu}) \\
 &\quad \boxed{f(D)x = x f(D+N)} \\
 &= q^{-\nu} - q^{-\nu} q^{\nu/2} x \prod_{i=1}^{\infty} (1 - 2q^{i+\nu})^{-1} \prod_{i=1}^{\infty} (1 - 2q^{i-1+\nu}) \\
 &= q^{-\nu} - q^{-\nu} (1 - 2q^{\nu}) \\
 &= q^{-\nu} (1 - q^{\nu/2} x (1 - 2q^{\nu})),
 \end{aligned}$$

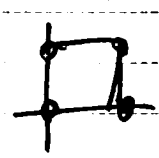
③ $B = (1 - 2q^{\nu}) \cdot x^{-1}$ — non-trivial

④ $A\Phi_0(x) = \Phi_0(x) \Leftrightarrow (1 - q^{\nu/2} x (1 - 2q^{\nu})) \Phi_0(x) = q^{\nu} \Phi_0(x)$

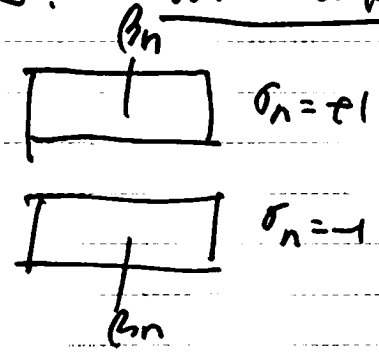
$\Leftrightarrow (1 - q^{\nu/2} x + q^{\nu/2} 2q^{\nu} - q^{\nu}) \Phi_0(x) = 0$

Haynes Li, Kasner-Poon

classical limit: $1 - x + 2xy - y = 0$



3. General strip geometry



$\tau(x) = \langle 0 | \phi_+(x) h \tau_-(q^{-\nu}) | 0 \rangle$

Essential part is $\mathfrak{g} \rightarrow G \rightarrow A, B$

$\Phi_j(x) = q$ -hypergeometric

$A = q^{-\nu} (1 - q^{\nu/2} x R),$

$B = R^{-1} x^{-1},$

$$R = \prod_{m \leq n} (1 - 2q^{\sigma_n \sigma_m})^{-\sigma_m \sigma_n} \cdot \prod_{m > n} (1 - 2q^{\sigma_n \sigma_m})^{-\sigma_n \sigma_m}$$

入学年度	回生	学部	学組	学科	コース	504400 (文)
※ 氏名等の記入欄は、ペン又はボールペンで記入すること。 4. <u>学期試験</u> 年度 全学共通科目 前・後期試験 曜日・時間						