Dispersionless Hirota equations of multi-component integrable hierarchies

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- K.T. and T. Takebe, Universal Whitham hierarchy, dispersionless Hirota equations and multi-component KP hierarchy, nlin.SI/0608068.
- K.T., Dispersionless Hirota equations of two-component BKP hierarchy, SIGMA **2** (2006), Paper 057.
- work in progress ...

1. Dispersionless Hirota equations of KP hierarchy

τ-function of KP hierarchy

$\tau = \tau(\boldsymbol{t})$, $\boldsymbol{t} = (t_1, t_2, \ldots)$, $t_1 = x$

- Relation to Lax operator L:

$$L = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + \cdots,$$
$$u_2 = (\log \tau)_{t_1 t_1},\ u_3 = \frac{1}{2}(\log \tau)_{t_1 t_2} - \frac{1}{2}(\log \tau)_{t_1 t_1 t_1},\ \ldots$$

- Relation to wave functions:

$$\Psi(\boldsymbol{t}, z) = \frac{\tau(\boldsymbol{t} - [z^{-1}])}{\tau(\boldsymbol{t})} e^{\xi(\boldsymbol{t}, z)}, \quad \Psi^*(\boldsymbol{t}, z) = \frac{\tau(\boldsymbol{t} + [z^{-1}])}{\tau(\boldsymbol{t})} e^{-\xi(\boldsymbol{t}, z)},$$
$$\xi(\boldsymbol{t}, z) = \sum_{n=1}^{\infty} t_n z^n, \quad [z^{-1}] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots, \frac{z^n}{n}, \ldots\right)$$

An infinite number of Hirota equations

$$(3D_{t_2}^2 - 4D_{t_3}D_{t_1} + D_{t_1}^4)\tau \cdot \tau = 0, \ \ldots$$

where

$$P(D_{t_t}, D_{t_2}, \cdots)\tau \cdot \tau = P(\partial_{t'_1} - \partial_{t_1}, \partial_{t'_2} - \partial_{t_2}, \ldots)\tau(\boldsymbol{t}') \cdot \tau(\boldsymbol{t})|_{\boldsymbol{t}'=\boldsymbol{t}}$$

Generating functional form of Hirota equations (DJKM 81)

$$\oint \frac{dz}{2\pi i} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)}\tau(\boldsymbol{t}' - [z^{-1}])\tau(\boldsymbol{t} + [z^{-1}]) = 0$$

here

$$\xi(\boldsymbol{t}, z) = \sum_{n=1}^{\infty} t_n z^n, \quad [z^{-1}] = \Big(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots, \frac{z^n}{n}, \ldots\Big).$$

The contour integral is along $|z| = R$ with R sufficiently large (or interpreted as an algebraic operator extracting the coefficient of z^{-1}).

Dispersionless limit = quasi-classical limit

Assume that an $\hbar$-dependent τ-function $\tau = \tau(\hbar, \boldsymbol{t})$ behaves in classical ($\hbar \to 0$) limit as

$$\tau(\hbar, \hbar^{-1}\boldsymbol{t}) = \exp(\hbar^{-2}F(\boldsymbol{t}) + O(\hbar^{-1}))$$

Passage to dispersionless Hirota equations

$$\begin{array}{c} \text{DJKM bilinear equations for } \tau \\ \downarrow \\ \text{differential Fay identity for } \tau \\ \downarrow\ \hbar \to 0 \\ \text{dispersionless Hirota equations for } F \end{array}$$

Differential Fay identity (Adler & van Moerbeke 92)

$$\frac{\tau(\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}])\tau(\boldsymbol{t})}{\tau(\boldsymbol{t}+[\lambda^{-1}])\tau(\boldsymbol{t}+[\mu^{-1}])} = 1 - \frac{1}{\lambda-\mu}\partial_{t_1}\log\frac{\tau(\boldsymbol{t}+[\lambda^{-1}])}{\tau(\boldsymbol{t}+[\mu^{-1}])}$$

holds for arbitrary λ, μ.

