94-09-05

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量子化

Quantization

幾何学

Geometry

可積分系

Integrable Systems

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と

# Exact calculation of path integrals

$$Z = \int \mathcal{D}\Phi \, e^{A[\Phi]}$$

typical mechanism · trick:–

- Gaussian integral ($\leftrightarrow$ free fields)

  $$\Phi : \Sigma \longrightarrow M$$

  ($\longrightarrow$ nontrivial results if $\Sigma$, $M$ are nontrivial)

- cancellation

  (index Th, SUSY, …)

  $\rightarrow$ localization

- symmetries

  $$\left( \begin{array}{l} G \curvearrowright \mathcal{D}\Phi, \, A[\Phi] \\ \int_{\mathcal{F}} \mathcal{D}\Phi \, e^{A[\Phi]} \simeq \int_{\mathcal{F}/G} \cdots \end{array} \right)$$

# Gaussian integrals

(boson) $$\int dx\, e^{-\sum A_{ij} x_i x_j} = \text{cte} (\det A)^{-1/2}$$

symmetric, pos. def.

$$\int dz\, d\bar{z}\, e^{-\sum A_{ij} z_i \bar{z}_j} = \text{cte} (\det A)^{-1}$$

Hermitian, pos. def

(fermion) $$\int d\psi\, e^{-\sum A_{ij} \psi_i \psi_j} = \text{cte}\ \text{Pfaff} A$$

antisymmetric

$$\int d\psi\, d\bar{\psi}\, e^{-\sum A_{ij} \psi_i \bar{\psi}_j} = \text{cte} \det A$$

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$$\sum A_{ij} x_i x_j \longrightarrow -\int \phi \Delta \phi + m^2 \phi^2 \sim \int (D\phi)^2 + m^2 \phi^2$$

$$\sum A_{ij} z_i \bar{z}_j \longrightarrow \int \partial\phi \cdot \bar{\partial}\phi + \cdots$$

$$\sum A_{ij} \psi_i \bar{\psi}_j \longrightarrow \int \psi \not{\partial} \bar{\psi} + \cdots \quad (\not{\partial} = \gamma^\mu \partial_\mu)$$

Dirac

# ☆ Non-trivial Gaussian integrals — 

(1.1) $\Phi : \Sigma \to T$ = torus $\Phi = (\phi_1, \cdots, \phi_n)$, $\phi_i \in S^1$

e.g. $\Sigma = \overset{\text{2-dim}}{\text{torus}} \to T = n\text{-dim torus}$

topological invariants ( winding # )

$$\int \mathcal{D}\Phi\, e^S = \sum_{\text{winding \#}} \int_{\text{fluctuations}} \mathcal{D}\Phi\, e^S$$

instanton #

$\Theta$ - functions

$$\left( \frac{\det'(-\Delta g)}{\mathrm{vol}_g \Sigma} \right)^{-n/2}$$

(1.2) $\Phi : \Sigma \to \mathbb{C}$ $\Sigma$ = Riemann surface

$\psi \sim e^{-i\phi}$, $\bar{\psi} \sim e^{i\phi}$ (free fermion/boson)

$$\langle \prod e^{+i\phi(\xi_i)} \prod e^{-i\phi(\eta_j)} \rangle$$

$$\sim \frac{\prod E(\xi_i, \xi_j) \prod E(\eta_i, \eta_j)}{\prod E(\xi_i, \eta_j)} \times \frac{\Theta(\sum \xi_i - \sum \eta_j)}{(\det \bar{\partial}_0)^{1/2}}$$

prime forms

theta function

det (Dirac)

## ② WZW model as free field theory (Gerasimov et al..)

$\mathrm{WZ}\underbrace{\mathrm{W}}_{\mathrm{N}}$

path integral interpretation of Wakimoto repr.

$$L = -\frac{k}{4\pi}\left[\frac{1}{2}\mathrm{Tr}|g^{-1}\partial g|^2 + \frac{\sqrt{-1}}{3}d^{-1}\mathrm{Tr}(g^{-1}dg)^3\right]$$

SU(2) $$g = \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix} \quad \begin{pmatrix}\text{Gauss} \\ \text{decomp.}\end{pmatrix}$$

$$\downarrow \; W := k e^{-2\varphi}\partial\psi \quad ((1,0)\text{-form})$$

$$L = -\frac{1}{4\pi}\left(W\bar\partial\chi + k|\partial\varphi|^2\right)$$

$$\mathcal{D}\varphi\,\mathcal{D}\chi\,\mathcal{D}\psi$$

$$\approx \det(e^{-2\varphi}\partial)^{-1}\,\mathcal{D}\varphi\,\mathcal{D}\chi\,\mathcal{D}W$$

