京大理学部
研究会

Integrable Systems
and
Branes

K. Takasaki

Jan 28 1998

Plan

- Integrable Hamiltonian Systems (C^∞/analytic/algebraic)
- D=4 N=2 SUSY gauge theories and integrable systems
- D=4 N=2 SUSY gauge theories and M-theory (M5-branes)
- Mirror maps and integrable systems

(• What are branes?)

I. $\boxed{\text{Integrable Hamilton Systems}}$

Ref. Donagi & Markman, alg-geom/9505009
Donagi, alg-geom/9705010

1. integrable Hamiltonian systems (C^∞)

(C^∞) integrable Hamiltonian system
is a symplectic manifold (X, ω) (or Poisson)
with Lagrangian fibration $\pi : X \to U$

$X_u = \pi^{-1}(u)$: Lagrangian submanifold of X

- $H_1, H_2 : U \to \mathbb{R} \Rightarrow \{\pi^* H_1, \pi^* H_2\} = 0$
- $X_u \sim \mathbb{R}^n, \mathbb{T}^n$

invariant tori

2. algebraic/analytic integrable Hamiltonian systems

analytic : X_u = complex torus ($\simeq \mathbb{C}^n/L$)

algebraic : X_u = (polarized) abelian variety

$\downarrow$

holomorphic flat structure :

polarization

$H_1(X_u, \mathbb{Z}) \cong \mathbb{Z}^{2n}$, skew-symmetric form

smoothly depends on u

flat bundle over U

α_j, β_j $(j = 1..n)$: symplectic basis

$V_{\mathbb{Z}}^* := \sum \mathbb{Z}\alpha_j \subset H_1(X_u, \mathbb{Z})$

symplectic form

flat structure of U

modelled on $V = \mathrm{Hom}(V_{\mathbb{Z}}^*, \mathbb{C})$

$$dz_j = \int_{\alpha_j} \omega$$

$\{z_j\}$: flat coordinates

($\sim \{a_i\}$ in SW)

3. Cubic and Prepotential of algebraic IHS

period $\tau : U \to \mathbb{H}_g$ (Siegel upper half plane)

$$(\tau_{ij})$$

$$d\tau_{ij} = \sum \frac{\partial \tau_{ij}}{\partial z_k} dz_k$$

algebraic integrable system

cubic :

$$c_{ijk} = \frac{\partial \tau_{ij}}{\partial z_k} \text{ is totally symmetric}$$

potentials :

$$\tau_{ij} = \frac{\partial w_i}{\partial z_j}$$

$\{w_i\}$: dual coordinates

($\sim \{a_i^D\}$ in SW)

prepotential $\mathcal{F}$:

$$w_i = \frac{\partial \mathcal{F}}{\partial z_i}$$

* Similar construction applies to an analytic integrable Hamiltonian system constructed from deformations of Calabi-Yau 3-folds.

4. Isospectral Problem (Lax representation)

Example (due to Garnier; Moser; Adams et al; Beauville)

total space:

$$X = \left\{ L(t) = \sum_{i=1}^{N} \frac{A_i}{t-\alpha_i} \;\middle|\; A_i \in sl(n,\mathbb{C}) \right\}$$

$$\cong sl(n,\mathbb{C}) \times \cdots \times sl(n,\mathbb{C})$$

Poisson structure: Kostant-Kirillov structure

Hamiltonians:

$$\det(vI - L(t)) = v^n + \sum_{\alpha=2}^{n} g_\alpha(t)\, v^{n-\alpha}$$

$\prod_{i=1}^{N}(t-\alpha_i)^{r_i} \cdot g_\alpha(t)$: polynomial in t

two types of coefficients:

$\begin{cases} \text{Casimirs (determined by coad. orbit)} \\ \text{isospectral Hamiltonians} \end{cases}$

