Hamiltonian structure of higher flows in string equation

a soliton-theoretic perspective of isomonodromic deformations

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Contents

1. String equation

String equation (also called Douglas equation)

- General form

$$[Q, P] = 1$$

where P and Q are differential operators ($\partial_x = \partial/\partial x$)

$$Q = \partial_x^q + u_2\partial_x^{q-2} + \cdots + u_q,$$
$$P = \partial_x^p + v_2\partial_x^{p-2} + \cdots + v_p.$$

- This talk is focussed on the special case

$$q = 2, \quad p = 2g + 1,$$
$$Q = \partial_x^2 + u, \quad P = \partial_x^{2g+1} + \cdots$$

- g is the genus of an underlying spectral curve.

Construction of operator P

$$P = B_{2g+1} + c_1 B_{2g-1} + \cdots + c_{g-1} B_3 + c_g B_1$$

- $c_1, \ldots, c_g$ are constants.
- B_{2n+1}'s are the differential operators

$$B_{2n+1} = (Q^{n+1/2})_{\geq 0}$$

used in the construction of the KdV hierarchy.

- $Q^{1/2}$ is the square root

$$Q^{1/2} = \partial_x + g_2 \partial_x^{-1} + g_3 \partial_x^{-2} + \cdots$$

of $Q = \partial_x^2 + u$. The coefficients $g_2, g_3, \ldots$, are differential polynomials of u.

Gelfand-Dickey polynomials

$$R_0 = 1, \quad R_1 = \frac{u}{2}, \quad R_2 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2,$$
$$R_3 = \frac{1}{32}u_{xxxx} + \frac{5}{16}uu_{xx} + \frac{5}{32}u_x^2 + \frac{5}{16}u^3, \ \ldots$$

- "Residue" of $Q^{n-1/2}$

$$Q^{n-1/2} = B_{2n+1} + R_n \partial_x^{-1} + \cdots$$

- Recurrence formula (Lenard relations)

$$\partial_x R_{n+1} = (\frac{1}{4}\partial_x^3 + u\partial_x + \frac{1}{2}u_x)R_n$$

- Q-adic expansion of B_{2n+1}

$$B_{2n+1} = \sum_{m=0}^{n} \left(R_m \partial_x - \frac{1}{2} R_{m,x} \right) Q^{n-m}$$

String equation in terms of Gelfand-Dickey polynomials

• The commutator $[B_{2n+1}, Q]$ can be written as

$$[B_{2n+1}, Q] = 2R_{n+1,x}$$

• The commutator equation $[Q, P] = 1$ reduces to the equation

$$2(R_{g+1} + c_1 R_g + \cdots + c_g R_1)_x + 1 = 0.$$

• Integrating it once and removing the integral constant by $x \to x + c$, one obtains the equation

$$2R_{g+1} + 2c_1 R_g + \cdots + 2c_g R_1 + x = 0,$$

which is the final form of the string equation.

• c_1 is suppressed in the following.

Examples

- $g = 1,\ P = B_3$: 1st Painlevé equation (PI)

$$2R_2 + x = 0 \iff \frac{1}{4}u_{xx} + \frac{3}{4}u^2 + x = 0$$

- $g = 2,\ P = B_5 + c_2 B_1$:

$$2R_3 + 2c_2 R_1 + x = 0 \iff$$
$$\frac{1}{16}u_{xxxx} + \frac{5}{8}uu_{xx} + \frac{5}{16}u_x^2 + \frac{5}{8}u^3 + c_2 u + x = 0$$

- $g = 3,\ P = B_7 + c_2 B_3 + c_3 B_1$:

$$2R_4 + 2c_2 R_2 + 2c_3 R_1 + x = 0 \iff$$
$$\frac{1}{64}u_{xxxxxx} + \cdots + x = 0$$

2. String equation as isomonodromic deformations

• $[Q,P]=1 \iff$ integrability of linear system

$$Q\psi = \lambda\psi, \quad P\psi = \partial_\lambda\psi.$$

• Equivalent 2×2 matrix linear system for $\Psi = {}^{\mathrm{t}}(\psi \;\; \partial_x\psi)$:

$$\partial_x\Psi = U(\lambda)\Psi, \quad \partial_\lambda\Psi = V(\lambda)\Psi,$$

where $U(\lambda)$ and $V(\lambda)$ are 2×2 matrices

$$U(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \; V(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}$$

of polynomials in λ.

