# 3D Young diagrams and Gromov-Witten theory of $\mathbb{CP}^1$

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## 1. Motivation and goal

Gromov-Witten theory of all genera

— integrable Hamiltonian systems (Dubrovin & Zhang)

— quantization of symplectic geometry (Givental)

— computable cases (point, $\mathbb{CP}^1$, etc.)

Gromov-Witten theory of $\mathbb{CP}^1$ (Okounkov & Pandharipande)

— relation to Hurwitz numbers

— fermionic (semi-infinite wedge product) formalism

— relation to Toda lattice

— several versions (absolute; relative; equivariant)

My interest

— search for combinatorial and integrable structures in computable cases

This talk presents an approach to Gromov-Witten theory of $\mathbb{CP}^1$ from a statistical model of 3D Young diagrams. This melting crystal model is a kind of $q$-deformation of the Gromov-Witten invariants of $\mathbb{CP}^1$, and also the simplest case of instanton partition functions of 5D supersymmetric gauge theories on $\mathbb{R}^4 \times S^1$.

I will show

— prescrption of $q \to 1$ (5D $\to$ 4D) limit

— quantum spectral curve in this limit

— integrable structure in this limit

Reference: K.T., Quantum curve and 4D limit of melting crystal model, arXiv:1704.02750 [math-ph] (to be revised)

## 2. $\mathbb{CP}^1$ Gromov-Witten theory in a nutshell

### GW invariants

The correlators of the descendants $\tau_k(\omega)$, $k = 0, 1, \ldots$, of the Kähler class $\omega$ are defined by intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d)$ of stable maps to $\mathbb{CP}^1$:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_{g,d}^{\circ} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\omega)$$

where

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d) \to \mathbb{CP}^1, \ (f, C, p_1, \ldots, p_n) \mapsto f(p_i),$$

$$\psi_i = c_1(L_i), \ (L_i)_{(f,C,p_1,\cdots,p_n)} = T^*_{p_i} C$$

## GW/Hurwitz correspondence (Okounkov-Pandharipande)

The (disconnected) correlators of $\tau_k(\omega)$'s can be expressed in terms of Hurwitz numbers of $\mathbb{CP}^1$:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_{g,d}^{\bullet} = \sum_{\lambda \vdash d} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 \prod_{i=1}^{n} \frac{\boldsymbol{p}_{k_i+1}(\lambda)}{(k_i+1)!}$$

where

$$\lambda = (\lambda_1, \lambda_2, \ldots) \text{ (partition)}, \ \lambda \vdash d \iff |\lambda| = \sum_{i \geq 1} \lambda_i = d,$$

$$\boldsymbol{p}_k(\lambda) = \sum_{i \geq 1} \Big( (\lambda_i - i + 1/2)^k - (-i + 1/2)^k \Big) + (1 - 2^{-k}) \zeta(-k),$$

$$\frac{\dim \lambda}{|\lambda|!} = \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)}, \quad h(i,j) = \lambda_i + {}^{\mathrm{t}}\lambda_j - i - j + 1$$

## Generating function of GW invariants

$$\left\langle \exp\left(\sum_{k=0}^{\infty} \tau_k(\omega) t_k\right)\right\rangle^{\bullet} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots,k_n=0}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega)\right\rangle^{\bullet} \prod_{i=1}^{n} t_{k_i}$$

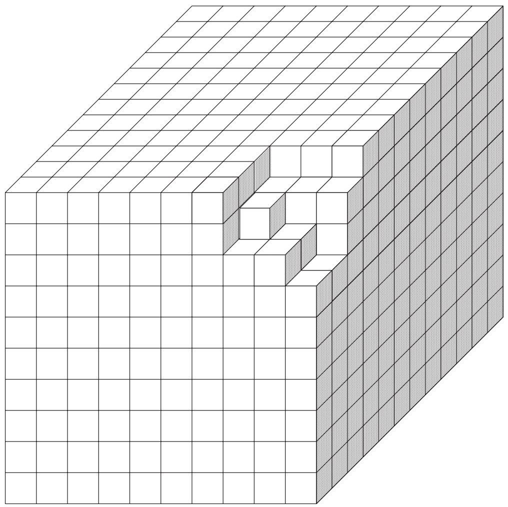
$$= \sum_{\lambda\in\mathcal{P}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \exp\left(\sum_{k=0}^{\infty} \frac{\boldsymbol{p}_{k+1}(\lambda)}{(k+1)!} t_k\right)$$

where $\mathcal{P}$ is the set of all partitions (of arbitrary length).

Remark: This sum resembles the instanton partition functions of 4D $\mathcal{N}=2$ supersymmetric gauge theories (Losev, Marshakov & Nekrasov).