Symplectic leaves $\cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_N$

$\mathcal{O}_i \subset sl(n,\mathbb{C})$: coad. orbit

$X \supset X_{\mathcal{O}_1,\cdots,\mathcal{O}_N} \longrightarrow U$: algebraic integrable system

fiber $=$ Jac(Σ), Σ: $\det(vI - L(t)) = 0$

II. $D=4$ $N=2$ SUSY Gauge Theories and Integrable systems

Ref. Donagi & Witten, hep-th/9510101

1. $\mathcal{N}=2$ SUSY YM without matter, $G = SU(n)$

spectral curve

$$t^2 - 2P(v)t + \Lambda^{2n} = 0$$

$$P(v) = v^n + \sum_{j=2}^{n} u_j v^{n-j}$$

moduli $\{u_j\}$ $\longleftrightarrow$ vevs of scalar ϕ

$$\det(v+\phi) = P(v)$$

genus $= n-1$ $=$ # moduli

$$t = \frac{P + \sqrt{P^2 - \Lambda^{2n}}}{\Lambda^n}$$

hyperelliptic curve

$$w^2 = P^2(v) - \Lambda^{2n}.$$

$$t = \frac{P+w}{\Lambda^n}$$

Seiberg-Witten differential

$$dS = v \, d\log t$$

periods

$$a_i = \oint_{A_i} dS$$

$$a_i^D = \oint_{B_i} dS \qquad (A_i \cdot B_j = \delta_{ij})$$

prepotential $\mathcal{F} = \mathcal{F}(a)$

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i}$$

$$\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \tau_{ij} = \text{period matrix}$$

2. <u>integrable system</u>: <u>n-periodic Toda chain</u>

$$t^2 - 2P(v)t + \Lambda^{2n} = \det(t - M(v))$$
$$= \tfrac{?}{} \det(v - L(t))$$

$$\left.\begin{array}{ll} M(v): & 2 \times 2 \text{ matrix} \\ L(t): & n \times n \text{ matrix} \end{array}\right\} \text{"Lax operators"}$$

$$P(v) = v^n + \sum_{j=2}^{n} u_j v^{n-j}$$

$\{u_j\}$: constants of motion (Hamiltonians)

$$L(t) = \begin{pmatrix} b_1 & c_1 & & & c_n t \\ c_1 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ c_n t^{-1} & & & c_{n-1} & b_n \end{pmatrix}$$

$b_i = p_i$, $c_i = e^{q_{i-1} - q_i}$ $\qquad \{p_i, q_j\} = \delta_{ij}$

$$\begin{array}{ccc} X & & (p_1, \cdots, p_{n-1}, q_1, \cdots, q_{n-1}) \\ \downarrow & & \downarrow \\ U & & (u_2, \cdots, u_n) \end{array}$$

fiber $X_u \hookrightarrow$ Jacobian of spectral curve C

e.g. $M_{12}(v) = 0 \rightarrow$ $n-1$ roots
$v = v_1, \cdots, v_{n-1}$

$t_i := M_{11}(v_i) \rightarrow (t_i, v_i) \in C$

$n-1$ points $\{(t_i, v_i)\}_{i=1}^{n-1} \rightarrow$ diviser $D \in \mathrm{Jac}^{g}(C)$

3. Other Cases

(a) $G = A, D, E$ $\rightarrow$ $\hat{G}$ - Toda chain, Prym variety

In general: Langlands dual ?

(b) $\mathcal{N}=2$ SYM + adjoint matter $\left(\xrightarrow{m\to 0} \begin{array}{c}\mathcal{N}=4\\ \text{SYM}\end{array}\right)$
$G = SU(n)$ mass m

$\rightarrow$ elliptic Calogero-Moser system

spectral curve $C \rightarrow E$ (elliptic covering of curve)

cf. (a) $C \rightarrow \mathbb{C}P^1$

For G other than $SU(n)$: ?

(c) $\mathcal{N}=2$ SYM + fundamental matters

$$t^2 - 2P(v)\,t + \Lambda^{2n-n'} Q(v) = 0$$

$$Q(v) = \prod_{j=1}^{n'} (v + m_j) \qquad n' = \#\text{ matters}$$

integrable systems: spin chain ?
(not satisfactory)

4. Why Integrable systems?

Donagi & Witten : (also Freed, hep-th/9712042)

① effective gauge theory is abelian —

$G \to T$ e.g. $G = SU(n) \to T = U(1)^{n-1}$.

$$L_{eff} \sim \mathrm{Im}\,\tau_{ij} F^i_{\mu\nu} F^j_{\mu\nu} + \mathrm{Re}\,\tau_{ij} F^i_{\mu\nu} \tilde{F}^j_{\mu\nu} + \cdots$$

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j} \quad (\leftarrow N=2 \text{ prepotential})$$

② abelian gauge theory has Electric-Magnetic duality, and EM charges are quantized to a Lattice with principal polarization