Jacobian = conformal anomaly

$$L \to L_{tot} = -\frac{1}{4\pi}\left(W\bar\partial\chi + (k+\underline{\underline{2}})|\partial\varphi|^2 + \underline{\underline{R\varphi}}\right)$$

$$R = \partial\bar\partial \log\det g_{ab}$$

# Cancellation

## ① Dirac index $\quad \mathrm{Index}(\not{D})$

$$\text{Spinors} = \{\Gamma_{D+1}\psi = \psi\} \oplus \{\Gamma_{D+1}\psi = -\psi\}$$

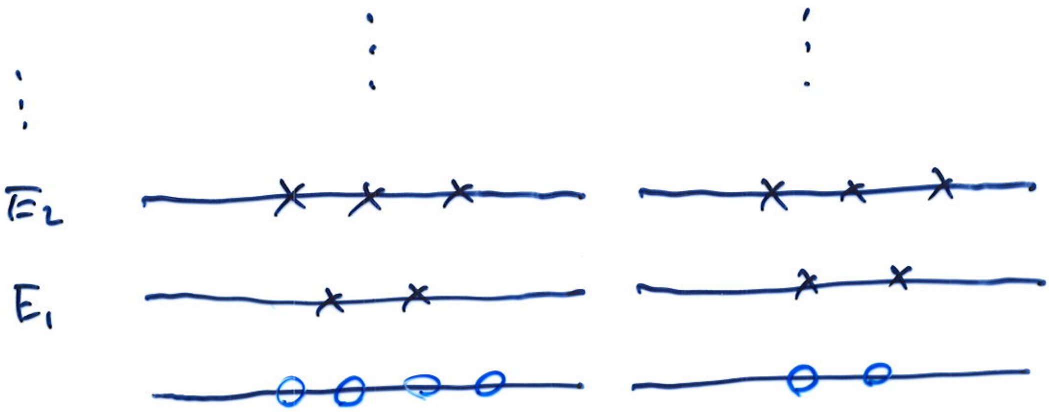
positive chirality $\qquad$ negative chirality

$$\boxed{\mathrm{Index}(\not{D}) = \dim \mathrm{Ker}\, \not{D}_+ - \dim \mathrm{Ker}\, \not{D}_-}$$

$$\mathrm{Ker}\, \not{D}_\pm = \{\psi \mid \Gamma_{D+1}\psi = \pm\psi,\ \not{D}\psi = 0\}$$

$$\boxed{\text{non-zero eigenvalue states occur in pair}}$$

$$(E \neq 0) \quad \not{D}\psi_+ = E\psi_+ \ \underset{1:1}{\longleftrightarrow} \ \not{D}\psi_- = E\psi_-$$




$$\mathrm{Index} = 4 - 2 = 2$$

$$\boxed{\mathrm{Index}(\not{D}) = \mathrm{Tr}\left(\Gamma_{D+1}\, e^{-\beta D^2}\right)}$$

($\beta > 0$ : arbitrary)

$$\mathrm{Index}(\not{D}) = \sum \mathrm{sign}(i)\, e^{-\beta E_i^2}$$

(chirality : $\mathrm{sign}(i) = \pm 1$

cancellation

$$\sum_{E_i^2>0} \mathrm{sign}(i)\, e^{-\beta E_i^2} = \sum_{E_i^2>0} e^{-\beta E_i^2} - \sum_{E_i^2>0} e^{-\beta E_i^2} = 0$$ )

$$\boxed{\mathrm{Tr}\left(\Gamma_{D+1}\, e^{-\beta D^2}\right) = \text{trace of heat kernel}}$$

heat kernel $K(t, x, y)$, $t = \beta$

$t \to 0$ asymptotic expansion

## ② Supersymmetry

A.) $$\left.\begin{aligned} \int dz\, d\bar{z}\; e^{-\sum A_{ij} z_i \bar{z}_j} &= \mathrm{cte}(\det A)^{-1} \\ \int d\psi\, d\bar{\psi}\; e^{-\sum A_{ij} \psi_i \bar{\psi}_j} &= \mathrm{cte}(\det A) \end{aligned}\right\}$$ cancellation may occur!

# B) Counting of physical states in $\underline{\underline{N=2}}$ string theory (Ooguri Takebe)

$$
\begin{aligned}
X, \bar{X} \to \begin{cases} x^\mu, \bar{x}^\mu & \prod (1-q^n)^{-4} \\ \psi^\mu_R, \bar{\psi}^\mu_R & \times \prod (1+q^n)^4 \end{cases} \\
L_n \to (b, c) \quad & \times \prod (1-q^n)^2 \\
G_r \to (\beta, \gamma) \quad & \times \prod (1+q^n)^{-2} \\
\bar{G}_r \to (\bar{\beta}, \bar{\gamma}) \quad & \times \prod (1+q^n)^{-2} \\
J_n \to (\tilde{b}, \tilde{c}) \quad & \times \prod (1-q^n)^2 = 1
\end{aligned}
$$

