Gaudin Model, KZ Equation
and
Isomonodromic Problem on Torus

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Goal: Construct an elliptic analogue of the Schlesinger equation

Schlesinger equation — Isomonodromic Problem of matrix system

\[ \frac{dY}{d\lambda} = M(u) \ Y, \quad n \times n \]

\[ M(u) = \sum_{i=1}^{N} \frac{A_i}{2 + i} : \text{rational} \ (X = \mathbb{C} \ P^1) \]

Elliptic analogue — Isomonodromic Problem with elliptic coefficients

\[ M(u) = ? \quad (X = \mathbb{C}/\mathbb{Z} + 2\pi i) \]

C.f. Scalar equation

\[ \frac{d^2y}{d\lambda^2} = q(x) y, \ \text{higher order, ...} \]

— Okamoto, Iwasaki, Kawai

(Complex analytic and geometric approach)
Approaches:

Complex analytic and geometric approach

Okamoto, Iwasaki, Kawai

\[ \text{twisted de Rham cohomology} \]

CFT, \( r \)-matrix approach

1. Reshetikhin, Harnad

\[ g = 0 \text{ (Schlesinger eq)} \]

KZ (Kuznecov-Zamolodchikov) equation

rational \( r \)-matrix

(rational Gaudin model)

2. Korotkin, Samtleben

Lavin, Olshanetsky

\[ g = 1 \]

KZB (KZ-Bernard) equation

dynamical \( r \)-matrix

Calogero-Gaudin model

My approach

elliptic KZ equation of Etingof

Belavin's \( r \)-matrix

elliptic \( SU(n) \) Gaudin model of Kunokai and Takebe
Road Map

Elliptic Gaudin model

\[ J(\lambda), \quad H_0, H_1, \ldots, H_N \]
quantum Hamiltonians

classical limit

Non-autonomous multi-time Hamiltonian system

\[ M(\lambda), \quad H_0, H_1, \ldots, H_N \]
classical
\[ \tau, t_1, \ldots, t_N \]

Lax representation

\[ \left[ \frac{\partial}{\partial \lambda} - M(\lambda), \quad \cdots \right] = 0 \]

Isomonodromic linear system

\[ \frac{\partial Y}{\partial \lambda} = M(\lambda) Y, \quad \cdots \]

Geometric interpretation of isomonodromic deformations

Symplectic form \( \Omega \) on moduli space \( M \) of zero connections with given constant local monodromy data

classical limit + picture changing (Schrödinger \( \rightarrow \) Heisenberg)
Elliptic $SU(n)$ Gaudin model

Models of 1-dim. magnet

Irreps of $SU(n)$: \((V_1, P_1) \quad (V_2, P_2) \quad \cdots \quad (V_N, P_N)\)

Lattice: \(1 \quad 2 \quad \cdots \quad N\)

Space of states \(V = V_1 \otimes V_2 \otimes \cdots \otimes V_N\)

Hamiltonian \(H: V \to V\) (linear operator)

Problem: Find eigenvalues and eigenvectors of \(H\) (and diagonalize \(H\))

Methods for "Solvable Models":

- Bethe Ansatz
- QISM (quantum inverse scattering method)
- SOV (separation of variables)

Solvable models:

- Heisenberg model (XXX, XXZ, XYZ)
- Haldane-Shastry model.
- Gaudin model (XXX \& XYZ)
Elliptic Gaudin model

$SU(2)$ : Gaudin

$SU(n)$ : Kuroki, Takebe

Auxiliary operator : $C^n \otimes V \to C^n \otimes V$

$$J(x) = \sum_{i=1}^{N} \sum_{(ab) \neq (10)} \sum_{t_i} \mathcal{J}_{ab}^{(\lambda_i-t_i)} \mathcal{J}_{ab} \otimes \mathcal{P}_i \, (\mathcal{J}_{ab})$$

$\mathcal{J}_{ab} = g^a h^b$, $g, h \in SU(n)$, $g h = e^{2\pi i/n} h g$

$\mathcal{J}_{ab} = \mathcal{J}_{ab}^{-1} / n$.

$g^n = h^n = I$

$W_{ab}(\xi) = \frac{\Theta_{ab}(\xi) \Theta_{00}(\xi)}{\Theta_{ab}(0) \Theta_{00}(0)}$, $(ab) \neq (00)$, $a, b \in \mathbb{Z}^n$

$\Theta_{ab}(\xi) = \Theta \frac{a - \frac{i}{2}, b - \frac{i}{n}}{2 \pi i}$ characteristic

Hamiltonians $H_0, H_1, \ldots, H_N : V \to V$

$$\frac{1}{2} \text{Tr}_{C^n} J(x)^2 = \sum_{i=1}^{N} C_i \mathcal{J}_{ab}^{(\lambda_i-t_i)} + \sum_{i=1}^{N} H_i \mathcal{J}_{ab}^{(t_i-t_i)} + H_0$$