$L \subset t^*$

$$q^i = (\mathrm{Im}\,\tau^{-1})^{ij}\left((\mathrm{Re}\,\tau)_{jk} m^k + n_j\right)$$

$$g^i = m^i \qquad (m^i, n_j \in \mathbb{Z})$$

$$q^i + \sqrt{-1}\, g^i = (\mathrm{Im}\,\tau^{-1})^{ij} \left(\tau_{jk} m^k + n_j\right)$$

integral symplectic form

$$\langle (q,g), (q',g') \rangle = m \cdot n' - n \cdot m' \rightarrow \text{principal polarization}$$

③ t^*/L is an abelian variety X_u that depends holomorphically on moduli $\{u_i\}$

④ a natural symplectic form exists on the total space of $X \to U$ (induced from T^*U)

III. N=2 SUSY gauge theories and M-theory (M5-branes)

Ref. Witten, hep-th/9703166

11D $\boxed{\text{M-theory}}$ $\qquad \mathbb{R}^4 \times \mathbb{R}^3 \times X$

X: complex symplectic manifold (actually hyperKähler)

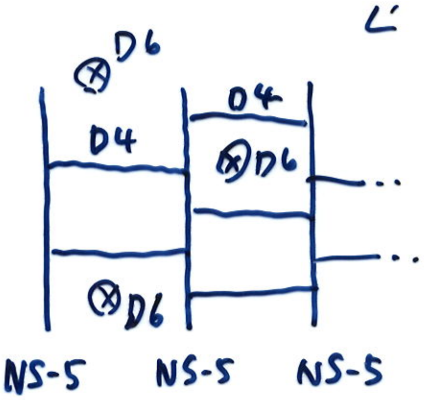
multi-Taub-NUT (KK-monopole)

S^1 compactification (radius R, R: small)

10D $\boxed{\text{IIA}}$

$\mathbb{R}^4 \times \Sigma$ 5-brane world volum

Σ: complex curve (Riemann surface)



D4 : IIA D4-brane

NS5 : IIA NS 5-brane

D6 : IIA D6-brane

(D: Dirichlet
NS: Neven-Schwarz)

web of branes

Witten's proposal:

Gauge theory is induced on $\mathbb{R}^4$ from dynamics of M5-brane.

1. <u>N=2 SYM without matter</u> $\quad G = SU(n)$

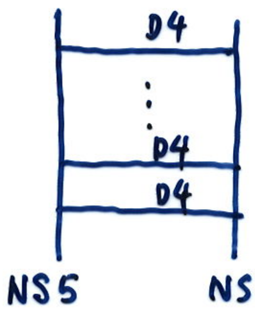
<u>11D</u> : $\quad \mathbb{R}^{10} \times S^1$ $\qquad x^{10} \sim x^{10} + 2\pi R$

$$\underbrace{x^0\ x^1\ x^2\ x^3}_{\mathbb{R}^4}\ \ x^4\ x^5\ x^6\ x^7\ x^8\ x^9\ x^{10}$$

$$v = \frac{x^4 + i x^5}{R} \qquad s = \frac{x^6 + i x^{10}}{R}$$

$$t = e^{-s}$$

<u>10D</u> Brane configuration : (conceptual, not accurate)



