Fay-type identities and dispersionless limit of integrable hierarchies
(joint work with Takashi Takebe)

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Introduction

Dispersionless limit of integrable systems

• Dispersionless limit of KdV equation

\[
\epsilon \rightarrow 0 \quad u_t + uu_x + \epsilon^2 u_{xxx} = 0
\]

\[
\epsilon \rightarrow 0 \quad u_t + uu_x = 0 \quad \text{(dispersionless KdV eqn)}
\]

• Dispersionless limit of 2D Toda equation

\[
\epsilon \phi(s)_{xy} + e^{(\phi(s+\epsilon)-\phi(s))/\epsilon} - e^{(\phi(s)-\phi(s-\epsilon))/\epsilon} = 0
\]

\[
\epsilon \rightarrow 0 \quad \phi(s)_{xy} + (e^{\phi(s)}s)_s = 0 \quad \text{(dispersionless Toda eqn)}
\]

N.B. This equation is also known in general relativity as the Boyer-Finley equation.
Motivation of this talk

Recent (since 2000) applications of dispersionless integrable hierarchies (dKP, dToda, etc.):

- Laplacian growth
- Associativity (WDVV) equations
- Light-cone string field theory
- Löwner equation
- Large-$N$ normal matrix model
- Nonlinear optics
- Seiberg-Witten curve of 4D/5D $U(1)$ gauge theories
- Limit shape of 2D/3D Young diagrams (really??)

Fay-type identities play a significant role in some of these applications. This talk is concerned with theoretical aspects of Fay-type identities.
2. Differential Fay identity for KP hierarchy

\( \tau \)-function of KP hierarchy

- Time variables \( t = (t_1, t_2, \ldots) \)
- Spatial variable \( x = t_1 \)
- Fermionic formula of tau function (DJKM 81)

\[
\tau = \tau(t) = \langle 0 | e^{H(t)} g | 0 \rangle
\]

where \( H(t) \) and \( g \) are made from free fermion operators

\[
[\psi_j, \psi_k^*]_+ = \delta_{jk}, \quad [\psi_j, \psi_k]_+ = [\psi_j^*, \psi_k^*]_+ = 0
\]

as

\[
H(t) = \sum_{n=1}^{\infty} H_n t_n, \quad H_n = \sum_{j \in \mathbb{Z}} \psi_j \psi_j^* + n,
\]

\[
g = \exp \sum_{j, k \in \mathbb{Z}} a_{jk} \psi_j \psi_k^*, \ldots
\]
An infinite number of Hirota equations (Sato & Sato; DJKM)

\[(3D_{t2}^2 - 4D_{t3}D_{t1} + D_{t1}^4)\tau \cdot \tau = 0, \ldots\]

where

\[P(D_{t1}, D_{t2}, \ldots)\tau \cdot \tau = P(\partial_{t1}' - \partial_{t1}, \partial_{t2}' - \partial_{t2}, \ldots)\tau(t') \cdot \tau(t)|_{t'=t}\]

Generating functional form of Hirota equations (DJKM 81)

\[\oint \frac{dz}{2\pi i} e^{\xi(t' - t, z)}\tau(t' - [z^{-1}])\tau(t + [z^{-1}]) = 0\]

where

\[\xi(t, z) = \sum_{n=1}^{\infty} t_n z^n, \quad [z^{-1}] = \left(z^{-1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots, \frac{z^{-n}}{n}, \ldots\right)\]

The contour integral is along a sufficiently large circle \(|z| = R| \).
**Fay identity** (derived by Sato from Plücker relation)

\[
(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\tau(t + [\lambda_1] + [\lambda_2])\tau(t + [\lambda_3] + [\lambda_4])
- (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)\tau(t + [\lambda_1] + [\lambda_3])\tau(t + [\lambda_2] + [\lambda_4])
+ (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)\tau(t + [\lambda_1] + [\lambda_4])\tau(t + [\lambda_2] + [\lambda_3]) = 0
\]

**Differential Fay identity** (Adler & van Moerbeke 92)

\[
\frac{\tau(t + [\lambda^{-1}] + [\mu^{-1}])\tau(t)}{\tau(t + [\lambda^{-1}])\tau(t + [\mu^{-1}])} = 1 - \frac{1}{\lambda - \mu} \partial_{t_1} \log \frac{\tau(t + [\lambda^{-1}])}{\tau(t + [\mu^{-1}])}
\]
Some features of differential Fay identity

- The original derivation due to Adler & van Moerbeke starts from the Fay identity and carries out a degeneration procedure. One can derive it directly from the DJKM bilinear equation as well.