## 2. Melting crystal model

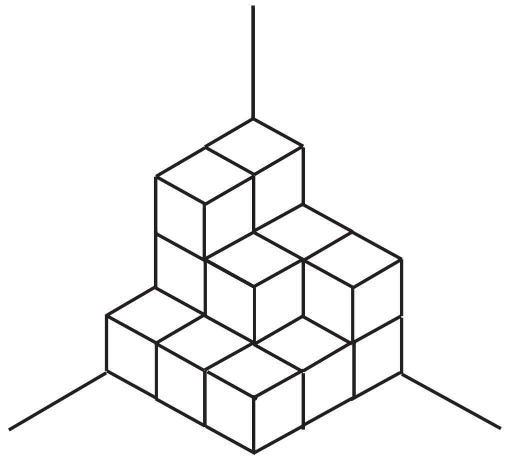
Statistical model (Okounkov, Reshetikhin & Vafa)



crystal corner

complement

$\longleftrightarrow$



3D Young diagram

The melting crystal model is a statistical model of random 3D Young diagrams. The 3D Young diagrams are represented by the plane partitions $\pi = (\pi_{ij})_{i,j=1}^{\infty}$ ($\pi_{ij}$ = height of $(i, j)$-th column).

## Undeformed partition function

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|} \qquad \text{(sum over plane partitions)}$$

$\downarrow$ diagonal slicing of 3DYD

$$= \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 \quad \text{(sum over ordinary partitions)}$$

$q$ is a parameter in the range $0 < q < 1$,

$$|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij} \quad \text{(volume of 3DYD)},$$

and $s_\lambda(q^{-\rho})$ is the special value of the infinite-variate Schur function $s_\lambda(\boldsymbol{x})$ at $\boldsymbol{x} = q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{i-1/2}, \ldots)$.

## Hook-length formula

$$s_\lambda(q^{-\rho}) = \frac{\dim_q \lambda}{|\lambda|!} = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda}(q^{-h(i,j)/2} - q^{h(i,j)/2})},$$

$$\kappa(\lambda) = \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i + 1).$$

This is a $q$-deformation of the quantity

$$\frac{\dim \lambda}{|\lambda|!} = \frac{1}{\prod_{(i,j)\in\lambda} h(i,j)}$$

that emerges in the GW/Hurwitz correspondence.

## Deformed partition function

$$Z(\boldsymbol{t}) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\boldsymbol{t},\lambda)},$$

$$\phi(\boldsymbol{t},\lambda) = \sum_{k=1}^{\infty} t_k \phi_k(\lambda),$$

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left( q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

$Q$ is a new parameter, and $\boldsymbol{t} = (t_1, t_2, \ldots)$ are coupling constants of the external potentials $\phi_k(\lambda)$.

## Deformed partition function

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$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left( q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

$Z(\boldsymbol{t})$ is a tau function of the KP hierarchy (Nakatsu & T.).

Proof: Use fermions and quantum torus algebra.

## 4D partition function

$$Z_{4\mathrm{D}}(\boldsymbol{t}) = \sum_{\lambda\in\mathcal{P}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \left(\frac{\Lambda}{\hbar}\right)^{|\lambda|} e^{\phi_{4\mathrm{D}}(\boldsymbol{t},\lambda)},$$

$$\phi_{4\mathrm{D}}(\boldsymbol{t},\lambda) = \sum_{k=1}^{\infty} t_k \phi_k^{4\mathrm{D}}(\lambda),$$

$$\phi_k^{4\mathrm{D}}(\lambda) = \sum_{i=1}^{\infty} \Big((\lambda_i - i + 1)^k - (-i+1)^k\Big)$$

This is essentially the same as the aforementioned generating function of the GW invariants of $\mathbb{CP}^1$, though slightly modified for comparison with $Z(\boldsymbol{t})$.

## 3. Formulation of 4D limit

What is 4D limit?

Nekrasov's instanton partition functions of 5D gauge theories are derived for theories on $\mathbb{R}^4 \times S^1$. The partition functions are expected to turn into those of 4D gauge theories on $\mathbb{R}^4$ as the radius $R$ of $S^1$ tends to 0.

## $R$-dependent parametrization of $q, Q$

The 4D limit $Z(\mathbf{0}) \to Z_{\mathrm{4D}}(\mathbf{0})$ of the undeformed partition function can be achieved in the following well-known manner:

— $q$ and $Q$ are parametrized as

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2.$$

— As $R \to 0$, the Boltzmann weights behave nicely:

$$\lim_{R\to 0} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} = \left(\frac{\dim \lambda}{|\lambda|}\right)^2 \left(\frac{\Lambda}{\hbar}\right)^{2|\lambda|}.$$

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We want to extend this prescription to the deformed partition function $Z(\boldsymbol{t})$.