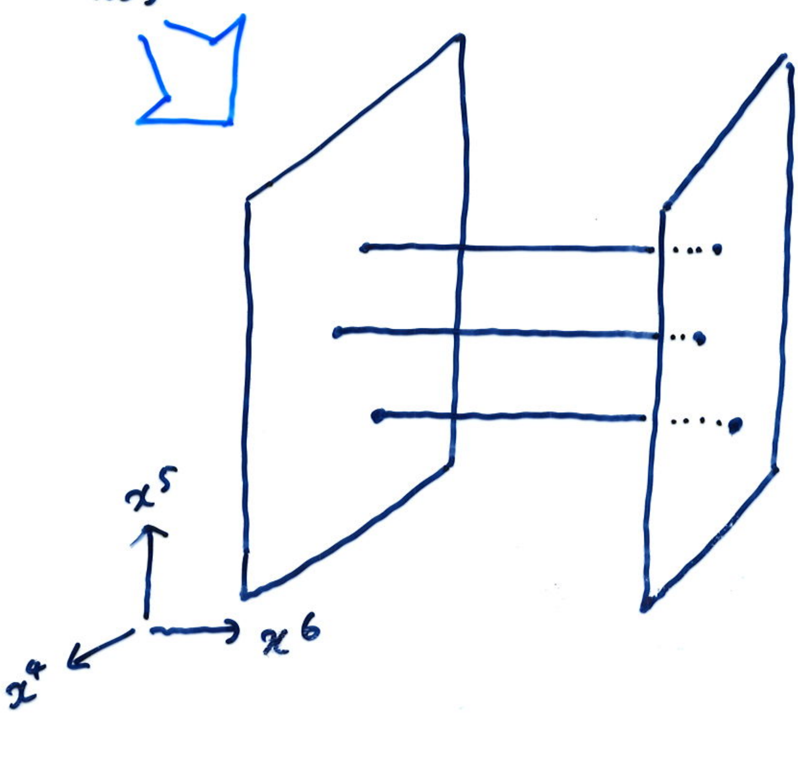
$\rightarrow v = \phi_1$

$\rightarrow v = \phi_{n-1}$

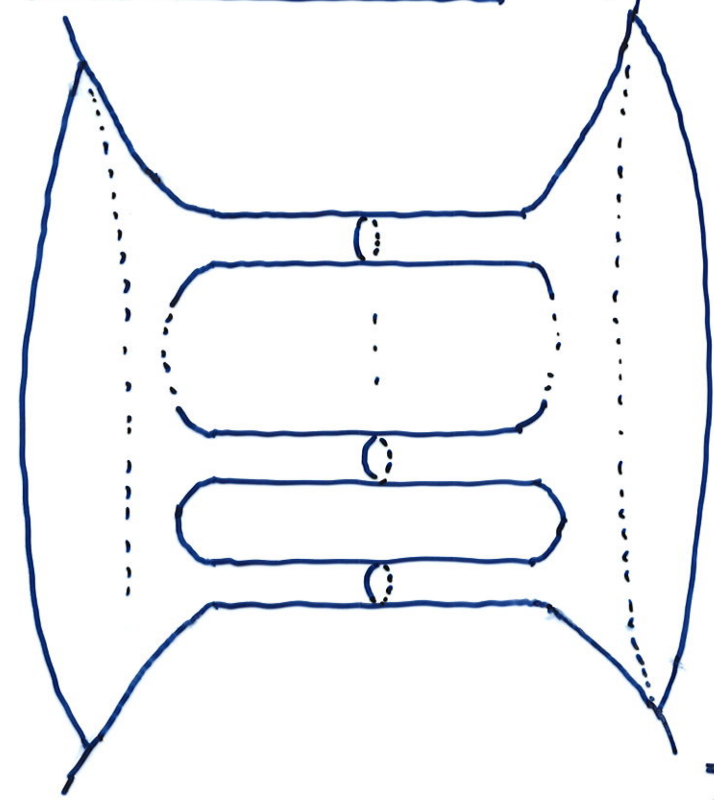
$\rightarrow v = \phi_n$

$$P(v) = \prod_{a=1}^{n} (v - \phi_a)$$

ϕ_a : vev of scalars



11D M5-brane configuration : (true shape of branes)



$\mathbb{R}^4 \times \Sigma$

Σ : Riemann surface

M5-brane : $\mathbb{R}^4 \times \Sigma \hookrightarrow \mathbb{R}^4 \times \mathbb{R}^6 \times S^1$

$$t^2 - 2P(v)t + \Lambda^{2n} = 0.$$

Σ is embedded into a complex symplectic manifold :

$$\Sigma \hookrightarrow \mathbb{R}^3 \times S^1 = \underset{(v,\ t)}{\mathbb{C} \times \mathbb{P}^1}$$

$$\omega = dv \wedge d\log t$$

$\downarrow$

SW differential $= v\, d\log t$

<u>Test</u> :

Ref ① Howe, Lambert & West hep-th/9710034

② Lambert & West hep-th/9712040

① M5-brane kinetic energy (← quadratic part of Born-Infeld action)

$$I = \int_{\mathbb{R}^4 \times \Sigma} d^6x \; \partial_\mu s \cdot \partial^\mu \bar{s}$$

$\downarrow$ Riemann's bilinear relation, etc.

$$= \text{const.} \operatorname{Im} \int_{\mathbb{R}^4} d^4x \; \tau_{ij} \, \partial_\mu a_i \, \partial^\mu \bar{a}_j$$

$$a_i = \oint_{A_i} ds \, , \quad \tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$$