• $[Q,P]=1 \iff$ isomonodromic Lax equation

$$[\partial_x - U(\lambda), \, \partial_\lambda - V(\lambda)] = 0.$$

• $\alpha(\lambda), \beta(\lambda), \gamma(\lambda)$ are polynomials in λ as follows:

$$\beta(\lambda) = R_g(\lambda) + c_2 R_2(\lambda) + \cdots + c_g R_0(\lambda),$$
$$\alpha(\lambda) = -\frac{1}{2}\partial_x \beta(\lambda),\ \gamma(\lambda) = (\lambda - u)\beta(\lambda) - \frac{1}{2}\partial_x^2 \beta(\lambda)$$

where

$$R_n(\lambda) = \lambda^n + R_1 \lambda^{n-1} + \cdots + R_n.$$

• In particular, deg $\alpha(\lambda) = g-1$, deg $\beta(\lambda) = g$, deg $\beta(\lambda) = g+1$, and $\beta(\lambda)$ and $\gamma(\lambda)$ are monic.
• $V(\lambda)$ is the same matrix as used in the Lax form of the stationary higher KdV equation

$$[Q, P] = 0.$$

The string equation is thus an isomonodromic analogue of the stationary higher KdV equation.

3. Higher KdV flows as isomonodromic deformations

Higher KdV flows

- The (Q, P) pair can be further deformed by the first $g-1$ flows t_{2n+1}, $n = 1, \ldots, g-1$ of the KdV hierarchy:

$$\partial_{t_{2n+1}} = [B_{2n+1}, Q], \quad \partial_{t_{2n+1}} = [B_{2n+1}, P].$$

- The "constants" $c_2, \ldots, c_g$ now depend on the time variables as

$$c_2 = \frac{2g-1}{2} t_{2g-1}, \quad \ldots, \quad c_g = \frac{3}{2} t_3.$$

- Relation to the Lax and Orlov-Schulman operators L, M of the KP hierarchy:

$$Q = L^2, \quad P = \frac{1}{2} M L^{-1}.$$

These flows are also isomonodromic

• The higher KdV flows are obtained by adding the linear equations

$$\partial_{t_{2n+1}} \psi = B_{2n+1} \psi$$

to the previous linear system $Q\psi = \lambda\psi$, $P\psi = \partial_\lambda \psi$.

• These equations are equivalent to the matrix form

$$\partial_{t_{2n+1}} \Psi = U_n(\lambda) \Psi$$

where $U_n(\lambda)$ are 2×2 matrices of polynomials in λ.

• The higher KdV flows can be thus converted to the isomonodromic Lax equations

$$[\partial_{t_{2n+1}} - U_n(\lambda), \partial_\lambda - V(\lambda)] = 0.$$

• The t_1-flow can be identified with the deformations with respect to x: $U_0(\lambda) = U(\lambda)$.

Technical remarks

- The matrix elements of

$$U_n(\lambda) = \begin{pmatrix} a_n(\lambda) & b_n(\lambda) \\ c_n(\lambda) & -a_n(\lambda) \end{pmatrix}$$

have a structure similar to $V(\lambda)$:

$$b_n(\lambda) = R_n(\lambda), \quad a_n(\lambda) = -\frac{1}{2}\partial_x R_n(\lambda),$$

$$c_n(\lambda) = (\lambda - u)R_n(\lambda) - \frac{1}{2}\partial_x^2 R_n(\lambda).$$

- $V(\lambda)$ is a linear combination of $U_n(\lambda)$'s:

$$V(\lambda) = U_g(\lambda) + c_2 U_{g-2}(\lambda) + \cdots + c_g U_0(\lambda).$$

