超幾何系ワークショップ in 神戸 '97

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Gaudin Model, KZ Equation and Isomonodromic Problem on Torus

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Goal: Construct an elliptic analogue of the Schlesinger equation

Schlesinger equation — Isomonodromic Problem of matrix system

$$\frac{dY}{d\lambda} = M(\lambda) Y, \qquad n \times n$$

$$M(\lambda) = \sum_{i=1}^{N} \frac{A_i}{\lambda - t_i} \ : \text{rational} \ (X = \mathbb{CP}^1)$$

Elliptic analogue — Isomonodromic Problem with elliptic coefficients

$$M(\lambda) = \ ? \qquad (X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z})$$

cf. Scalar equation

$$\frac{d^2 y}{d\lambda^2} = q(\lambda) y, \ \text{higher order}, \ldots$$

— Okamoto, Iwasaki, Kawai

(Complex analytic and geometric approach)

Approaches.

Complex analytic and geometric approach

Okamoto, Iwasaki, Kawai

twisted de Rham cohomology

CFT, r-matrix approach

① Reshetikhin, Harnad) $g = 0$ (Schlesinger eq)

KZ (Knizhnik-Zamolodchikov) equation

rational r-matrix

(rational Gaudin model)

② Korotkin, Samtleben, Lavin, Olshanetsky) $g = 1$

KZB (K-Z-Bernard) equation

dynamical r-matrix

Calogero-Gaudin model

My approach

elliptic KZ equation of Etingof

Belavin's r-matrix

elliptic $SU(n)$ Gaudin model of Kuroki and Takebe

Road Map

Elliptic Gaudin model

$J(\lambda)$, $\underbrace{H_0, H_1, \cdots, H_N}_{\text{quantum Hamiltonians}}$

↓ classical limit

Non-autonomous multi-time Hamiltonian system ← **geometric interpretation of isomonodromic deformations**

symplectic form Ω on moduli space $\mathcal{M}$ of mero. connections with given constant local monodromy data

$M(\lambda)$, $\underbrace{H_0, H_1, \cdots, H_N}_{\text{classical}}$

$\tau, t_1, \cdots, t_N$

↓

Lax representation

$\left[\frac{\partial}{\partial\lambda} - M(\lambda), \cdots\right] = 0$

↓

Isomonodromic linear system

$\frac{\partial Y}{\partial\lambda} = M(\lambda)Y, \cdots$

↑ classical limit + picture changing (Schrödinger → Heisenberg)

Elliptic KZ equation

Elliptic $SU(n)$ Gaudin model

Models of 1-dim. magnet

irreps of $SU(n)$: (V_1, ρ_1) (V_2, ρ_2) $\cdots$ (V_N, ρ_N)

lattice :

Space of states $V = V_1 \otimes V_2 \otimes \cdots \otimes V_N$

Hamiltonian $H : V \to V$ (linear operator)

Problem : Find eigenvalues and eigenvectors of H (and diagonalize H)

Methods for "Solvable Models" :

Bethe Ansatz
QISM (quantum inverse scattering method)
SOV (Separation of variables)
⋮

Solvable models :

Heisenberg model (XXX, XXZ, XYZ)
Haldane - Shastry model.
Gaudin model (XXX ~ XYZ)
⋮

$\boxed{\text{Elliptic Gaudin model}}$

$SU(2)$: Gaudin

$SU(n)$: Kuroki, Takebe

auxiliary operator : $\mathbb{C}^n \otimes V \to \mathbb{C}^n \otimes V$

$$\mathcal{J}(\lambda) = \sum_{i=1}^{N} \sum_{(ab) \neq (0,0)} w_{ab}(\lambda - \underbrace{t_i}_{}) J_{ab} \otimes \rho_i(J^{ab})$$

t_i: parameters

$$J_{ab} = g^a h^b, \quad g, h \in SU(n), \quad gh = e^{2\pi i/n} hg$$

$$J^{ab} = J_{ab}^{-1}/n, \qquad g^n = h^n = I$$

$$w_{ab}(\lambda) = \frac{\theta_{[ab]}(\lambda)\, \theta'_{[00]}(0)}{\theta_{[ab]}(0)\, \theta_{[00]}(\lambda)}, \quad (ab) \neq (00), \quad a, b \in \mathbb{Z}_n$$