Equivalent form

$$\exp\Big((e^{D(\lambda)}-1)(e^{D(\mu)}-1)\log\tau\Big) = 1 - \frac{1}{\lambda-\mu}\partial_{t_1}(e^{D(\lambda)}-e^{D(\mu)})\log\tau,$$

where

$$D(z) = \sum_{n=1}^{\infty}\frac{z^{-n}}{n}\partial_{t_n}.$$

Derivation of differential Fay identity

1. Differentiate the DJKM bilinear equation by t'_1.
2. Set $\boldsymbol{t}' = \boldsymbol{t} + [\lambda^{-1}] + [\mu^{-1}]$ $(|\lambda|, |\mu| > R)$.

This yields an equation of the form

$$\oint \frac{dz}{2\pi i} \frac{\lambda\mu}{(z-\lambda)(z-\mu)} \Big((\partial_{t_1}\tau)(\boldsymbol{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}]) \\ + z\tau(\boldsymbol{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}]) \Big) \tau(\boldsymbol{t} + [z^{-1}]) = 0.$$

The integral can be calculated by deforming the contour and collecting the residues at $z = \lambda, \mu, \infty$. The outcome is the differential Fay identity.

Dispersionless limit

• For the rescaled τ-function $\tau_\hbar = \tau(\hbar, \hbar^{-1}\boldsymbol{t})$, the differential Fay identity reads

$$\exp\Bigl((e^{\hbar D(\lambda)} - 1)(e^{\hbar D(\mu)} - 1)\log\tau_\hbar\Bigr) = 1 - \frac{1}{\lambda - \mu}\hbar\partial_{t_1}(e^{\hbar D(\lambda)} - e^{\hbar D(\mu)})\log\tau_\hbar.$$

• By the quasi-classical ansatz $\tau_\hbar \sim e^{\hbar^{-2}F(\boldsymbol{t})}$, this equation reduces to the dispersionless Hirota equation (T. & Takebe 95; Carroll & Kodama 95)

$$e^{D(\lambda)D(\lambda)F} = 1 - \frac{\partial_{t_1}(D(\lambda) - D(\mu))F}{\lambda - \mu}.$$

• In other words, the differential Fay identity is a dispersive counterpart of the dispersionless Hirota equation.

Relation to auxiliary linear equations

- The differential Fay identity can be rewritten as

$$\lambda e^{-D(\lambda)}\Psi(\mu) = -(\partial_{t_1} - \partial_{t_1} \log \Psi(\lambda))\Psi(\mu).$$

- An equivalent form (known?)

$$\lambda \Psi(\lambda)^{-1} e^{-D(\lambda)}\Psi(\mu) = -\partial_{t_1}\Big(\Psi(\lambda)^{-1}\Psi(\mu)\Big).$$

- This equation is a generating functional form of the auxiliary linear equation

$$\partial_{t_n}\Psi(\mu) = B_n(\partial_{t_1})\Psi(\mu), \quad n = 1, 2, \ldots,$$

of the KP hierarchy. This is a key to prove the equivalence of the differential Fay identity and the full KP hierarchy (T. & Takebe 95).

• In quasi-classical limit, $\Psi(z) \sim e^{\hbar^{-1}S(z)}$, the forgoing equation turns into an equation of the form

$$(\lambda - \mu)e^{D(\lambda)D(\mu)F} = \partial_{t_1}S(\lambda) - \partial_{t_1}S(\mu),$$

where

$$S(z) = \xi(\boldsymbol{t}, z) - D(z)F = \sum_{n=1}^{\infty} t_n z^n - \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{t_n}F.$$

• This is a generating functional form (Bogdanov, Konopelchenko & Martinez Alonso 03) of the Hamilton-Jacobi equations

$$\partial_{t_n}S(z) = \mathcal{B}_n(\partial_{t_1}S(z)), \quad n = 1, 2, \ldots,$$

of the dispersionless KP hierarchy.