$C_i = \frac{1}{2} \sum_{(ab)} P_i \, (\mathcal{J}_{ab}) \, P_i \, (\mathcal{J}_{ab})$: quadratic Casimir of $P_i$

$[H_i, H_j] = [H_i, H_0] = 0$

$\sum_{i=1}^{N} H_i = 0$

(simultaneously diagonalizable)
Non-autonomous multi-time Hamiltonian system

Classical limit

\[ p_i(J^{ab}) \rightarrow A_i^{ab} : \text{elements of commutative ring} \]

(\text{coordinates on } \mathfrak{sl}(N, \mathbb{C}))

\[ [ , , ] \rightarrow \{ , , \} : \text{Kostant-Kirillov bracket} \]

\[
[ J^{ab}, J^{cd} ] = \sum_{pq} f^{abcd}_{pq} J^{pq}
\]

\[
\{ A_i^{ab}, A_j^{cd} \} = \delta_{ij} \sum_{pq} f^{abcd}_{pq} A_j^{pq}
\]

\[
J(\lambda) \rightarrow M(\lambda) = \sum_{i=1}^{N} \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} A_i^{ab}
\]

\( \mathfrak{sl}(n, \mathbb{C}) \)-valued function of \( \lambda \)

\( H_i, H_0 \rightarrow H_i, H_0 : \text{classical Hamiltonians,} \)

\[ \{ H_i, H_j \} = \{ H_i, H_0 \} = 0 \]

\( C_i \rightarrow C_i : \text{central} \)

(\text{determined by coadjoint orbit})

\[ \rightarrow \text{local monodromy data} \]
Hamiltonian System

\( \{ A_i^{ab} \} : \) coordinates on \( \mathfrak{sl}(n, \mathbb{C})^N \)
(\( \mathfrak{sl}(n, \mathbb{C}) \) : \( n \times n \) complex matrices with trace zero, \( \mathbb{C} \) is the complex numbers)

\( \tau, t_1, \ldots, t_N : \) time variables

\( A_i^{ab} = A_i^{ab}(\tau, t_1, \ldots, t_N) \)

We consider the Hamiltonian equations

\[
\frac{\partial A_i^{ab}}{\partial t_i} = \{ A_j^{ab}, H_i \} \quad (i = 1 \ldots N),
\]

\[
\frac{\partial A_j^{ab}}{\partial \tau} = \{ A_j^{ab}, \frac{H_0 - \eta_1 \sum_{i=1}^{N} t_i H_i}{2\pi i} \}.
\]

where \( \eta_1 \) is the constant in the transformation formula

\( J(\tau + 1) = J(\tau) + \eta_1. \)

What is this?
Symplectic Structure on Moduli Space of Connections

\[ \frac{dY}{d\lambda} = M(\lambda) Y, \]
\[ M(\lambda+\tau) = e^{\tau} M(\lambda) e^{-\tau}, \quad M(\lambda+\tau) = g M(\lambda) g^{-1} \]

Lie algebra bundle \( \mathfrak{sl}(n,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{C}) \times \mathbb{C}/(\mathbb{C}+\mathbb{C}) \)
\[ X = \mathbb{C} \rightarrow Y = \mathbb{C}/(\mathbb{C}+\mathbb{C}) \]
\[ (A, \lambda) \sim (hAh^{-1}, \lambda + \tau) \sim (g^\tau A g, \tau + \tau) \]

Moduli Space of Mer. Connections

\[ M = \{ (M(t_i), t_1, ..., t_n, \tau) \mid \text{given constant local monodromy data around } t_1, ..., t_n \} \]

\[ \Omega \text{ is a symplectic form on } M \quad \text{(Iwasaki, Kawai)} \]

\[ \Omega = \sum_{i=1}^{N} dH_i \wedge d\lambda_i + d\left( \frac{H_0 - \eta}{2\pi i} \sum_{i=1}^{N} \chi_i H_i \right) \wedge \lambda - \sum_{i=1}^{N} \text{tr} dB_i \wedge dA_i \]

where \( dB_i = (a dA_i)^{\tau} dA_i \)

Hamiltonian for \( \tau \)

Kostant-Kirillov on coadj. orbits of \( \mathfrak{sl}(n,\mathbb{C}) \)
Hamiltonian system is equivalent to:

\[
\frac{\partial M(\lambda)}{\partial t_i} = - [A_i(\lambda), M(\lambda)] - \frac{\partial A_i(\lambda)}{\partial \lambda},
\]

\[
\frac{\partial M(\lambda)}{\partial \tau} = [2B(\lambda), M(\lambda)] + 2 \frac{\partial B(\lambda)}{\partial \lambda}.
\]

where

\[
A_i(\lambda) = \sum_{(a,b)} w_{ab} (\lambda - t_i) \text{Tr}_{ab} A_{i}^{ab},
\]

\[
B(\lambda) = \sum_{i=1}^{N} \sum_{(a,b)} Z_{ab} (\lambda - t_i) \text{Tr}_{ab} A_{i}^{ab},
\]

\[
Z_{ab}(\lambda) = \frac{w_{ab}(\lambda)}{4\pi^2} \left( \frac{\partial \Omega_{ab}(\lambda)}{\partial \Omega_{ab}(\lambda)} - \frac{\partial \Omega_{ab}^{(0)}}{\partial \Omega_{ab}^{(0)}} \right).
\]

building blocks of elliptic K8 equation!

