Toy models of separation of variables

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1. SOV in Hamilton-Jacobi theory

Hamilton-Jacobi theory

- Autonomous Hamiltonian system

$$\dot{q}_j = \frac{\partial H}{\partial p_j},\ \dot{p}_j = -\frac{\partial H}{\partial q_j},\ j = 1, \dots, N$$

with canonical coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \dots, q_n, p_1, \dots, p_n)$
and Hamiltonian $H = H(\boldsymbol{q}, \boldsymbol{p})$

- Hamilton-Jacobi equation

$$H\left(q_1, \dots, q_N, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_N}\right) = E$$

• A solution of the Hamilton-Jacobi equation is called a complete solution if it depends on $I_1 = E$ and extra arbitrary constants $I_2, \ldots, I_N$ as $S = S(\boldsymbol{q}, \boldsymbol{I})$, $\boldsymbol{I} = (I_1, \ldots, I_N)$, and satisfies the regularity condition

$$\text{rank}\left(\frac{\partial^2 S}{\partial q_j \partial I_k}\right)_{j,k=1,\ldots,N} = N.$$

• A complete solution of the Hamilton-Jacobi equation defines a canonical transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{\phi}, \boldsymbol{I})$,

$$p_j = \frac{\partial S}{\partial q_j},\ \phi_j = \frac{\partial S}{\partial I_j},$$

which sends the previous Hamiltonian system to an action-angle system

$$\dot{\phi}_j = \delta_{j1},\ \dot{I}_J = 0,\ j = 1, \ldots, N,$$

thereby solving the problem.

Separation of variables

• Separation of variables (SOV) in the Hamilton-Jacobi theory is to find a set of canonical coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu}) = (\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N)$ for which a complete solution of the Hamilton-Jacobi equation takes the separated form

$$S = \sum_{j=1}^{N} S_j(\lambda_j, \boldsymbol{I}).$$

• More precisely, this means that the Hamilton-Jacobi equation

$$F\left(\lambda_1, \dots, \lambda_N, \frac{\partial S}{\partial \lambda_1}, \dots, \frac{\partial S}{\partial \lambda_N}\right) = E$$

in the separation coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $F(\boldsymbol{\lambda}, \boldsymbol{\mu}) = H(\boldsymbol{q}, \boldsymbol{p})$, reduces to separated Hamilton-Jacobi equations

$$f_j(\lambda_j, S_j'(\lambda_j), \boldsymbol{I}) = 0, \; j = 1, \dots, N.$$

• Moreover, the separated Hamilton-Jacobi equations are required to depend on extra arbitrary constants $I_2, \ldots, I_N$ in such a way that the function S to be constructed below satisfies the regularity condition.

• Solving this equation for $S_j'(\lambda_j) = dS_j/d\lambda_j$ as

$$\frac{\partial S_j}{\partial \lambda_j} = g_j(\lambda_j, \boldsymbol{I})$$

and integrating by λ_j, one obtains a complete solution of the Hamilton-Jacobi equation:

$$S = \sum_{j=1}^{N} \int^{\lambda_j} g_j(\lambda, \boldsymbol{I}) d\lambda.$$

Thus a separable system can be solved by quadrature.

Relation to Liouville integrablity

Geometrically, a complete solution of the Hamilton-Jacobi equation defines a (local) Lagrangian foliation of the phase space. For the separable system in the aforementioned sense, the Lagrangian submanifolds are defined by the equations

$$f_j(\lambda_j, \mu_j, \boldsymbol{I}) = 0,\ j = 1, \ldots, N.$$

Another interpretation of these equations is that they define the I_j's as (implicit) functions $I_j = H_j(\boldsymbol{\lambda}, \boldsymbol{\mu})$ of the phase space coordinates. They are first integrals in involution, $\{H_j, H_k\} = 0$, of the Hamiltonian $F(\boldsymbol{\lambda}, \boldsymbol{\mu}) = H(\boldsymbol{q}, \boldsymbol{p})$. Thus separability almost implies Liouville integrability; "almost" means that H_j's might not be defined globally on the phase space.

