Fay-type identities and dispersionless limit of integrable hierarchies

(joint work with Takashi Takebe)

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Introduction

Dispersionless limit of integrable systems

- Dispersionless limit of KdV equation

$$
\begin{aligned}
& u_t + uu_x + \epsilon^2 u_{xxx} = 0 \\
\stackrel{\epsilon \to 0}{\longrightarrow} \ & u_t + uu_x = 0 \ \text{(dispersionless KdV eqn)}
\end{aligned}
$$

- Dispersionless limit of 2D Toda equation

$$
\begin{aligned}
& \epsilon\phi(s)_{xy} + e^{(\phi(s+\epsilon)-\phi(s))/\epsilon} - e^{(\phi(s)-\phi(s-\epsilon))/\epsilon} = 0 \\
\stackrel{\epsilon \to 0}{\longrightarrow} \ & \phi(s)_{xy} + (e^{\phi(s)_s})_s = 0 \ \text{(dispersionless Toda eqn)}
\end{aligned}
$$

N.B. This equation is also known in general relativity as the Boyer-Finley equation.

Motivation of this talk

Recent (since 2000) applications of dispersionless integrable hierarchies (dKP, dToda, etc.):

- Laplacian growth
- Associativity (WDVV) equations
- Light-cone string field theory
- Löwner equation
- Large-N normal matrix model
- Nonlinear optics
- Seiberg-Witten curve of 4D/5D $U(1)$ gauge theories
- Limit shape of 2D/3D Young diagrams (really??)

Fay-type identities play a significant role in some of these applications. This talk is concerned with theoretical aspects of Fay-type identities.

2. Differential Fay identity for KP hierarchy

τ-function of KP hierarchy

- Time variables $\boldsymbol{t} = (t_1, t_2, \ldots)$
- Spatial variable $x = t_1$
- Fermionic formula of tau function (DJKM 81)

$$\tau = \tau(\boldsymbol{t}) = \langle 0|e^{H(\boldsymbol{t})} g|0\rangle$$

where $H(\boldsymbol{t})$ and g are made from free fermion operators

$$[\psi_j, \psi_k^*]_+ = \delta_{jk},\ [\psi_j, \psi_k]_+ = [\psi_j^*, \psi_k^*]_+ = 0$$

as

$$H(\boldsymbol{t}) = \sum_{n=1}^{\infty} H_n t_n, \quad H_n = \sum_{j\in\mathbf{Z}} \psi_j \psi_{j+n}^*,$$

$$g = \exp \sum_{j,k\in\mathbf{Z}} a_{jk}\psi_j\psi_k^*,\ \ldots$$

An infinite number of Hirota equations (Sato & Sato; DJKM)

$$(3D_{t_2}^2 - 4D_{t_3}D_{t_1} + D_{t_1}^4)\tau \cdot \tau = 0, \ \ldots$$

where

$$P(D_{t_t}, D_{t_2}, \cdots)\tau \cdot \tau = P(\partial_{t'_1} - \partial_{t_1}, \partial_{t'_2} - \partial_{t_2}, \ldots)\tau(\boldsymbol{t}') \cdot \tau(\boldsymbol{t})|_{\boldsymbol{t}'=\boldsymbol{t}}$$

Generating functional form of Hirota equations (DJKM 81)

$$\oint \frac{dz}{2\pi i} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} \tau(\boldsymbol{t}' - [z^{-1}])\tau(\boldsymbol{t} + [z^{-1}]) = 0$$

where

$$\xi(\boldsymbol{t}, z) = \sum_{n=1}^{\infty} t_n z^n, \quad [z^{-1}] = \left(z^{-1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots, \frac{z^{-n}}{n}, \ldots\right).$$

The contour integral is along a sufficiently large circle $|z| = R$.

Fay identity (derived by Sato from Plücker relation)

$$
\begin{aligned}
&(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)\tau(\boldsymbol{t}+[\lambda_1]+[\lambda_2])\tau(\boldsymbol{t}+[\lambda_3]+[\lambda_4])\\
&-(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)\tau(\boldsymbol{t}+[\lambda_1]+[\lambda_3])\tau(\boldsymbol{t}+[\lambda_2]+[\lambda_4])\\
&+(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)\tau(\boldsymbol{t}+[\lambda_1]+[\lambda_4])\tau(\boldsymbol{t}+[\lambda_2]+[\lambda_3])=0
\end{aligned}
$$

Differential Fay identity (Adler & van Moerbeke 92)

$$
\frac{\tau(\boldsymbol{t}+[\lambda^{-1}]+[\mu^{-1}])\tau(\boldsymbol{t})}{\tau(\boldsymbol{t}+[\lambda^{-1}])\tau(\boldsymbol{t}+[\mu^{-1}])}=1-\frac{1}{\lambda-\mu}\partial_{t_1}\log\frac{\tau(\boldsymbol{t}+[\lambda^{-1}])}{\tau(\boldsymbol{t}+[\mu^{-1}])}
$$

Some features of differential Fay identity

• The original derivation due to Adler & van Moerbeke starts from the Fay identity and carries out a degeneration procedure. One can derive it directly from the DJKM bilinear equation as well.