Summary

Hirota formalism		Lax formalism
DJKM bilinear eqn	$\Longleftrightarrow$	auxiliary linear eqns (usual form)
$\downarrow$		$\uparrow$
differential Fay identity	$\Longleftrightarrow$	auxiliary linear eqns (generating form)
$\downarrow \hbar \to 0$		$\downarrow \hbar \to 0$
dispersionless Hirota eqn	$\overset{(*)}{\Longleftrightarrow}$	Hamilton-Jacobi eqns

$(*)$ This part can also be explained in the langauge of complex analysis (Faber polynomials and Grunsky coefficients) — cf. L.P. Teo's work (2003) for the dispersionless Toda hierarchy.

2. Toward multi-component generalization of dispersionless Hirota equations

Toda hierarchy = "charged" two-component KP hierarchy

- Two sets of continuous variables: $\boldsymbol{t} = (t_1, t_2, \ldots)$, $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$
- Discrete variable: $s \in \mathbf{Z}$
- s is interpreted as charge of a state in the Fock space of charged fermions (Jimbo & Miwa 83; Takebe 85):

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \langle s | e^{H(\boldsymbol{t})} g^{-\bar{H}(\bar{\boldsymbol{t}})} | s \rangle$$

- Lax formalism based on scalar wave functions $\Psi(z), \bar{\Psi}(z)$

$\longrightarrow$ straightforward generalization of prescription for KP

Dispersionless Hirota equations of Toda hierarchy(Zabrodin & Wiegmann 00, 01)

$$(1) \quad (\lambda-\mu)e^{D(\lambda)D(\mu)F} = \lambda e^{-\partial_s D(\lambda)F} - \mu e^{-\partial_s D(\mu)F},$$

$$(2) \quad (\lambda-\mu)e^{\bar{D}(\lambda)\bar{D}(\mu)F} = \lambda e^{\partial_s \bar{D}(\mu)F} - \mu e^{\partial_s \bar{D}(\lambda)F},$$

$$(3) \quad e^{D(\lambda)\bar{D}(\mu)F} = 1 - \frac{\mu}{\lambda} e^{(\partial_s^2 + \partial_s D(\lambda) - \partial_s \bar{D}(\mu))F},$$

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{t_n}, \quad \bar{D}(z) = \sum_{n=1}^{\infty} \frac{z^{n}}{n} \partial_{\bar{t}_n}.$$

• Recently, Teo derived "Fay-type identities" as a dispersive counterpart of these equations, and proved the equivalence with the full Toda hierarchy (nlin.SI/0606059).

Generalizations

1. "charged" multi-component KP hierarchy (N discrete variables $s_1, \ldots, s_N$ show up in $N+1$-component case)

2. two-component BKP hierarchy (neutral fermions)

3. Pfaff lattice, "charged" BKP hierarchy, coupled KP hierarchy, and their "neutral" counterparts (D'_∞ family)

1 — T. & Takebe, nlin.SI/0608068.
2 — T., SIGMA **2** (2006), Paper 057.
3 — work in progress

3. Dispersionless Hirota equations of multi-component KP hierarchy

"Charged" $N+1$-component KP hierarchy

• $N+1$ sets of continuous variables $\boldsymbol{t}_0 = (t_{01}, t_{02}, \ldots)$, $\boldsymbol{t}_1 = (t_{11}, t_{12}, \ldots)$, $\ldots$ $\boldsymbol{t}_N = (t_{N1}, t_{N2}, \ldots)$
• N discrete variables $s_1, \ldots, s_N \in \mathbf{Z}$, auxiliary variable $s_0 = -s_1 - \ldots - s_N$
• $\boldsymbol{s} = (s_0, s_1, \ldots, s_N)$ is the charge vector (total charge $= 0$) of a state in the Fock space of $N+1$-component charged fermions $\psi_{\alpha j}, \psi^*_{\alpha j}$, $0 \le \alpha \le N$, $j \in \mathbf{Z}$ (DJKM 81; Kac & van de Leur 93):