2. First toy model

Calogero's solvable particle system

$$H = \sum_{j=1}^{N} \frac{e^{\mu_j}}{\prod_{k \neq j}(\lambda_j - \lambda_k)} = \sum_{j=1}^{N} \frac{e^{\mu_j}}{B'(\lambda_j)},$$

where $B(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j)$, and $\lambda_1, \ldots, \lambda_N, \mu_1, \ldots, \mu_N$ are canonical coordinates.

• (F. Calogero, 1978) This Hamiltonian system describes the motion of zeroes of $B(\lambda)$ as $B(\lambda)$ varies "linearly":

$$\ddot{B}(\lambda) = 0 \quad \text{i.e.,} \quad B(\lambda) = B_0(\lambda) + tB_1(\lambda).$$

• (F. Calogero) This system can be generalized in several different ways. Some of them are "solvable".

• (C. Morosi and G. Tondo, 1998) This system is "separable".

Procedure of SOV

- If S is assumed to be in the separated form $S = \sum_{j=1}^{N} S_j(\lambda_j)$, the Hamilton-Jacobi equation reads

$$\sum_{j=1}^{N} \frac{\exp S_j'(\lambda_j)}{B'(\lambda_j)} = E.$$

- The Lagrange interpolation formula implies the identity

$$\sum_{j=1}^{N} \frac{\lambda_j^{N-n}}{B'(\lambda_j)} = \delta_{n,1} \text{ for } n = 0, \ldots, N-1.$$

• In particular, the polynomial

$$A(\lambda) = E\lambda^{N-1} + u_2\lambda^{N-2} + \cdots + u_N,$$

where $u_2, \ldots, u_N$ are arbitrary constants, satisfies the equation

$$\sum_{j=1}^{N} \frac{A(\lambda_j)}{B'(\lambda_j)} = E.$$

• Consequently, the foregoing Hamilton-Jacobi equation can be separated to

$$\exp S_j'(\lambda_j) = A(\lambda_j),\ j = 1, \ldots, N.$$

• One thus obtains a complete solution of the Hamilton-Jacobi eqution in the form

$$S = \sum_{j=1}^{N} \int^{\lambda_j} \log A(\lambda) d\lambda.$$

Interpretation

• By SOV, one can see that this system is a system of moving points (λ_j, μ_j), $j = 1, \ldots, N$, on the fixed curve

$$C = \{(\lambda, \mu) \mid e^{\mu} = A(\lambda)\}.$$

This curve amounts to the spectral curve in the theory of integrable system.

• One can obtain a set of first integrals $H_n(\boldsymbol{\lambda}, \boldsymbol{\mu})$ in involution by solving the equations

$$e^{\mu_j} = A(\lambda_j),\ j = 1, \ldots, N$$

for u_n's. They can be written explicitly as

$$u_n = H_n = -\sum_{j=1}^{N} \frac{e^{\mu_j}}{B'(\lambda_j)} \frac{\partial v_n}{\partial \lambda_j},$$

where $v_1, \ldots, v_N$ are the coefficients of $B(\lambda)$,

$$B(\lambda) = \lambda^N + v_1 \lambda^{N-1} + \cdots + v_N.$$

Since $v_1 = -\lambda_1 - \cdots - \lambda_N$, we have $H_1 = H$ as expected.