• Expression à la Wiegmann & Zabrodin

$$\exp\Big((e^{D(\lambda)}-1)(e^{D(\mu)}-1)\log\tau\Big) = 1 - \frac{\partial_{t_1}(e^{D(\lambda)}-e^{D(\mu)})\log\tau}{\lambda-\mu}$$

where

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{t_n}.$$

This is convenient for deriving the dispersionless limit (presented later on).

• Equivalent expression

$$e^{-D(\lambda)}\Psi(\mu) = -\lambda^{-1}(\partial_{t_1} - \partial_{t_1}\log\Psi(\lambda))\Psi(\mu) \qquad \cdots (\diamondsuit)$$

in terms of the standard wave function

$$\Psi(z) = \frac{\tau(\boldsymbol{t} - [z^{-1}])}{\tau(\boldsymbol{t})} e^{\xi(\boldsymbol{t},z)}.$$

• Theorem: The differential Fay identity is equivalent to the KP hierarchy (T. & Takebe 95).

• Key: ($\diamondsuit$) is a generating functional form of the auxiliary linear equations

$$\partial_{t_n}\Psi(z) = B_n\Psi(z)$$

of the KP hierarchy.

3. Relation to dispersionless KP hierarchy

Dispersionless limit = quasi-classical limit

• Assume that an $\hbar$-dependent τ-function $\tau = \tau(\hbar, \boldsymbol{t})$ behaves in classical ($\hbar \to 0$) limit as

$$\tau(\hbar, \hbar^{-1}\boldsymbol{t}) = \exp(\hbar^{-2}F(\boldsymbol{t}) + O(\hbar^{-1}))$$

• For the rescaled τ-function $\tau_\hbar = \tau(\hbar, \hbar^{-1}\boldsymbol{t})$, the differential Fay identity reads

$$\exp\Big((e^{\hbar D(\lambda)} - 1)(e^{\hbar D(\mu)} - 1)\log\tau_\hbar\Big) = 1 - \frac{\hbar\partial_{t_1}(e^{\hbar D(\lambda)} - e^{\hbar D(\mu)})\log\tau_\hbar}{\lambda - \mu}$$

N.B. Rescaling rule: $\partial_{t_n} \to \hbar\partial_{t_n}$, $D(z) \to \hbar D(z)$

• As $\hbar \to 0$, the rescaled differential Fay identity

$$\exp\Big((e^{\hbar D(\lambda)}-1)(e^{\hbar D(\mu)}-1)\log\tau_\hbar\Big) = 1 - \frac{\hbar\partial_{t_1}(e^{\hbar D(\lambda)}-e^{\hbar D(\mu)})\log\tau_\hbar}{\lambda-\mu}$$

turns into the dispersionless Hirota equation (T. & Takebe 95; Carroll & Kodama 95)

$$e^{D(\lambda)D(\mu)F} = 1 - \frac{\partial_{t_1}(D(\lambda)-D(\mu))F}{\lambda-\mu}.$$

• Systematic expansion in powers of $\hbar$ is possible.

• Theorem: The dispersionless Hirota equation is equivalent to the dispersionless KP (dKP) hierarchy (Boyarsky et al. 01; Teo 03).

Dispersionless KP hierarchy

• Lax equations of KP hierarchy (Sato)

$$\partial_{t_n} L = [B_n, L],\ L = \partial_x + \sum_{n=1}^{\infty} u_{n+1}\partial_x^{-n},\ x = t_1,$$
$$B_n = (L^n)_{\geq 0} \text{ (nonnegative powers of } \partial_x)$$