$$\theta_{[ab]}(\lambda) = \theta_{\underbrace{\frac{a}{n} - \frac{1}{2}}_{}, \underbrace{\frac{1}{2} - \frac{b}{n}}_{}}(\lambda | \tau).$$

characteristic

Hamiltonians $H_0, H_1, \cdots, H_N : V \to V$

$$\frac{1}{2} \mathrm{Tr}_{\mathbb{C}^n} \mathcal{J}(\lambda)^2 = \sum_{i=1}^{N} C_i \wp(\lambda - t_i) + \sum_{i=1}^{N} H_i \zeta(t - t_i) + H_0$$

$$C_i = \frac{1}{2} \sum_{(ab)} \rho_i(J_{ab}) \rho_i(J^{ab})$$: quadratic Casimir of ρ_i

$$\boxed{\begin{gathered}[H_i, H_j] = [H_i, H_0] = 0 \\ \sum_{i=1}^{N} H_i = 0\end{gathered}}$$

(simultaneously diagonalizable)

Non-autonomous multi-time Hamiltonian system

Classical limit

$\rho_i(J^{ab}) \longrightarrow A_i^{ab}$: elements of commutative ring (coordinates on $sl(n,\mathbb{C})$)

$[\ ,\ ] \longrightarrow \{\ ,\ \}$: Kostant-Kirillov bracket

$$[J^{ab}, J^{cd}] = \sum_{pq} \underline{\underline{f^{abcd}_{pq}}}\, J^{pq}$$

same structure constants

$$\{A_i^{ab}, A_j^{cd}\} = \delta_{ij} \sum_{pq} \underline{\underline{f^{abcd}_{pq}}}\, A_j^{pq}$$

$$\mathcal{J}(\lambda) \longrightarrow M(\lambda) = \sum_{i=1}^{N} \sum_{(ab)} w_{ab}(\lambda - t_i) J_{ab} A_i^{ab}$$

$sl(n,\mathbb{C})$-valued function of λ

$H_i, H_0 \longrightarrow H_i, H_0$: classical Hamiltonians,

$$\boxed{\{H_i, H_j\} = \{H_i, H_0\} = 0}$$

$C_i \longrightarrow C_i$: central

(determined by coadjoint orbit)

$\longrightarrow$ local monodromy data

Hamiltonian System

$\{A_i^{ab}\}$: coordinates on $\mathfrak{sl}(n,\mathbb{C})^N$

(or on $\mathcal{O}_1 \times \cdots \times \mathcal{O}_N$, $\mathcal{O}_i \subset \mathfrak{sl}(n,\mathbb{C})$)
coad. orbit

$\tau, t_1, \cdots, t_N$: time variables

$$A_i^{ab} = A_i^{ab}(\tau, t_1, \cdots, t_N)$$

We consider the Hamiltonian equations

$$\frac{\partial A_j^{ab}}{\partial t_i} = \{A_j^{ab}, H_i\} \quad (i = 1 \ldots N),$$

$$\frac{\partial A_j^{ab}}{\partial \tau} = \{A_j^{ab}, \boxed{\frac{H_0 - \eta_1 \sum_{i=1}^N t_i H_i}{2\pi\sqrt{-1}}}\}.$$

where η_1 is the constant in the transformation formula

$$\zeta(z+1) = \zeta(z) + \eta_1.$$

What is this?

Symplectic Structure on Moduli Space of Connections

$\boxed{\text{ODE}}$ $\quad \dfrac{dY}{d\lambda} = M(\lambda) Y.$

$$M(\lambda+1) = h^{-1} M(\lambda) h, \quad M(\lambda+\tau) = g M(\lambda) g^{-1}$$

Lie algebra bundle $sl(n,\mathbb{C})^{tw} = sl(n,\mathbb{C}) \times \tilde{X}/\sim$

$$\tilde{X} = \mathbb{C} \to X = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}),$$

$$(A,\lambda) \sim (hAh^{-1}, \lambda+1) \sim (g^{-1}Ag, \lambda+\tau)$$

$\boxed{\text{Moduli Space of Mero. Connections}}$

$\mathcal{M} = \{ (M(\lambda), t_1, \cdots, t_N, \tau) \mid$ given constant local monodromy data around $t_1, \cdots, t_N \}$

$\downarrow$ Poincaré - Lefschetz duality (twisted de Rham cohom.. with coeff. in $\underline{sl(n,\mathbb{C})^{tw}}$)