$$\tau(\boldsymbol{s}, \boldsymbol{t}) = \langle \boldsymbol{s} | e^{H(\boldsymbol{t})} g | \mathbf{0} \rangle,$$

$$H(\boldsymbol{t}) = \sum_{\alpha=0}^{N} \sum_{n=1}^{\infty} t_{\alpha n} J_{\alpha n}, \qquad J_{\alpha n} = \sum_{j \in \mathbf{Z}} \psi_{\alpha j} \psi^*_{\alpha, j+n}$$

Lax formalism with scalar wave functions (T., talk at MISGRAM workshop at SISSA, 2005/09)

• Scalar-valued wave functions (= matrix elements in the 1st row of a matrix-valued wave function)

$$\Psi_0(z) = z^{s_0} e^{\xi(\boldsymbol{t}_0,z)} \frac{e^{-D_0(z)}\tau(\boldsymbol{s},\boldsymbol{t})}{\tau(\boldsymbol{s},\boldsymbol{t})},$$

$$\Psi_\beta(z) = \tilde{\epsilon}_\beta(\boldsymbol{s}) z^{s_\beta - 1} e^{\xi(\boldsymbol{t}_\beta,z)} \frac{e^{-D_\beta(z)}\tau(\boldsymbol{s}+\boldsymbol{e}_0-\boldsymbol{e}_\beta,\boldsymbol{t})}{\tau(\boldsymbol{s},\boldsymbol{t})}, \quad 1 \le \beta \le N,$$

where $\boldsymbol{e}_\alpha = (\ldots, 0, 1, 0, \ldots)$ (1 in the α-th component),

$$D_0(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{0n}, \quad D_\beta(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{\beta n}, \quad \partial_{\beta n} = \partial_{t_{\beta n}},$$

and $\tilde{\epsilon}_\beta(\boldsymbol{s}) = (-1)^{s_1+\cdots+s_\beta}$ (these sign factors were overlooked in last year's talk.)

• Auxiliary linear equations for scalar wave functions

(1) $\partial_{0n}\Psi_\beta(z) = B_{0n}(\partial_{01})\Psi_\beta(z)$,

(2) $\partial_{\alpha n}\Psi_\beta(z) = B_{\alpha n}(\partial_{\alpha 1})\Psi_\beta(z)$,

(3) $\partial_{01}\Psi_\beta(z) = (e^{-\partial_{\alpha 0}} + q_\alpha)\Psi_\beta(z)$,

(4) $\partial_{\alpha 1}\Psi_\beta(z) = r_\alpha e^{\partial_{\alpha 0}}\Psi_\beta(z)$,

(5) $((\partial_{01} - q_\alpha)\partial_{\alpha 1} - r_\alpha)\Psi_\beta(z) = 0, \quad 1 \le \alpha \le N,\ 0 \le \beta \le N$,

where $e^{\partial_{\alpha 0}}$ stand for the shift operators $e^{\partial/\partial s_\alpha}$ in $\boldsymbol{s}$,

$$e^{\pm\partial_{\alpha 0}} f(\boldsymbol{s}) = f(\boldsymbol{s} \mp \boldsymbol{e}_0 \pm \boldsymbol{e}_\alpha) = f(s_0 \mp 1, \ldots, s_\alpha \pm 1, \ldots).$$

• Interpretation —

(1),(2): KP hierarchy in $\boldsymbol{t}_0$- and $\boldsymbol{t}_\beta$-sectors

(3),(4): auxiliary linear equation of Toda field equation in $(s_\alpha, t_{01}, t_{\alpha 1})$-space

(5): two-dimensional Schrödinger equation with magnetic field (Veselov & Novikov 84) in $(t_{01}, t_{\alpha 1})$-space

• The Hamilton-Jacobi equations of these equations in dispersionless (= quasi-classical) limit

$$\Psi_0(z) \sim e^{\hbar^{-1} S_0(z)}, \ \ldots, \ \Psi_N(z) \sim e^{\hbar^{-1} S_N(z)}$$

are exactly the evolution equations of the S-functions in the universal Whitham hierarchy of genus zero with $N+1$ marked points at $\infty, q_1, \ldots, q_N$.