= low energy effective action of <u>scalar</u> fields in $N = 2$ multiplet

② eqs of motion of tensor fields on M5-brane

$\downarrow$

eqs of motion of <u>gauge</u> fields and <u>scalar</u> fields in low energy effective theory

$$S = \text{const.} \operatorname{Im} \int d^4x \; \tau_{ij} \left(\partial_\mu a_i \, \partial^\mu \bar{a}_j + F_{i\mu\nu} F^{i\mu\nu} \right)$$

2. Other Cases

① $N=2$ $SU(n)$ with fundamental matters $\left(\begin{smallmatrix}\text{Coulomb}\\ \text{branch}\end{smallmatrix}\right)$

— adding D6-branes

$$X = Q_0 = \mathbb{R}^3 \times S^1 \quad \cdots\cdots\to \quad X = Q \ (\text{multi-Taub NUT})$$

$$= \mathbb{C} \times \mathbb{P}^1 \qquad\qquad (v, y) \quad \text{ALE space}$$

$$(v, t)$$

$$yz = \Lambda^{2n-n'} \prod_{i=1}^{n'} (v - e_i)$$

$$\omega = dv \wedge d\log y$$

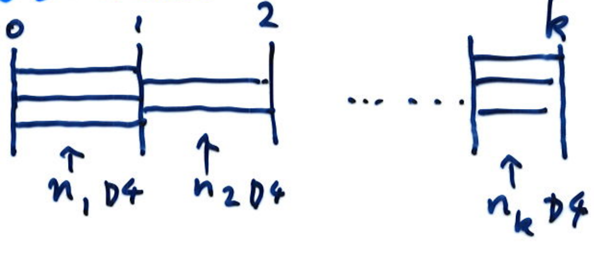
$\Sigma \hookrightarrow Q$:

$$y^2 - 2\prod_{a=1}^{n}(v - \phi_a)\cdot y + \Lambda^{2n-n'} \prod_{i=1}^{n'} (v - e_i) = 0$$

$-e_i = m_i$: mass of i-th fundamental matter multiplet

② $N=2$ $\quad G = \prod_{\alpha=1}^{k} SU(n_\alpha)$

— $k+1$ NS5-branes



$\Sigma \hookrightarrow Q_0$:

$$t^{k+1} + g_1(v)\, t^k + g_2(v)\, t^{k-1} + \cdots + g_k(v)\, t + 1 = 0$$

$$\deg(g_\alpha) = n_\alpha$$

physical contents:

$N=2$ gauge fields + bi-fundamental matter

③ $N=2$ $G=\prod_{\alpha=1}^{k} SU(n_\alpha)$ with additional fundamental matters

— adding D6-branes to case ②

$\Sigma \hookrightarrow Q$

④ elliptic model

$X \to E$ (elliptic curve)

$\left(\text{cf. ①, ② ③} \quad X \to \mathbb{P}^1 \right)$

typically,

$$X = X_m = \underset{(v,\ x^6,\ x^{10})}{\mathbb{C} \times \mathbb{R} \times S^1} / \sim$$

$$(v, x^6, x^{10}) \sim (v+m,\ x^6 + 2\pi L,\ x^{10} + \theta)$$

m, L, θ : parameters

$$\begin{array}{ccc} X & \longrightarrow & E = \mathbb{R} \times S^1/\sim \\ (v, x^6, x^{10}) & \longrightarrow & (x^6, x^{10}) \end{array}$$

3. Where is an integrable system?

special M5-branes

$\Updownarrow$

embedded Riemann surfaces $\Sigma \subset X$

with boundary conditions and
sheet structure conditions } ← brane structure

→ same number of moduli as genus (= rank G)

(∃ some other parameters which are
NOT moduli (of Coulomb branch)
...... bare coupling constant,
bare mass m_i
RG mass scale Λ)

This is also a characteristic of integrable systems!

- $X \to U$ X_u = Lagrangian submanifolds of (X, ω)
  u : moduli of invariant tori
- Toda system p_i, q_i $(i = 1 \dots n-1)$ → Hamiltonian H_i $(i = 1 \dots n-1)$ = moduli
- etc ...

However, the M5-brane picture shows only the spectral Riemann surface Σ (or the moduli space U). Where is the total phase space of an integrable system?