- ψ satisfies the linear constraint

$$\partial_\lambda \psi = \sum_n \frac{(2n+1)}{2} t_{2n+1} \partial_{t_{2n-1}} \psi.$$

4. Darboux coordinates and Hamiltonian system

Spectral Darboux coordinates

Recall that $V(\lambda)$ is a matrix of the form

$$V(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}.$$

• We introduce a collection of new dynamical variables λ_j, μ_j $(j = 1, \ldots, g)$ as follows:

$$\beta(\lambda) = \prod_{j=1}^{g} (\lambda - \lambda_j), \quad \mu_j = \alpha(\lambda_j).$$

• They play the role of Darboux coordinates in our Hamiltonian formulation of the string equation and the higher KdV flows.

Spectral curve

• λ_j and μ_j satisfy the algebraic relation

$$\mu_j^2 = p(\lambda_j),$$

where $p(\lambda)$ is a polynomial of degree $2g+1$ defined by

$$p(\lambda) = \alpha(\lambda)^2 + \beta(\lambda)\gamma(\lambda) = -\det V(\lambda).$$

• In other words, $(\lambda_j, \mu_j)_{j=1}^{g}$ are a collection of points (or divisor) on the hyperelliptic curve (spectral curve)

$$\mu^2 = p(\lambda) = \lambda^{2g+1} + \cdots,$$

which, too, deforms in isomonodromic deformations.

Structure of $p(\lambda)$

$p(\lambda)$ is a polynomial of the form

$$p(\lambda) = I_0(\lambda) + I_1\lambda^{g-1} + \cdots + I_g.$$

• $I_0(\lambda)$ is a linear combination of quadratic polynomials of $c_2 = (2g-1)t_{2g-1}/2$, …, $c_g = 3t_3/2$ and $x = t_1$:

$$\begin{aligned} I_0(\lambda) &= \lambda^{2g+1} + 2c_2\lambda^{2g-1} + 2c_3\lambda^{2g-2} + (2c_4 + c_2^2)\lambda^{2g-3} \\ &\quad + \cdots + (x + c_2c_{g-2} + \cdots + c_{g-2}c_2)\lambda^g. \end{aligned}$$

Note that this part does not contain true dynamical variables (i.e., $u, u_x, u_{xx}, \ldots$).

• $I_1, \ldots, I_g$ are differential polynomials of u. In the stationary KdV equation, they are conserved quantities; in the present setting, they are not conserved.

Relation to 2nd order ODE

The 1st order system $\frac{d\Psi}{d\lambda} = V(\lambda)\Psi$ can be reduced to the 2nd order scalar equation

$$\frac{d^2\psi}{d\lambda^2} + p_1(\lambda)\frac{d\psi}{d\lambda} + p_2(\lambda)\psi = 0.$$

The coefficients are given by

$$p_1(\lambda) = -\frac{\beta'(\lambda)}{\beta(\lambda)} = -\sum_{j=1}^{g} \frac{1}{\lambda - \lambda_j},$$

$$p_2(\lambda) = -p(\lambda) - \alpha'(\lambda) + \alpha(\lambda)\frac{\beta'(\lambda)}{\beta(\lambda)}.$$

Consequently, λ_j's are apparent singularities, and μ_j's are the residue of $p_2(\lambda)$ at these points.

Hamiltonian form of string equation

Theorem 1 The string equation is equivalent to the non-autonomous Hamiltonian system

$$\frac{\partial \lambda_j}{\partial x} = \frac{\partial H}{\partial \mu_j}, \qquad \frac{\partial \mu_j}{\partial x} = -\frac{\partial H}{\partial \lambda_j}$$

with the Hamiltonian

$$H = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)}.$$

Proof is done by naive calculations: First derive differential equations of λ_j and μ_j from the Lax equations of $V(\lambda)$, and rewrite these equations into the Hamiltonian form. Then show that the Lax equations can be reconstructed from the Hamiltonian system.