• (1), (2) and (5) may be thought of as auxiliary linear equations for rational reductions of the KP hierarchy (Krichever; Dickey; Enriques, Orlov, Rubtsov), which is another approach to the zero-genus Whitham hierarchy and its Benney-type reductions.

• Many problems remain open (additional symmetries, string equations, q-analogues, etc).

Bilinear equations for τ-function

$$\oint \frac{dz}{2\pi i} z^{s'_0 - s_0} e^{\xi(\boldsymbol{t}'_0 - \boldsymbol{t}_0, z)} (e^{-D_0(z)}\tau)(\boldsymbol{s}', \boldsymbol{t}')(e^{D_0(z)}\tau)(\boldsymbol{s}, \boldsymbol{t})$$
$$+ \sum_{\gamma=1}^{N} \tilde{\epsilon}_\gamma(\boldsymbol{s}')\tilde{\epsilon}_\gamma(\boldsymbol{s}) \oint \frac{dz}{2\pi i} z^{s'_\gamma - s_\gamma - 2} e^{\xi(\boldsymbol{t}'_\gamma - \boldsymbol{t}_\gamma, z)} \times$$
$$\times (e^{-D_\gamma(z)}\tau)(\boldsymbol{s}' + \boldsymbol{e}_0 - \boldsymbol{e}_\gamma, \boldsymbol{t}')(e^{D_\gamma(z)}\tau)(\boldsymbol{s} - \boldsymbol{e}_0 + \boldsymbol{e}_\gamma, \boldsymbol{t}) = 0.$$

• These equations are obtained from the bilinear equations

$$\sum_{\gamma=0}^{N} \oint \frac{dz}{2\pi i} \Psi_\gamma(z)\Psi^*_\gamma(z) = 0$$

of the scalar wave functions $\Psi_\gamma(z)$ and their duals $\Psi^*_\gamma(z)$.
• Though being part of the full bilinear equations of matrix wave functions (DJKM 81; Kac & van de Leur 93), they actually contain all relevant equations (because of $\boldsymbol{s}$-dependence).

Differential Fay identities

$$(1)\quad \exp\Big((e^{D_0(\lambda)}-1)(e^{D_0(\mu)}-1)\log\tau(\boldsymbol{s},\boldsymbol{t})\Big) = 1-\frac{\partial_{01}\Big(e^{D_0(\lambda)}-e^{D_0(\mu)}\Big)\log\tau(\boldsymbol{s},\boldsymbol{t})}{\lambda-\mu},$$

$$(2)\quad \lambda\exp\Big((e^{D_0(\lambda)}-1)(e^{\partial_{\alpha 0}+D_\alpha(\mu)}-1)\log\tau(\boldsymbol{s},\boldsymbol{t})\Big) = \lambda-\partial_{01}(e^{D_0(\lambda)}-e^{\partial_{\alpha 0}+D_\alpha(\mu)})\log\tau(\boldsymbol{s},\boldsymbol{t}),$$

$$(3)\quad \exp\Big((e^{\partial_{\alpha 0}+D_\alpha(\lambda)}-1)(e^{\partial_{\alpha 0}+D_\alpha(\mu)}-1)\log\tau(\boldsymbol{s},\boldsymbol{t})\Big) = -\frac{\lambda\mu\partial_{01}\Big(e^{\partial_{\alpha 0}+D_\alpha(\lambda)}-e^{\partial_{\alpha 0}+D_\alpha(\mu)}\Big)\log\tau(\boldsymbol{s},\boldsymbol{t})}{\lambda-\mu},$$