## Prescription for external potentials

5D external potentials

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left( q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

4D external potentials

$$\phi_k^{4\mathrm{D}}(\lambda) = \sum_{i=1}^{\infty} \left( (\lambda_i - i + 1)^k - (-i + 1)^k \right)$$

Question: How $\phi_k^{4\mathrm{D}}(\lambda)$ can be derived from $\phi_k(\lambda)$ ($q = e^{-R\hbar}$) in the limit as $R \to 0$?

Hint: Take linear combinations of $\phi_k(\lambda)$'s to derive $\phi_k^{4\mathrm{D}}(\lambda)$.

# Key relation

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \phi_k^{\mathrm{4D}}(\lambda)(-R\hbar)^k + O(R^{k+1})$$

This relation implies the identity

$$\lim_{R \to 0} \sum_{k=1}^{\infty} \frac{T_k}{(-R\hbar)^k} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \sum_{k=1}^{\infty} T_k \phi_k^{\mathrm{4D}}(\lambda).$$

Since

$$\sum_{k=1}^{\infty} \frac{T_k}{(-R\hbar)^k} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k} \phi_j(\lambda),$$

the 4D limit $\phi(\boldsymbol{t}, \lambda) \to \phi^{4\mathrm{D}}(\boldsymbol{T}, \lambda)$ is achieved if $t_k$'s are parametrized by $T_k$'s as $t_j = \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k}$.

Letting $R \to 0$ under the $R$-dependent parametrization

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2, \quad t_j = \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k},$$

we have the correct 4D limit $Z(\boldsymbol{t}) \to Z_{4\mathrm{D}}(\boldsymbol{T})$.

## 4. Quantum spectral curve

### Single-variate specialization of $Z(\boldsymbol{t})$

Substituting

$$t_k = -\frac{q^{-k/2}x^k}{k},\ k = 1, 2, \ldots,$$

in $e^{\phi(\boldsymbol{t},\lambda)}$ gives

$$e^{\phi(\boldsymbol{t},\lambda)} = \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2}x}{1 - q^{-i+1/2}x},$$

thus $Z(\boldsymbol{t})$ turns into

$$Z(x) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2}x}{1 - q^{-i+1/2}x}.$$

## Quantum spectral curve of melting crystal model

$Z(x)$ satisfies the $q$-difference equation

$$A(x, q^D)Z(x) = Z(x),$$

where $D = x\frac{d}{dx}$ and

$$A = (1 - q^{1/2}x)q^D + (1 + Q)q^{1/2}x + Qx^2(1 - q^{-1/2}x)^{-1}q^{-D}.$$

Remarks:

— $q^D = e^{D \log q}$ is a $q$-shift operator: $q^D f(x) = f(qx)$.

— In the classical limit as $q \to 1$, $q^D \to y$, this equation turns into the equation $A_{\mathrm{cl}}(x, y) = 1$ of an ordinary curve.

## 4D limit of $Z(x)$

As $R \to 0$ under the $R$-dependent parametrization

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2, \quad x = e^{R(X-\hbar/2)},$$

$Z(x)$ converges to

$$Z_{\mathrm{4D}}(X) = \sum_{\lambda \in \mathcal{P}} \left( \frac{\dim \lambda}{|\lambda|} \right)^2 \left( \frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{i=1}^{\infty} \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i+1)\hbar}.$$

Proof:

$$\frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i+1/2} x} = \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i+1)\hbar} (1 + O(R)).$$

## 4D limit of quantum spectral curve

$Z_{4\mathrm{D}}(X)$ satisfies the difference equation

$$\left((X-\hbar)(e^{-\hbar d/dX}-1)+\frac{\Lambda^2}{X}e^{\hbar d/dX}\right)Z_{4\mathrm{D}}(X)=0.$$

Remark: Dunin-Barkowski et al. pointed out that

$$\Psi(X)=\exp\left(B\left(-\hbar\frac{d}{dX}\right)\frac{X-X\log X}{\hbar}\right)Z_{4\mathrm{D}}(X+\hbar)$$

$(B(z)=z/(e^z-1))$ satisfies the equation

$$\left(e^{-\hbar d/dX}+\Lambda^2 e^{\hbar d/dX}-X\right)\Psi(X)=0$$

of the quantum spectral curve of $\mathbb{CP}^1$ GW theory.