• Dispersionless Lax equations of dKP hierarchy (Kodama & Gibbons; Krichever)

$$\partial_{t_n} \mathcal{L} = \{\mathcal{B}_n, \mathcal{L}\},\ \mathcal{L} = p + \sum_{n=1}^{\infty} u_{n+1} p^{-n},\ x = t_1,$$
$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0} \text{ (nonnegative powers of } p)$$

based on Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}$$

Hamilton-Jacobi equations and dispersionless Hirota equation

• Quasi-classical approximation $\Psi(z) \sim e^{\hbar^{-1}S(z)}$ (Zakharov; Kodama & Gibbons) to the rescaled auxililiary linear equations

$$\hbar\partial_{t_n}\Psi(z) = B_n(\hbar\partial_x)\Psi(z)$$

yields the Hamilton-Jacobi equations

$$\partial_{t_n}S(z) = \mathcal{B}_n(\partial_x S(z))$$

for the S-function

$$S(z) = \xi(\boldsymbol{t}, z) - D(z)F = \sum_{n=1}^{\infty} t_n z^n - \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{t_n}F.$$

• These Hamilton-Jacobi equations are equivalent to the dispersionless Lax equations through inversion of the map $z \mapsto p(z) = \partial_x S(z)$:

$$p = p(z) \longleftrightarrow z = \mathcal{L}(p).$$

• The dispersionless Hirota equation can be rewritten as

$$e^{D(\lambda)D(\mu)F} = \frac{\partial_x S(\lambda) - \partial_x S(\mu)}{\lambda - \mu}.$$

This is a generating functional form of the Hamilton-Jacobi equation (Bogdanov, Konopelchenko & Martinez Alonso 03).

• $\mathcal{B}_n$'s are identified with the Faber polynomials of the (conformal) mapping $z \to p(z)$ (Teo 03).

3. Difference Fay identities for Toda hierarchy

Tau function

• Two sets of continuous variables: $\boldsymbol{t} = (t_1, t_2, \ldots)$, $\tilde{\boldsymbol{t}} = (\tilde{t}_1, \tilde{t}_2, \ldots)$
• Discrete variable: $s \in \mathbf{Z}$
• Fermionic formula of tau function (Jimbo & Miwa 83; Takebe 85):

$$\tau(s, \boldsymbol{t}, \tilde{\boldsymbol{t}}) = \langle s| e^{H(\boldsymbol{t})} g e^{-\tilde{H}(\tilde{\boldsymbol{t}})} |s\rangle$$

where

$$H(\boldsymbol{t}) = \sum_{n=1}^{\infty} H_n t_n, \quad \tilde{H}(\tilde{\boldsymbol{t}}) = \sum_{n=1}^{\infty} \tilde{H}_n \tilde{t}_n,$$
$$H_n = \sum_{j \in \mathbf{Z}} \psi_j \psi_{j+n}^*, \quad \tilde{H}_n = H_{-n}.$$

• Generating functional form of Hirota equations (Ueno & T 83)

$$\oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)} \tau(s', \boldsymbol{t}' - [z^{-1}], \tilde{\boldsymbol{t}}') \tau(s, \boldsymbol{t} + [z^{-1}], \tilde{\boldsymbol{t}})$$
$$= \oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(\tilde{\boldsymbol{t}}'-\tilde{\boldsymbol{t}},z^{-1})} \tau(s'+1, \boldsymbol{t}', \tilde{\boldsymbol{t}}' - [z]) \tau(s-1, \boldsymbol{t}, \tilde{\boldsymbol{t}} + [z])$$

• Three distinct Fay-type identities (so to speak, difference Fay identities) are obtained (Zabrodin 01; Teo 06). They can be derived from the foregoing bilinear equation by letting
(1) $\boldsymbol{t}' = \boldsymbol{t} + [\lambda^{-1}] + [\mu^{-1}]$, $\tilde{\boldsymbol{t}}' = \tilde{\boldsymbol{t}}$, $s' = s + 1$
(2) $\boldsymbol{t}' = \boldsymbol{t}$, $\tilde{\boldsymbol{t}}' = \tilde{\boldsymbol{t}} + [\lambda] + [\mu]$, $s' = s - 3$
(3) $\boldsymbol{t}' = \boldsymbol{t} + [\lambda^{-1}]$, $\tilde{\boldsymbol{t}}' = \tilde{\boldsymbol{t}} + [\mu]$, $s' = s$