$$(4)\quad \epsilon_{\alpha\beta}\exp\Big((e^{\partial_{\alpha 0}+D_\alpha(\lambda)}-1)(e^{\partial_{\beta 0}+D_\beta(\mu)}-1)\log\tau(\boldsymbol{s},\boldsymbol{t})\Big) = -\partial_{01}\Big(e^{\partial_{\alpha 0}+D_\alpha(\lambda)}-e^{\partial_{\beta 0}+D_\beta(\mu)}\Big)\log\tau(\boldsymbol{s},\boldsymbol{t}),\qquad \alpha\neq\beta,$$

where

$$\epsilon_{\alpha\beta} = \begin{cases} +1 & (\alpha \le \beta), \\ -1 & (\alpha > \beta). \end{cases}$$

Dispersionless Hirota equations

(1) $$e^{D_0(\lambda)D_0(\mu)F} = 1 - \frac{\partial_{01}(D_0(\lambda) - D_0(\mu))F}{\lambda - \mu},$$

(2) $$\lambda e^{D_0(\lambda)(\partial_{\alpha 0} + D_\alpha(\mu))F} = \lambda - \partial_{01}(D_0(\lambda) - \partial_{\alpha 0} - D_\alpha(\mu))F,$$

(3) $$e^{(\partial_{\alpha 0} + D_\alpha(\lambda))(\partial_{\alpha 0} + D_\alpha(\mu))F} = -\frac{\lambda\mu\partial_{01}(D_\alpha(\lambda) - D_\alpha(\mu))F}{\lambda - \mu},$$

(4) $$\epsilon_{\alpha\beta} e^{(\partial_{\alpha 0} + D_\alpha(\lambda))(\partial_{\beta 0} + D_\beta(\mu))F} = -\partial_{01}(\partial_{\alpha 0} + D_\alpha(\lambda) - \partial_{\beta 0} - D_\beta(\mu))F.$$

Auxiliary linear equations in generating functional form

• Differential Fay identities (1)–(4) are equivalent to

$$\lambda e^{-D_0(\lambda)}\Psi_\beta(\mu) = -(\partial_{01} - \partial_{01}\log\Psi_0(\lambda))\Psi_\beta(\mu),$$
$$e^{-\partial_{\alpha 0}-D_\alpha(\lambda)}\Psi_\beta(\mu) = (\partial_{01} - \partial_{01}\log\Psi_\alpha(\lambda))\Psi_\beta(\mu),$$
$$1 \le \alpha \le N,\ 0 \le \beta \le N.$$

• These equations are equivalent to the auxiliary linear equations

$$\partial_{0n}\Psi_\beta(z) = B_{0n}(\partial_{01})\Psi_\beta(z),$$
$$\partial_{\alpha n}\Psi_\beta(z) = B_{\alpha n}(\partial_{\alpha 1})\Psi_\beta(z),$$
$$\partial_{01}\Psi_\beta(z) = (e^{-\partial_{\alpha 0}} + q_\alpha)\Psi_\beta(z),$$
$$\partial_{\alpha 1}\Psi_\beta(z) = r_\alpha e^{\partial_{\alpha 0}}\Psi_\beta(z)$$

of the usual form. In other words, the differential Fay identities are auxiliary linear equations in disguise.

4. Some other cases

2-component BKP hierarchy

• Two sets of continuous variables: $\boldsymbol{t} = (t_1, t_3, t_5, \ldots)$, $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_3, \bar{t}_5, \ldots)$ (labeled by odd indices)
• No discrete variable
• Formula of τ-function in the language of neutral fermions (DJKM 81–82; Kac & van de Leur 98):

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \langle 0|e^{H(\boldsymbol{t})} g^{-\bar{H}(\bar{\boldsymbol{t}})}|0\rangle = \langle 1|e^{H(\boldsymbol{t})} g^{-\bar{H}(\bar{\boldsymbol{t}})}|1\rangle$$

• Lax formalism based on scalar wave functions $\Psi(z), \bar{\Psi}(z)$ (Shiota 89; Krichever 05) $\longrightarrow$ straightforward generalization of prescription for KP and Toda

• Dispersionless Hirota equations and associated differential Fay identities have been obtained (T. 06; Chen & Tu 06). They generalize the result for the case of the one-component BKP hierarchy (Bogdanov & Konopelcheno 05).