- Expression à la Wiegmann & Zabrodin

\[
\exp\left((e^{D(\lambda)} - 1)(e^{D(\mu)} - 1) \log \tau\right) = 1 - \frac{\partial_{t_1}(e^{D(\lambda)} - e^{D(\mu)}) \log \tau}{\lambda - \mu}
\]

where

\[D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{t_n}.\]

This is convenient for deriving the dispersionless limit (presented later on).
• Equivalent expression

\[ e^{-D(\lambda)}\psi(\mu) = -\lambda^{-1}(\partial_{t_1} - \partial_{t_1} \log \psi(\lambda))\psi(\mu) \cdots (\diamond) \]

in terms of the standard wave function

\[ \psi(z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\xi(t,z)}. \]

• Theorem: The differential Fay identity is equivalent to the KP hierarchy (T. & Takebe 95).

• Key: (\diamond) is a generating functional form of the auxiliary linear equations

\[ \partial_{t_n} \psi(z) = B_n \psi(z) \]

of the KP hierarchy.
3. Relation to dispersionless KP hierarchy

Dispersionless limit = quasi-classical limit

- Assume that an $\hbar$-dependent $\tau$-function $\tau = \tau(\hbar, t)$ behaves in classical ($\hbar \to 0$) limit as

$$\tau(\hbar, \hbar^{-1}t) = \exp(\hbar^{-2}F(t) + O(\hbar^{-1}))$$

- For the rescaled $\tau$-function $\tau_{\hbar} = \tau(\hbar, \hbar^{-1}t)$, the differential Fay identity reads

$$\exp\left((e^{\hbar D(\lambda)} - 1)(e^{\hbar D(\mu)} - 1) \log \tau_{\hbar}\right) = 1 - \frac{\hbar \partial_t(\exp(\hbar D(\lambda)) - \exp(\hbar D(\mu))) \log \tau_{\hbar}}{\lambda - \mu}$$

N.B. Rescaling rule: $\partial_t \to \hbar \partial_t$, $D(z) \to \hbar D(z)$
As $\hbar \to 0$, the rescaled differential Fay identity
\[
\exp \left( (e^{\hbar D(\lambda)} - 1)(e^{\hbar D(\mu)} - 1) \log \tau_\hbar \right) = 1 - \frac{\hbar \partial_{t_1}(e^{\hbar D(\lambda)} - e^{\hbar D(\mu)}) \log \tau_\hbar}{\lambda - \mu}
\]
turns into the dispersionless Hirota equation (T. & Takebe 95; Carroll & Kodama 95)
\[
e^{D(\lambda)D(\mu)}F = 1 - \frac{\partial_{t_1}(D(\lambda) - D(\mu))F}{\lambda - \mu}.
\]

• Systematic expansion in powers of $\hbar$ is possible.

• Theorem: The dispersionless Hirota equation is equivalent to the dispersionless KP (dKP) hierarchy (Boyarsky et al. 01; Teo 03).
Dispersionless KP hierarchy

- Lax equations of KP hierarchy (Sato)

\[ \partial t_n L = [B_n, L], \quad L = \partial_x + \sum_{n=1}^{\infty} u_{n+1} \partial_x^{-n}, \quad x = t_1, \]

\[ B_n = (L^n)_{\geq 0} \text{ (nonnegative powers of } \partial_x) \]

- Dispersionless Lax equations of dKP hierarchy (Kodama & Gibbons; Krichever)

\[ \partial t_n \mathcal{L} = \{B_n, \mathcal{L}\}, \quad \mathcal{L} = p + \sum_{n=1}^{\infty} u_{n+1} p^{-n}, \quad x = t_1, \]

\[ B_n = (\mathcal{L}^n)_{\geq 0} \text{ (nonnegative powers of } p) \]

based on Poisson bracket

\[ \{F, G\} = \frac{\partial F}{\partial p} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial p} \]
Hamilton-Jacobi equations and dispersionless Hirota equation