More familiar form:

\[
\left[ \frac{3}{\partial t_i} + A_i(\lambda), \frac{\partial}{\partial \lambda} - M(\lambda) \right] = 0,
\]

\[
\left[ \frac{\partial}{\partial \tau} - 2B(\lambda), \frac{\partial}{\partial \lambda} - M(\lambda) \right] = 0.
\]

(The other Lax equations can be derived from these special equations.)
Isomonodromic Property

Lax equations (\(\Leftrightarrow\) Hamiltonian system) are Frobenius integrability condition of following linear system:

\[
\begin{align*}
\left(\frac{\partial}{\partial \lambda} - M(\lambda)\right) Y &= 0 \\
\left(\frac{\partial}{\partial t} + A;_t(\lambda)\right) Y &= 0 \\
\left(\frac{\partial}{\partial \bar{t}} - 2B(\lambda)\right) Y &= 0
\end{align*}
\]

Local monodromy

\[Y(\lambda) \rightarrow Y(\lambda) \Gamma_i\]

Global monodromy

\[Y(\lambda + 1) = h Y(\lambda) \Gamma_\alpha\]
\[Y(\lambda + 2) = g^{-1} Y(\lambda) \Gamma_\beta\]

\(\Gamma_i, \Gamma_\alpha, \Gamma_\beta\) constant: isomonodromic deformations
Relation to Elliptic KZ Equation

Etingof's twisted trace of vertex operators?
Kuroki and Takebe's twisted WZW models

\[ \Rightarrow \text{elliptic KZ equation for } N \text{-point conformal blocks} \]

\[ F(t_1, ..., t_N) \in \text{End } V : \]

\[ \left( \kappa \frac{\partial}{\partial t_i} + \sum_{j \neq i} \sum_{(ab)} w_{ab} (t_i - t_j) P_i (J_{ab}) P_j (J_{ab}) \right) F = 0 \]

\[ \left( \kappa \frac{\partial}{\partial t} + \sum_{i, j} \sum_{(ab)} Z_{ab} (t_i - t_j) P_i (J_{ab}) P_j (J_{ab}) \right) F = 0. \]

where

\[ \kappa = k + n, \quad k : \text{level of } \widehat{sl}(n) \]

\[ V = V_1 \otimes \cdots \otimes V_N, \quad (V_i, P_i) : \text{irreps} \]
Add one more point at $\lambda$ with fundamental representation $(C^n, id)$. Equations for $(N+1)$-point conformal blocks $G(\lambda, t_1, \ldots, t_N) \in \text{End}(C^n \otimes V)$

\begin{align*}
(k \frac{\partial^2}{\partial \lambda^2} + \sum_j \sum_{(ab)} w_{ab} (\lambda - t_j) \delta_{ab} p_j (J^{ab})) G &= 0, \\
(k \frac{\partial^2}{\partial t_i^2} + \sum_{(ab)} w_{ab} (t_i - \lambda) \delta_{ab} \tilde{p}_i (J^{ab}) \\
&\quad + \sum \sum_{ji} w_{ab} (t_i - t_j) \tilde{p}_i (J^{ab}) p_j (J^{ab})) G = 0, \\
(k \frac{\partial^2}{\partial \lambda^2} + \ldots) G &= 0
\end{align*}

2. Apply "gauge transformation" $G \rightarrow X = (F^{-1} \otimes I) G$

by "invertible solution" $F = \exp S$ of $N$-point elliptic $KZ$ equations. (Schrödinger → Heisenberg)

$\Rightarrow$ $X$ satisfies (almost) the same equation as $Y$,

upon defining

$$A_i^{ab} = F(t_1 \ldots t_n)^{-1} \tilde{p}_i (J^{ab}) F(t_1 \ldots t_n)$$

and taking the classical limit

$$A_i^{ab} \in \text{End} V \rightarrow A_i^{ab} : \text{function on } (\text{cod. orbit})$$

(choose $\lambda = -1$, but this is not essential)
• An elliptic analogue of the Schlesinger equation can be constructed from the building blocks WA, Zab, etc of the elliptic Gaudin model and the associated elliptic KZ equation.

• Without such a link with solvable spin models and CFT, constructing such a matrix system of isomonodromic deformation would be extremely difficult. (Possible? Okamoto's notes)

• This issue is also deeply related with "Hitchin systems" (2-dim. reduced SYM equations, or deformations of Higgs pairs).

• What about difference/q-difference analogue? q-difference analogues of PT, PTV of Jimbo and Sakai might be connected with a qKZ equation in this way.