Witten says:

total space of integrable system is given by

$$\mathcal{W} = \{ (\Sigma, \mathcal{L}) \mid \Sigma \subset X,\ \mathcal{L} \text{ is a line bundle (over } \Sigma) \text{ of specific degree} \}$$

$$\downarrow$$

$$U = \{ \Sigma \mid \exists \mathcal{L},\ (\Sigma, \mathcal{L}) \in \mathcal{W} \}$$

Physical role of $\mathcal{L}$:

splitting of a 2-form field on $\mathbb{R}^4 \times \Sigma$ induces g harmonic forms on Σ

$\rightarrow$ a point of $\mathrm{Jac}(\Sigma)$

Further generalization:

$(\Sigma, \mathcal{L}) \rightsquigarrow$ coherent sheaf $\mathcal{E}$ over X with $\operatorname{supp} \mathcal{E} = \Sigma$ and $\mathcal{E}|_\Sigma = \mathcal{L}$

$\rightarrow$ Link with the work of

Mukai, Donagi, Ein & Lazarsfeld

$\rightarrow$ brane as sheaf ?

IV. Mirror Maps and Integrable Systems

1. "IIA – IIB duality" via integrable systems

Marshakov, Martellini & Morozov,
hep-th/9706050

IIA/M		IIB/F
Whitham (= SW) integrable structure	$\xleftarrow{\text{link}}$	Hitchin integrable structure
⋮		⋮
M5-brane effective action of 4D gauge fields etc…		F → IIB compactification over torus, dimensional reduction of 10D SYM etc…

Logic of integrable systems only.

None of string/field theories.

unsatisfactory!

2. integrable systems from intermediate Jacobians

Donagi & Markman, alg-geom/9408004
alg-geom/9507017

① X : Calabi-Yau 3-fold

$$J(X) = H^3(X,\mathbb{C}) / (F^2H^3 + H^3(X,\mathbb{Z}))$$

$$F^p = \bigoplus_{i \geq p} H^{i,3-i}$$

complex torus, but NOT abelian

② $\mathcal{X} \to U$: holomorphic family of CY 3-folds
U
X_u : fiber at u

$J(X_u)$, $u \in U$, fit together to form a holomorphic family $\mathcal{J} \to U$ of intermediate Jacobians.

③ $\mathcal{X} \to U$: complete family, & $T_{[X]} U \cong H^1(TX)$

$\Rightarrow$ fatten family $\tilde{\mathcal{J}} \to \tilde{U}$ is an analytic integrable system. Here:

$\tilde{U} = \{ (X, \Omega) \mid \Omega \text{ nonzero } (3,0)\text{-form} \}$

$\tilde{\mathcal{J}} \to \tilde{U}$ is the pullback of $\mathcal{J} \to U$ by the map $\tilde{U} \to U$.

④ "Hamiltonians" $\tilde{t}_i = \int_{\Gamma_i} \Omega$ Γ_i: 3-cycles $(i = 0, \cdots, h^{21})$

can be rewritten into action integrals:

$$\tilde{t}_i = \int_{\gamma_i} \theta \qquad \begin{pmatrix} d\theta = \text{symplectic 2-form on } \mathcal{X} \\ \gamma_i : \text{1-cycles on } J(X_u) \end{pmatrix}$$

analogues of $a_i = \oint_{A_i} dS$ in SW

⑤ This is essentially the mirror map: -

$$t_i = \tilde{t}_i / \tilde{t}_0$$

$$\begin{tikzcd} \mathcal{X} \arrow[d] & \\ U \arrow[r, "t"] & K' \end{tikzcd}$$

K' : Kähler moduli of mirror CY 3-fold X'

B model ⋮ integrable system

A model ⋮ Hodge theory

parallel

---

cf. Marshakov, Mantellini & Morozov

$$\text{IIB} \xrightarrow{a} \text{IIA}$$

Hitchin system

Whitham system

$$a_i = a_i(u) = \oint_{A_i} dS$$

$$\text{Im} \int d^4x \; \tau_{ij} \, \partial_\mu a_i \cdot \partial^\mu \bar{a}_j$$

What is the point ?

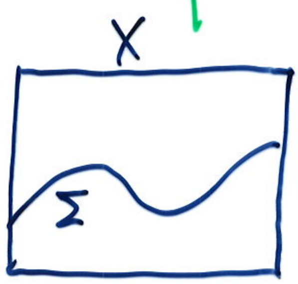
— integrable system provides a "mirror map" as action variables.

What is mirror symmetry ?

— (according to Strominger, Yau and Zaslow) a kind of "Radon transformation" ?

reminiscent of twistor theory !?

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ \Sigma \subset X \end{array}$$



What is brane ?

— "brane" is a sheaf

— branes form an integrable system ?

Open End ...