## 5. Bilinear equations of Fay type

### Bilinear equations for $Z(\boldsymbol{t}, x_1, \ldots, x_N)$

The shifted partition function

$$Z(\boldsymbol{t}, x_1, \ldots, x_N) = Z\Bigl(\ldots, t_k - \sum_{j=1}^{N} \frac{q^{-k/2} x_j^k}{k}, \ldots\Bigr)$$

satisfy the following (and many other) bilinear equations of the Fay type that characterize the tau functions of the KP hierarchy:

$$\begin{aligned} &(x_1 - x_2)(x_3 - x_4) Z(\boldsymbol{t}, x_1, x_2) Z(\boldsymbol{t}, x_3, x_4) \\ &- (x_1 - x_3)(x_2 - x_4) Z(\boldsymbol{t}, x_1, x_3) Z(\boldsymbol{t}, x_2, x_4) \\ &+ (x_1 - x_4)(x_2 - x_3) Z(\boldsymbol{t}, x_1, x_4) Z(\boldsymbol{t}, x_2, x_3) = 0 \end{aligned}$$

## 4D limit of $Z(\boldsymbol{t}, x_1, \ldots, x_N)$

$$
\begin{aligned}
&Z(\boldsymbol{t}, x_1, \ldots, x_N) \\
&= \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\boldsymbol{t},\lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x_j}{1 - q^{-i+1/2} x_j}.
\end{aligned}
$$

The 4D limit $Z(\boldsymbol{t}, x_1, \ldots, x_N) \to Z_{\mathrm{4D}}(\boldsymbol{T}, X_1, \ldots, X_N)$ can be achieved by letting $R \to 0$ under the same $R$-dependent parametrization of $q, Q, \boldsymbol{t}$ as for $Z(\boldsymbol{t})$ and the substitution

$$
x_j = e^{R(X_j - \hbar/2)}
$$

of the same form as used for $Z(x)$.

$$
\begin{aligned}
&Z_{\mathrm{4D}}(\boldsymbol{T}, X_1, \ldots, X_N) \\
&= \sum_{\lambda \in \mathcal{P}} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 \left( \frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{\mathrm{4D}}(\boldsymbol{T}, \lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{X_j - (\lambda_i - i + 1)\hbar}{X_j - (-i + 1)\hbar} \\
&= Z_{\mathrm{4D}}\Big( \ldots,\ T_k - \sum_{j=1}^{N} \frac{h^k}{k {X_j}^k},\ \ldots \Big).
\end{aligned}
$$

Remark: $X_j$'s are contained in the denominator in contrast to the case of $Z(\boldsymbol{t}, x_1, \ldots, x_N)$:

$$
Z(\boldsymbol{t}, x_1, \ldots, x_N) = Z\Big( \ldots,\ t_k - \sum_{j=1}^{N} \frac{q^{-k/2} {x_j}^k}{k},\ \ldots \Big).
$$

## 4D limit of three-term bilinear equation

$$
\begin{aligned}
&(X_1 - X_2)(X_3 - X_4) Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, X_2) Z_{4\mathrm{D}}(\boldsymbol{T}, X_3, X_4) \\
&- (X_1 - X_3)(X_2 - X_4) Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, X_3) Z_{4\mathrm{D}}(\boldsymbol{T}, X_2, X_4) \\
&+ (X_1 - X_4)(X_2 - X_3) Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, X_4) Z_{4\mathrm{D}}(\boldsymbol{T}, X_2, X_3) = 0
\end{aligned}
$$

Proof: As $R \to 0$, $Z(\boldsymbol{t}, x_i, x_j)$ converges to $Z_{4\mathrm{D}}(\boldsymbol{T}, X_i, X_j)$ and

$$x_i - x_j = R(X_i - X_j) + O(R).$$

Corollary:

$Z_{4\mathrm{D}}(\boldsymbol{T})$ is a tau function of the KP hierarchy.

## Concluding remarks

Both $Z(\boldsymbol{t})$ and $Z_{\mathrm{4D}}(\boldsymbol{t})$ can be extended to $Z(s,\boldsymbol{t})$ and $Z_{\mathrm{4D}}(s,\boldsymbol{t})$, $s \in \mathbb{Z}$. The following facts are known:

— $Z(s,\boldsymbol{t})$ is, up to simple factors, a tau function of the Toda hierarchy (Nakatsu and T., 2009). This is proven with the aid of fermions and the quantum torus algebra.

— A similar statement for $Z_{\mathrm{4D}}(s,\boldsymbol{t})$ was conjectured around 2000 (the Toda conjecture) and proved later on by several different methods (Getzler; Dubrovin & Zhang; Milanov).

It will be possible to reprove the Toda conjecture by the 4D limit.