Modified model

The same procedure of SOV works for the modified Hamiltonian (also due to Calogero)

$$H = \sum_{j=1}^{N} \frac{e^{\mu_j} - c\lambda_j^N}{B'(\lambda_j)} \quad (c \text{ is a constant})$$

as well. The polynomial $A(\lambda)$ now takes the form

$$A(\lambda) = c\lambda^N + \sum_{n=1}^{N} u_n \lambda^{N-n}.$$

3. Variation I

Three variants of the foregoing toy model have been proposed (K.T. and T. Takebe, J. Geom. Phys. 47 (2003), 1–20). They are formulated by the following pair $A(\lambda), B(\lambda)$ of functions or, more precisely, by the quotient $B(\lambda)/A(\lambda)$:

- rational model
- hyperbolic model
- ellitpic model

Rational model

$$A(\lambda) = \lambda^N + \sum_{n=2}^{N} u_n \lambda^{N-n},$$

$$B(\lambda) = \lambda^{N-1} + \sum_{n=2}^{N} v_n \lambda^{N-n}.$$

- Solving the equations

$$e^{\mu_j} = A(\lambda_j)$$

for $u_2, \ldots, u_N$ yields the first integrals

$$H_n = -\sum_{j=1}^{N-1} \frac{e^{\mu_j} - \lambda_j^N}{B'(\lambda_j)} \frac{\partial v_n}{\partial \lambda_j}.$$

• In particular, the lowest Hamiltonian $H = H_2$ reads

$$H = \sum_{j=1}^{N-1} \frac{e^{\mu_j} - \lambda_j^N}{B'(\lambda_j)}.$$

• This Hamiltonian and the higher ones are simultaneously seprabable by the same method as the previous case. (This is obvious from the construction.)

• This system is actually not new; it is equivalent to the open Toda chain. Moreover, writing $A(\lambda)$ and $B(\lambda)$ in the factorized form

$$A(\lambda) = \prod_{j=1}^{N} (\lambda - \alpha_j),\ B(\lambda) = \prod_{j=1}^{N-1} (\lambda - \lambda_j),$$

one can see a link with Donaldson's moduli space of SU(2) monopoles. This factorized form inspires the following generalizations.

Hyperbolic and elliptic models

These models are formulated by the following pair of functions $A(\lambda), B(\lambda)$:

1. Hyperbolic model

$$A(\lambda) = \prod_{j=1}^{N} \sinh(\lambda - \alpha_j), \; B(\lambda) = \prod_{j=1}^{N} \sinh(\lambda - \lambda_j).$$

2. Elliptic model

$$A(\lambda) = \prod_{j=1}^{N} \sigma(\lambda - \alpha_j), \; B(\lambda) = \prod_{j=1}^{N} \sigma(\lambda - \lambda_j).$$

The Hamiltonians of these models are obtained by epand-ing $A(\lambda)$ with a suitable set of functions $f_n(\lambda)$ as

$$A(\lambda) = \sum_n u_n f_n(\lambda)$$

and solving the equations $e^{\mu_j} = A(\lambda_j)$ for u_n's.

Open (?) problem

One will naturally expect to derive the hyperbolic and elliptic analogues

$$1. \qquad H = \sum_{j=1}^{N} \frac{e^{\mu_j}}{\prod_{k \neq j} \tanh(\lambda_j - \lambda_k)},$$

$$2. \qquad H = \sum_{j=1}^{N} \frac{e^{\mu_j}}{\prod_{k \neq j} \sigma(\lambda_j - \lambda_k)}$$

in this framework or its suitable modification. This, however, has been unsuccessful; separability of these Hamiltonians seems to be an open problem. On the other hand, these two Hamiltonian are known to be "solvable" in the sense that they can be derived from the motion of zeroes of a function with simple t-dependence (see, e.g. F. Calogero and J.-P. Françoise, IMRN 15 (2000), 775–786).

4. Variation II

A slightly different version of the elliptic model is proposed by A. Odesskii and V. Rubtsov (arXiv:math.QA/0404159). Actually, they consider a quantum integrable system with a commuting set of Hamiltonians.