• These differential Fay identities can be converted to the following equations for the wave functions:

$$
\begin{aligned}
-(\partial_{t_1} - \partial_{t_1} \log \Psi(\lambda))\Psi(\mu) &= (\partial_{t_1} + \partial_{t_1} \log \Psi(\lambda))e^{-2D(\lambda)}\Psi(\mu), \\
-(\partial_{\bar{t}_1} - \partial_{\bar{t}_1} \log \bar{\Psi}(\lambda))\bar{\Psi}(\mu) &= (\partial_{\bar{t}_1} + \partial_{\bar{t}_1} \log \bar{\Psi}(\lambda))e^{-2\bar{D}(\lambda)}\bar{\Psi}(\mu), \\
(\partial_{t_1} - \partial_{t_1} \log \bar{\Psi}(\lambda))\Psi(\mu) &= (\partial_{t_1} + \partial_{t_1} \log \bar{\Psi}(\lambda))e^{-2\bar{D}(\lambda)}\Psi(\mu), \\
(\partial_{\bar{t}_1} - \partial_{\bar{t}_1} \log \Psi(\lambda))\bar{\Psi}(\mu) &= (\partial_{\bar{t}_1} + \partial_{\bar{t}_1} \log \Psi(\lambda))e^{-2D(\lambda)}\bar{\Psi}(\lambda).
\end{aligned}
$$

They give a generating functional form of auxiliary linear equations.

Hierarchies of D'_∞ family

• Integrable hierarchies of D'_∞ type (Jimbo & Miwa, Publ. RIMS, Kyoto Univ., **19** (1983), 943–1001):

1. Charged version with a set of continuous variables $\boldsymbol{t} = (t_1, t_2, \ldots)$ and a discrete variable s. Charged fermions are used to express the τ-functions $\tau(s, \boldsymbol{t})$, $s \in \mathbf{Z}$.

2. Neutral version with two sets of continuous variables $\boldsymbol{t} = (t_1, t_3, t_5, \ldots)$, $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_3, \bar{t}_5, \ldots)$. Neutral fermions are used to express the τ-functions $\tau_s(\boldsymbol{t}, \bar{\boldsymbol{t}})$, $s = 0, 1$.

• The charged version, also called "charged BKP hierarchy" by Kac and van de Leur, contains two copies of the Pfaff lattice (Adler, Horozov, Shiota & van Moerbeke). These two Pfaff lattices are described by the τ-functions $\{\tau(2s, \boldsymbol{t})\}_{s \in \mathbf{Z}}$ and $\{\tau(2s+1, \boldsymbol{t})\}_{s \in \mathbf{Z}}$ on the even and odd sublattices. Thus the charged D'_∞ hierarchy may be thought of as a "coupled Pfaff lattice". This hierarchy also contains the so called "coupled KP hierarchy".

• The neutral version contains two copies of the two-component BKP (2-BKP) hierarchy. The two 2-BKP hierarchies are described by the two τ-functions $\tau_0(\boldsymbol{t}, \bar{\boldsymbol{t}})$ and $\tau_1(\boldsymbol{t}, \bar{\boldsymbol{t}})$. Thus the neutral D'_∞ hierarchy may be called a "coupled (2-)BKP hierarchy."