Hamiltonian form of higher KdV flows

Theorem 2 The higher KdV flows are equivalent to the non-autonomous Hamiltonian system

$$\frac{\partial \lambda_j}{\partial t_{2n+1}} = \frac{\partial H_n}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_{2n+1}} = -\frac{\partial H_n}{\partial \lambda_j}$$

with the Hamiltonians

$$H_n = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_n(\lambda_j) - \sum_{j=1}^{g} \frac{\mu_j}{\beta'(\lambda_j)} R_n'(\lambda_j).$$

Here R_n's are understood to be functions of λ_j's that are implicitly defined by the relation between $\beta(\lambda)$ and R_n's:

$$\beta(\lambda) = R_g(\lambda) + c_2 R_{g-2}(\lambda) + \cdots + c_g R_0(\lambda).$$

Proof is again by calculations (far more complicated than the proof of Theorem 1).

4. Examples

$g = 1$ (PI)

- $2R_2 + x = \frac{1}{4}u_{xx} + \frac{3}{4}u^2 + x = 0.$

- $\beta(\lambda) = \lambda + \beta_1,\ \beta_1 = R_1 = \frac{u}{2}.$

- $\lambda_1 = -\beta_1,\ \mu_1 = -\frac{1}{2}\beta_{1,x}.$

- $p(\lambda) = I_0(\lambda) + I_1,\ I_0(\lambda) = \lambda^3 + x\lambda.$

- $H = \mu_1^2 - \lambda_1^3 - x\lambda_1.$

g = 2 (degenerate 2D Garnier system)

- $2R_3 + 2c_2R_1 + x = 0$, $c_2 = \frac{3}{2}t_3$, $x = t_1$.

- $\beta(\lambda) = \lambda^2 + \beta_1\lambda + \beta_2 = (\lambda - \lambda_1)(\lambda - \lambda_2)$,
$\beta_1 = R_1$, $\beta_2 = R_2 + c_2$.

- $\alpha(\lambda) = -\frac{1}{2}(\beta_{1,x}\lambda + \beta_{2,x})$, $\mu_j = \alpha(\lambda_j)$.

- $p(\lambda) = I_0(\lambda) + I_1\lambda + I_2$, $I_0(\lambda) = \lambda^5 + 2c_2\lambda^3 + x\lambda^2$.

- $H_0 = \frac{\mu_1^2 - I_0(\lambda_1)}{\lambda_1 - \lambda_2} + \frac{\mu_2^2 - I_0(\lambda_2)}{\lambda_2 - \lambda_1}$,
$H_1 = -\frac{\mu_1^2 - I_0(\lambda_1)}{\lambda_1 - \lambda_2}\lambda_2 - \frac{\mu_2^2 - I_0(\lambda_2)}{\lambda_2 - \lambda_1}\lambda_1 - \frac{\mu_1}{\lambda_1 - \lambda_2} - \frac{\mu_2}{\lambda_2 - \lambda_1}$.

(cf. H. Kimura; S. Shimomura)

$g = 3$ (degenerate 3D Garnier system?)

- $2R_4 + 2c_2R_2 + 2c_3R_1 + x = 0$, $c_2 = \frac{5}{2}t_5$, $c_3 = \frac{3}{2}t_3$, $x = t_1$.

- $\beta(\lambda) = \lambda^3 + \beta_1\lambda^2 + \beta_2\lambda + \beta_3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$,
$\beta_1 = R_1$, $\beta_2 = R_2 + c_2$, $\beta_3 = R_3 + c_2R_1 + c_3$.

- $\alpha(\lambda) = -\frac{1}{2}(\beta_{1,x}\lambda^2 + \beta_{2,x}\lambda + \beta_{3,x})$, $\mu_j = \alpha(\lambda_j)$.

- $p(\lambda) = I_0(\lambda) + I_1\lambda^2 + I_2\lambda + I_3$,
$I_0(\lambda) = \lambda^7 + 2c_2\lambda^5 + 2c_3\lambda^4 + (x + c_2^2)\lambda^3$.