$\exists\, \Omega$: symplectic form on $\mathcal{M}$ (Iwasaki Kawai)

Residue calculus gives :

$$\Omega = \sum_{i=1}^{N} dH_i \wedge dt_i + d\left(\boxed{\frac{H_0 - \eta_1 \sum_{i=1}^{N} t_i H_i}{2\pi\sqrt{-1}}} \right) \wedge d\tau - \sum_{i=1}^{N} \underline{\mathrm{tr}\, dB_i \wedge dA_i}$$

where $dB_i = (\mathrm{ad}_{A_i})^{-1} dA_i$

Hamiltonian for τ.

Kostant-Kirillov on coad. orbits of $sl(n,\mathbb{C})$

Lax Representation

Hamiltonian system is equivalent to:

r-matrix machinary

$$\boxed{\begin{aligned}\frac{\partial M(\lambda)}{\partial t_i} &= -[A_i(\lambda), M(\lambda)] - \frac{\partial A_i(\lambda)}{\partial \lambda},\\ \frac{\partial M(\lambda)}{\partial \tau} &= [2B(\lambda), M(\lambda)] + 2\frac{\partial B(\lambda)}{\partial \lambda}.\end{aligned}}$$

where

$$A_i(\lambda) = \sum_{(ab)} w_{ab}(\lambda - t_i)\, J_{ab} A_i^{ab},$$

$$B(\lambda) = \sum_{i=1}^{N} \sum_{(ab)} Z_{ab}(\lambda - t_i)\, J_{ab} A_i^{ab}.$$

$$Z_{ab}(\lambda) = \frac{w_{ab}(\lambda)}{4\pi\sqrt{-1}}\left(\frac{\theta'_{[ab]}(\lambda)}{\theta_{[ab]}(\lambda)} - \frac{\theta'_{[ab]}(0)}{\theta_{[ab]}(0)}\right).$$

building blocks of elliptic KZ equation!

More familiar form:

$$\left[\frac{\partial}{\partial t_i} + A_i(\lambda),\ \frac{\partial}{\partial \lambda} - M(\lambda)\right] = 0,$$

$$\left[\frac{\partial}{\partial \tau} - 2B(\lambda),\ \frac{\partial}{\partial \lambda} - M(\lambda)\right] = 0.$$

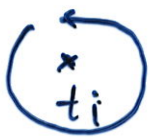
(The other Lax equations can be derived from these special equations.)

Isomonodromic Property

Lax equations ($\Longleftrightarrow$ Hamiltonian system) are Frobenius integrability condition of following linear system:

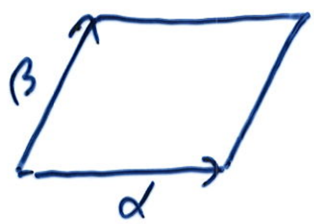
$$\boxed{\begin{aligned} &\left(\frac{\partial}{\partial \lambda} - M(\lambda)\right) Y = 0 \\ &\left(\frac{\partial}{\partial t_i} + A_i(\lambda)\right) Y = 0 \\ &\left(\frac{\partial}{\partial \tau} - 2B(\lambda)\right) Y = 0 \end{aligned}}$$

Local monodromy



$$Y(\lambda) \to Y(\lambda)\Gamma_i$$

Global monodromy



$$Y(\lambda+1) = h\, Y(\lambda)\, \Gamma_\alpha$$

$$Y(\lambda+\tau) = g^{-1} Y(\lambda)\, \Gamma_\beta$$

$\Gamma_i, \Gamma_\alpha, \Gamma_\beta$ constant : isomonodromic deformations

Relation to Elliptic KZ Equation

Etingof's twisted trace of vertex operators }
Kuroki and Takebe's twisted WZW models]