- Partial results on the dispersionless limit (work in progress)

1. Differential Fay identities for $\tau(2s, \boldsymbol{t})$ have been obtained:

$$\frac{\tau(2s, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}])\tau(2s, \boldsymbol{t})}{\tau(2s, \boldsymbol{t}+[\lambda^{-1}])\tau(2s, \boldsymbol{t}+[\mu^{-1}])} - 1 + \frac{1}{\lambda-\mu}\partial_{t_1} \log \frac{\tau(2s, \boldsymbol{t}+[\lambda^{-1}])}{\tau(2s, \boldsymbol{t}+[\mu^{-1}])}$$
$$= \frac{\tau(2s+2, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}])\tau(2s-2, \boldsymbol{t})}{\lambda^2\mu^2\tau(2s, \boldsymbol{t}+[\lambda^{-1}])\tau(2s, \boldsymbol{t}+[\mu^{-1}])},$$

$$\frac{\lambda^2\tau(2s-2, \boldsymbol{t}+[\lambda^{-1}])\tau(2s, \boldsymbol{t}+[\mu^{-1}]) - \mu^2\tau(2s, \boldsymbol{t}+[\lambda^{-1}])\tau(2s-2, \boldsymbol{t}+[\mu^{-1}])}{(\lambda-\mu)\tau(2s-2, \boldsymbol{t})\tau(2s, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}])}$$
$$= \lambda + \mu - \partial_{t_1} \log \frac{\tau(2s, \boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}])}{\tau(2s-2, \boldsymbol{t})}.$$

2. Under a suitable quasi-classical ansatz, they have a quasi-classical limit (i.e., dispersionless Hirota equations).

3. The differential Fay identities can be converted to the equations

$$
\begin{aligned}
&-(\partial_{t_1} - \partial_{t_1} \log \Psi(2s, \boldsymbol{t}, \lambda))\Psi(2s, \boldsymbol{t}, \mu) \\
&\qquad = \lambda e^{-D(\lambda)}\Psi(2s, \boldsymbol{t}, \mu) + \lambda^{-1}\frac{\tau(2s+2, \boldsymbol{t})}{\tau(2s, \boldsymbol{t})} e^{-D(\lambda)}\Phi(2s, \boldsymbol{t}, \mu), \\
&-(\partial_{t_1} - \partial_{t_1} \log \Phi^*(2s, \boldsymbol{t}, \lambda))\Phi(2s, \boldsymbol{t}, \mu) \\
&\qquad = -\lambda e^{D(\lambda)}\Phi(2s, \boldsymbol{t}, \mu) + \lambda^{-1}\frac{\tau(2s-2, \boldsymbol{t})}{\boldsymbol{t}(2s, \boldsymbol{t})} e^{D(\lambda)}\Psi(2s, \boldsymbol{t}, \mu),
\end{aligned}
$$

$\cdots$ (two more equations) $\cdots$

for matrix elements of the matrix-valued wave function

$$
W(2s, \boldsymbol{t}, z) = \begin{pmatrix} \Psi(2s, \boldsymbol{t}, z) & \Psi^*(2s, \boldsymbol{t}, z) \\ \Phi(2s, \boldsymbol{t}, z) & \Phi^*(2s, \boldsymbol{t}, z) \end{pmatrix}
$$

of the Pfaff lattice.

4. The foregoing equations are a generating functional expression of linear equations of the form

$$\partial_{t_n} W(2s, \boldsymbol{t}, z) = \begin{pmatrix} \partial_{t_1}^n + O(\partial_{t_1}^{n-1}) & O(\partial_{t_1}^{n-1}) \\ O(\partial_{t_1}^{n-1}) & -(-\partial_{t_1})^n + O(\partial_{t_1}^{n-1}) \end{pmatrix} W(2s, \boldsymbol{t}, z).$$

These linear equations seem to coincide with auxiliary linear equations derived by S. Kakei several years ago in an inverse scattering framework.

• Remark: It seems likely that this case does not have a natural scalar Lax formalism. Usually, achieving dispersionless limit of such a system is an extremely tough problem. In the present case, we can circumvent the difficulties by taking the route from dispersionless Hirota equations.