Cancellation

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$$\mathrm{tr}_{V(\text{one boson})}\, q^{L_0} = \prod (1-q^n)^{-1} \quad \cdots (\det A)^{-1}$$

$$\mathrm{tr}_{V(\text{one fermion})}\, q^{L_0} = \prod (1+q^n) \quad \cdots (\det A)$$

$$\mathrm{tr}_{V\left(\text{one } \cancel{\text{bosonic}}\ \underline{\underline{\text{ghost}}}\right)}\, q^{L_0} = \prod (1 \underline{\underline{+}} q^n)^{-1}$$

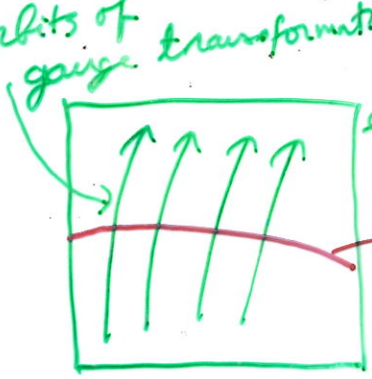
$$\mathrm{tr}_{V\left(\text{one fermionic } \underline{\underline{\text{ghost}}}\right)}\, q^{L_0} = \prod (1 \underline{\underline{-}} q^n)$$

# Symmetries

⓪ gauge symmetries — must be fixed for quantization

(Faddeev-Popov method, BRST quantization, Kugo-Ojima theory, ...)

orbits of gauge transformation



space of gauge fields $A_\mu$

e.g. Yang-Mills theory

gauge fixing condition

$$F[A_\mu, \partial A_\mu, \cdots] = 0$$

$\downarrow$

Faddeev-Popov determinant $\Delta_F[A]$

$$\int \mathcal{D}A\,\underbrace{\mathcal{D}\varphi}_{\text{matter}}\ \delta[F]\Delta_F[A]e^{S[A,\varphi]}$$

$$\approx \int \mathcal{D}A\,\underbrace{\mathcal{D}B}_{\text{Nakanishi-Lautrup field}}\,\underbrace{\mathcal{D}c\,\mathcal{D}\bar{c}}_{\text{Faddeev-Popov ghost}}\,\mathcal{D}\varphi\ \exp i\int \mathcal{L}(A,\varphi) + \underbrace{\mathcal{L}_{g.f.}(A,B) + \mathcal{L}_{F.P.}(c,\bar{c},A)}_{}$$

$$= \delta_{BRST}(V) \quad \text{(BRST symmetry)}$$

$$= \{Q, V\}$$

($Q$: BRST charge)

physical states $|\Psi\rangle$

$$\begin{cases} Q|\Psi\rangle = 0 \\ |\Psi\rangle \sim |\Psi\rangle + Q|\chi\rangle \end{cases}$$

# BRST quantization

$$\mathcal{L}_{tot} = \mathcal{L}(A,\varphi) + \underset{\delta_{BRST}(V)}{\underset{\shortparallel}{Q(V)}}, \quad V = \text{arbitrary}$$

($\rightarrow$ independent of gauge fixing)

$Q|\text{phys}\rangle = 0$ .... physical states are fixed points of BRST transformation

$\updownarrow$ same structure?

# Duistermaat-Heckman localization

$$Z = \int dz\,dc\,\exp\left(-\phi H - \omega - \beta\,\delta_H V\right)$$

$$= \text{indep. of } \beta \quad (V = \text{arbitrary})$$

$\beta \to \infty$: integral is localized to fixed points of $\delta_H$

$$Z = \sum_{dH=0} \frac{\sqrt{\det \omega_{ij}}}{\sqrt{\text{Hess}\,H}} \exp(-\phi H)$$

also similar to the relation of index and heat kernel

## ◎ 2-dim Yang-Mills theory

$A_\mu \quad \mu = 0, 1 \quad$ 2-fields

gauge fixing procedure removes 2 field components (not 1)

$$4\text{-dim} : 4 - 2 = 2$$
$$2\text{-dim} : 2 - 2 = 0$$ (no propagating fields)

2-dim YM $\approx$ quantum mechanical system

path integral reduced to sum over irreps.

partition function, Wilson loop correlators $\Big\} = \sum_{\text{irreps of } G}$ ( characters, branching coeff., etc )

## ◎ gauged WZW model

WZW + gauge fields ("G/G model")

--- similar to 2d Yang-Mills

... related to Verlinde formulas

## ◎ critical string theory (d=26, N=0, etc..)

$$\int \underline{\mathcal{D} g_{ab}} \cdots \approx \sum_g \int_{\mathcal{M}_g}$$

gauge fixing + absence of conformal anomaly

path integral is reduced to sum (g) and finite dim. integral ($\mathcal{M}_g$)