- Quasi-classical approximation $\Psi(z) \sim e^{\hbar^{-1}S(z)}$ (Zakharov; Kodama & Gibbons) to the rescaled auxiliary linear equations

$$\hbar \partial_{tn} \Psi(z) = B_n(\hbar \partial_x)\Psi(z)$$

yields the Hamilton-Jacobi equations

$$\partial_{tn} S(z) = B_n(\partial_x S(z))$$

for the $S$-function

$$S(z) = \xi(t, z) - D(z)F = \sum_{n=1}^{\infty} t_n z^n - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{tn} F.$$
These Hamilton-Jacobi equations are equivalent to the dispersionless Lax equations through inversion of the map $z \leftrightarrow p(z) = \partial_x S(z)$:

$$p = p(z) \quad \leftrightarrow \quad z = \mathcal{L}(p).$$

The dispersionless Hirota equation can be rewritten as

$$e^{D(\lambda)D(\mu)F} = \frac{\partial_x S(\lambda) - \partial_x S(\mu)}{\lambda - \mu}.$$

This is a generating functional form of the Hamilton-Jacobi equation (Bogdanov, Konopelchenko & Martinez Alonso 03).

$\mathcal{B}_n$’s are identified with the Faber polynomials of the (conformal) mapping $z \rightarrow p(z)$ (Teo 03).
3. Difference Fay identities for Toda hierarchy

**Tau function**

- Two sets of continuous variables: \( t = (t_1, t_2, \ldots), \tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots) \)
- Discrete variable: \( s \in \mathbb{Z} \)
- Fermionic formula of tau function (Jimbo & Miwa 83; Takebe 85):

\[
\tau(s, t, \tilde{t}) = \langle s | e^{H(t)} e^{-\tilde{H}(\tilde{t})} | s \rangle
\]

where

\[
H(t) = \sum_{n=1}^{\infty} H_n t_n, \quad \tilde{H}(\tilde{t}) = \sum_{n=1}^{\infty} \tilde{H}_n \tilde{t}_n,
\]

\[
H_n = \sum_{j \in \mathbb{Z}} \psi_j \psi_j^* + n, \quad \tilde{H}_n = H_{-n}.
\]
• Generating functional form of Hirota equations (Ueno & T 83)

\[
\oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(t'-t, z)} \tau(s', t' - [z^{-1}], \tilde{t}') \tau(s, t + [z^{-1}], \tilde{t})
= \oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(\tilde{t}'-\tilde{t}, z^{-1})} \tau(s' + 1, t', \tilde{t}' - [z]) \tau(s - 1, t, \tilde{t} + [z])
\]

• Three distinct Fay-type identities (so to speak, difference Fay identities) are obtained (Zabrodin 01; Teo 06). They can be derived from the foregoing bilinear equation by letting

1. \( t' = t + [\lambda^{-1}] + [\mu^{-1}], \tilde{t}' = \tilde{t}, s' = s + 1 \)
2. \( t' = t, \tilde{t}' = \tilde{t} + [\lambda] + [\mu], s' = s - 3 \)
3. \( t' = t + [\lambda^{-1}], \tilde{t}' = \tilde{t} + [\mu], s' = s \)
Difference Fay identities thus obtained

\begin{align*}
(1) \quad & \exp((e^D(\lambda) - 1)(e^D(\mu) - 1) \log \tau) - \frac{\lambda}{\lambda - \mu} \exp((e^D(\lambda) - 1)(e^{-\partial_s} - 1) \log \tau) \\
& + \frac{\mu}{\lambda - \mu} \exp((e^D(\mu) - 1)(e^{-\partial_s} - 1) \log \tau) = 0, \\
(2) \quad & \exp((e^{\tilde{D}}(\lambda) - 1)(e^{\tilde{D}}(\mu) - 1) \log \tau) + \frac{\mu}{\lambda - \mu} \exp((e^{\tilde{D}}(\lambda) - 1)(e^{\partial_s} - 1) \log \tau) \\
& - \frac{\lambda}{\lambda - \mu} \exp((e^{\tilde{D}}(\mu) - 1)(e^{\partial_s} - 1) \log \tau) = 0, \\
(3) \quad & \exp((e^{D}(\lambda) - 1)(e^{\tilde{D}}(\mu) - 1) \log \tau) - 1 + \frac{\mu}{\lambda} \exp((e^{D}(\lambda) - 1)(e^{\partial_s} - 1) \\
& + (e^{\tilde{D}}(\mu) - 1)(e^{-\partial_s} - 1) - (e^{\partial_s} - 1)(e^{-\partial_s} - 1)) \log \tau) = 0
\end{align*}

where

\[ D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{tn}, \quad \tilde{D}(z) = \sum_{n=1}^{\infty} \frac{z^{n}}{n} \partial_{\tilde{t}n}. \]
Dispersionless Hirota equations