• Translated to a classical Hamiltonian system, their construction is based on the equations

$$e^{\mu_j} = A(\lambda_j),$$

where $A(\lambda)$ is a function of the form

$$A(\lambda) = \sum_{n=1}^{N} u_n f_n(\lambda).$$

$f_n(\lambda)$'s are a basis of theta functions of degree N. Solving these equations for u_n's yields a Poisson-commuting set of Hamiltonians $u_n = H_n(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

• The choice of $f_n(\lambda)$'s is different from that of the joint work with Takebe. This obscures the relation of the two constructions.

• To go to the quauntm model, the phase space coordinates are replaced by operators as

$$\lambda_j \to \lambda_j,\ \mu_j \to \hbar\partial/\partial\lambda_j.$$

As Odesskii and Rutsov show, one can avoid the ordering problem and obtain a commuting set of quantum Hamiltonians, which give a quantum counterpart of the previous classical separable Hamiltonians. This is a toy model of quantum SOV.

5. Variations III

The previous two variations are concerned with the choice of $A(\lambda)$ and $B(\lambda)$. What about replacing e^{μ} by something different?

Yet another elliptic model?

A crazy, but intriguing idea is to take an elliptic function such as $\wp(\mu)$ in place of e^{μ}. Thus we now propose to consider

$$\wp(\mu_j) = A(\lambda_j),\ H = \sum_{j=1}^{N} \frac{\wp(\mu_j)}{B'(\lambda_j)}, \text{ etc.}$$

Moreover, if $A(\lambda)$ and $B(\lambda)$ are also elliptic functions, this may be called a "doubly elliptic model".

Possible degenerate models

An intermediate step towards a full understanding of this model will be to examine the cases that appear when $\wp(\mu)$ degenerates to hyperbolic, trigonometric or rational functions. Remarkably, the degenerate cases turn out to be related to many known separable (or integrable) systems.

• The point of departure is the relation between the elliptic function and an elliptic integral:

$$0) \quad \mu = \int_{\infty}^{z} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} \longleftrightarrow z = \wp(\mu)$$

As the ellitic curve $y^2 = 4z^3 - g_2 z - g_3$ degenerates to a singular (rational) curve, $\wp(\mu)$ tunrs into an elementary function. The following is a list of typical cases.

$$1) \quad \mu = \int^z \frac{dz}{2\sqrt{z}(z-1)} \quad \longleftrightarrow \quad z = \coth^2 \mu$$

$$2) \quad \mu = \int^z \frac{dz}{\sqrt{z^2-1}} \quad \longleftrightarrow \quad z = 2\cosh \mu$$

$$3) \quad \mu = \int^z \frac{dz}{2\sqrt{z}} \quad \longleftrightarrow \quad z = \mu^2$$

$$4) \quad \mu = \int^z \frac{dz}{z} \quad \longleftrightarrow \quad z = e^\mu$$

$$5) \quad \mu = \int^z dz \quad \longleftrightarrow \quad z = \mu$$

• 5) is the most degenerate case, for which SOV turns out to take a very simple form.

• 4) is exactly the case that has been considered in the previous models.

• 3) and 2) are rather familiar in the theory of int-grable systems. Curves of the form $2\cosh\mu = A(\lambda)$ and $\mu^2 = A(\lambda)$ take place in SOV of integrable systems with hyperelliptic spectral curves (the periodic Toda chain, the stationary higher KdV equations, etc.).

• 1) is as mysterious as the nondegenerate case 0) itself. What kind of separable (or integrable) systems arise in this case ?

6. Summary

• Calogero's particle system and its variants provide a simple model ("toy model") of SOV. They are all related to a curve of the special form $e^{\mu} = A(\lambda)$ (essentially, the graph of the function $A(\lambda)$). This simplifies the situation.

• This simplicity is inherited by a quantum system.

• The naive hyperbolic and elliptic deformations due to Calogero are "solvable", but separability seems to be an open problem.

• Replacing e^{μ} by $\wp(\mu)$ is a highly speculative idea. This, however, might be a unified framework that covers many known and unknown separable systems.