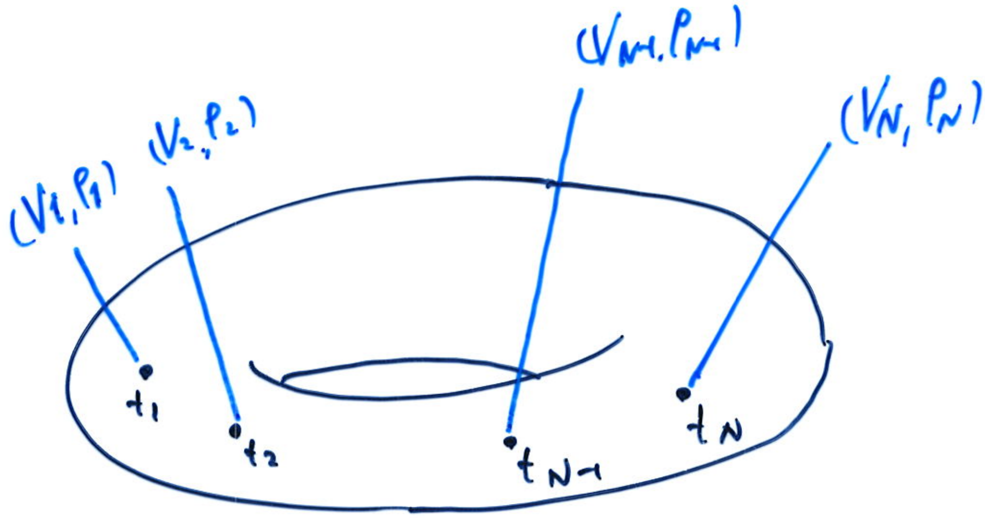
→ elliptic KZ equation for N-point conformal blocks
$F(t_1, \cdots, t_N) \in \mathrm{End}\, V$:

$$\left(K\frac{\partial}{\partial t_i} + \sum_{j(\neq i)} \sum_{(ab)} w_{ab}(t_i - t_j)\, \rho_i(J_{ab})\, \rho_j(J^{ab}) \right) F = 0$$

$$\left(K\frac{\partial}{\partial \tau} + \sum_{i,j} \sum_{(ab)} Z_{ab}(t_i - t_j)\, \rho_i(J_{ab})\, \rho_j(J^{ab}) \right) F = 0 .$$

where $K = k + n$, k: level of $\widehat{sl(n)}$

$V = V_1 \otimes \cdots \otimes V_N$, (V_i, ρ_i): irreps



① Add one more point at λ with fundamental representation $(\mathbb{C}^n, id)$. Equations for $(N+1)$-point conformal blocks $G(\lambda, t_1, \cdots, t_N) \in End(\mathbb{C}^n \otimes V)$

$$\left(\kappa \frac{\partial}{\partial \lambda} + \sum_j \sum_{(ab)} w_{ab}(\lambda - t_j)\, J_{ab}\, \rho_j(J^{ab})\right) G = 0,$$

$$\Big(\kappa \frac{\partial}{\partial t_i} + \sum_{(ab)} w_{ab}(t_i - \lambda)\, \rho_i(J_{ab}) J^{ab}$$

$$+ \sum_{j(\neq i)} \sum_{(ab)} w_{ab}(t_i - t_j)\, \rho_i(J_{ab})\, \rho_j(J^{ab})\Big) G = 0,$$

$$\left(\kappa \frac{\partial}{\partial \tau} + \cdots \qquad\qquad \right) G = 0$$

② Apply "gauge transformation" $G \to X = (F^{-1} \otimes I) G$ by "invertible solution" $F = \exp S$ of N-point elliptic KZ equation. (Schrödinger → Heisenberg)

$\Rightarrow$ X satisfies (almost) the same equation as Y, upon defining

$$A_i^{ab} = F(t_1 \cdots t_N)^{-1} \rho_i(J^{ab}) F(t_1 \cdots t_N)$$

and taking the classical limit

$$A_i^{ab} \in End\, V \longrightarrow A_i^{ab} : \text{function on } \mathcal{O}_i \subset sl(n, \mathbb{C})$$

(coad. orbit)

(also choose $\kappa = -1$, but this is not essential)

Conclusion

- An elliptic analogue of the Schlesinger equation can be constructed from the building blocks w_{ab}, z_{ab}, etc of the elliptic Gaudin model and the associated elliptic KZ equation.

- Without such a link with solvable spin models and CFT, constructing such a matrix system of isomonodromic deformation would be extremely difficult. (Possible? Okamoto's notes)

- This issue is also deeply related with "Hitchin systems" (2-dim. reduced SDYM Equations, or deformations of Higgs pair).

- What about difference/ q-difference analogue? q-difference analogues of P_{VI} & P_{V} of Jimbo and Sakai might be connected with a qKZ equation in this way.