In the dispersionless (= quasi-classical) limit

$$
\tau(\hbar, \hbar^{-1}s, \hbar^{-1}t, \hbar^{-1}\tilde{t}) \sim e^{\hbar^{-2}F(s,t,\tilde{t})},
$$

the difference Fay identities turn into the dispersionless Hirota equations of the dispersionless Toda (dToda) hierarchy (Wiegmann & Zabrodin 00; Kostov et al. 01):

\begin{align}
(1) & \quad e^{D(\lambda)D(\mu)F} = \frac{\lambda e^{-\partial_s D(\lambda)F} - \mu e^{-\partial_s D(\mu)F}}{\lambda - \mu}, \\
(2) & \quad e^{\tilde{D}(\lambda) \tilde{D}(\mu)F} = \frac{-\mu e^{\partial_s \tilde{D}(\lambda)F} + \lambda e^{\partial_s \tilde{D}(\mu)F}}{\lambda - \mu}, \\
(3) & \quad e^{D(\lambda)\tilde{D}(\mu)F} = 1 - \frac{\mu}{\lambda} e^{(\partial_s D(\lambda)-\partial_s \tilde{D}(\mu)+\partial^2_s)F}.
\end{align}

Theorem: They are equivalent to the dToda hierarchy (Teo 03).
Implications to wave functions

- Standard wave functions

\[
\psi(s, z) = \frac{\tau(s, t - [z^{-1}], \bar{t})}{\tau(s, t, \bar{t})} z^s e^{\xi(t, z)},
\]

\[
\tilde{\psi}(s, z) = \frac{\tau(s + 1, t, \bar{t} - [z])}{\tau(s, t, \bar{t})} z^s e^{\xi(\bar{t}, z^{-1})}
\]

- The difference Fay identities can be converted to equations for these wave functions:

\[
e^{-D(\lambda)} \psi(s, \mu) = \lambda^{-1} \left( \frac{\psi(s + 1, \lambda)}{\psi(s, \lambda)} - e^{\partial_s} \right) \psi(s, \mu),
\]

\[
e^{-\bar{D}(\lambda)} \psi(s, \mu) = \left( 1 - \frac{\tilde{\psi}(s + 1, \lambda)}{\tilde{\psi}(s, \lambda)} e^{-\partial_s} \right) \psi(s, \mu),
\]

\((\text{same equations for } \tilde{\psi}(s, \mu))\)
These equations give a generating functional expression of the auxiliary linear equations

\[ \partial_t \psi(s, z) = B_n(e^{\partial_s})\psi(s, z), \]
\[ \partial_{\tilde{t}} \psi(s, z) = \tilde{B}_n(e^{\partial_s})\psi(s, z), \]
\[ \ldots \text{(same equations for } \tilde{\psi}(s, z)\text{)} \ldots \]

of the Toda hierarchy.

Roughly speaking, this implies that the difference Fay identities are equivalent to the Toda hierarchy itself. (Details remain to be checked.)
6. Conclusion

- The dispersionless Hirota equations of the dKP/dToda hierarchies appear as the quasi-classical ($\hbar \to 0$) limit of the differential/difference Fay identities.

- These special Fay-type identities carry full information of the KP and Toda hierarchies.

- Systematic $\hbar$-expansion of these Fay-type identities may have application to, say, light-cone string field theory? (Bonora et al. 05)
Hirota formalism

DJKM bilinear eqn
\[\uparrow\]
Fay-type identity
\[\downarrow \hbar \to 0\]
dispersionless Hirota eqn

\[\iff\]

Lax formalism

auxiliary linear eqns (usual form)
\[\uparrow\]
auxiliary linear eqns (generating form)
\[\downarrow \hbar \to 0\]
Hamilton-Jacobi eqns
• Similar results have been obtained for

  ○ multi-component (charged) KP hierarchies (T. & Takebe 06)
  ○ 1- and 2-component BKP hierarchies (Bogdanov & Konopelchenko 05; T. 06; Chen & Tu 06)
  ○ DKP hierarchy (also called “Pfaff lattice” or “coupled KP hierarchy”) (T., work in progress)

• An even more interesting case will be the “elliptic” DKP hierarchy and the Landau-Lifshitz hierarchy (studied by DJKM in early 80’s using free fermions on an elliptic curve).

• Any relation to limit shape of 3D Young diagrams, Schur process, dimer models, etc. ??