Difference Fay identities thus obtained

$$(1)\qquad \exp\Big((e^{D(\lambda)}-1)(e^{D(\mu)}-1)\log\tau\Big)-\frac{\lambda}{\lambda-\mu}\exp\Big((e^{D(\lambda)}-1)(e^{-\partial_s}-1)\log\tau$$

$$+\frac{\mu}{\lambda-\mu}\exp\Big((e^{D(\mu)}-1)(e^{-\partial_s}-1)\log\tau\Big)=0,$$

$$(2)\qquad \exp\Big((e^{\tilde{D}(\lambda)}-1)(e^{\tilde{D}(\mu)}-1)\log\tau\Big)+\frac{\mu}{\lambda-\mu}\exp\Big((e^{\tilde{D}(\lambda)}-1)(e^{\partial_s}-1)\log\tau\Big)$$

$$-\frac{\lambda}{\lambda-\mu}\exp\Big((e^{\tilde{D}(\mu)}-1)(e^{\partial_s}-1)\log\tau\Big)=0,$$

$$(3)\qquad \exp\Big((e^{D(\lambda)}-1)(e^{\tilde{D}(\mu)}-1)\log\tau\Big)-1+\frac{\mu}{\lambda}\exp\Big(((e^{D(\lambda)}-1)(e^{\partial_s}-1)$$

$$+(e^{\tilde{D}(\mu)}-1)(e^{-\partial_s}-1)-(e^{\partial_s}-1)(e^{-\partial_s}-1))\log\tau\Big)=0$$

where

$$D(z)=\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\partial_{t_n},\qquad \tilde{D}(z)=\sum_{n=1}^{\infty}\frac{z^{n}}{n}\partial_{\tilde{t}_n}.$$

Dispersionless Hirota equations

In the dispersionless (= quasi-classical) limit

$$\tau(\hbar, \hbar^{-1}s, \hbar^{-1}\boldsymbol{t}, \hbar^{-1}\tilde{\boldsymbol{t}}) \sim e^{\hbar^{-2}F(s,\boldsymbol{t},\tilde{\boldsymbol{t}})},$$

the difference Fay identities turn into the dispersionless Hirota equations of the dispersionless Toda (dToda) hierarchy (Wiegmann & Zabrodin 00; Kostov et al. 01):

$$(1) \qquad e^{D(\lambda)D(\mu)F} = \frac{\lambda e^{-\partial_s D(\lambda)F} - \mu e^{-\partial_s D(\mu)F}}{\lambda - \mu},$$

$$(2) \qquad e^{\tilde{D}(\lambda)\tilde{D}(\mu)F} = \frac{-\mu e^{\partial_s \tilde{D}(\lambda)F} + \lambda e^{\partial_s \tilde{D}(\mu)F}}{\lambda - \mu},$$

$$(3) \qquad e^{D(\lambda)\tilde{D}(\mu)F} = 1 - \frac{\mu}{\lambda} e^{(\partial_s D(\lambda) - \partial_s \tilde{D}(\mu) + \partial_s^2)F}.$$

Theorem: They are equivalent to the dToda hierarchy (Teo 03).

Implications to wave functions

• Standard wave functions

$$\Psi(s,z) = \frac{\tau(s,\boldsymbol{t}-[z^{-1}],\tilde{\boldsymbol{t}})}{\tau(s,\boldsymbol{t},\tilde{\boldsymbol{t}})} z^s e^{\xi(\boldsymbol{t},z)},$$

$$\tilde{\Psi}(s,z) = \frac{\tau(s+1,\boldsymbol{t},\tilde{\boldsymbol{t}}-[z])}{\tau(s,\boldsymbol{t},\tilde{\boldsymbol{t}})} z^s e^{\xi(\tilde{\boldsymbol{t}},z^{-1})}$$

• The difference Fay identities can be converted to equations for these wave functions:

$$e^{-D(\lambda)}\Psi(s,\mu) = \lambda^{-1}\left(\frac{\Psi(s+1,\lambda)}{\Psi(s,\lambda)} - e^{\partial_s}\right)\Psi(s,\mu),$$

$$e^{-\tilde{D}(\lambda)}\Psi(s,\mu) = \left(1 - \frac{\tilde{\Psi}(s+1,\lambda)}{\tilde{\Psi}(s,\lambda)} e^{-\partial_s}\right)\Psi(s,\mu),$$

...(same equations for $\tilde{\Psi}(s,\mu)$)...

• These equations give a generating functional expression of the auxiliary linear equations

$$
\begin{aligned}
&\partial_{t_n}\Psi(s,z) = B_n(e^{\partial_s})\Psi(s,z),\\
&\partial_{\tilde{t}_n}\Psi(s,z) = \tilde{B}_n(e^{\partial_s})\Psi(s,z),\\
&\ldots\text{(same equations for } \tilde{\Psi}(s,z))\ldots
\end{aligned}
$$

of the Toda hierarchy.

• Roughly speaking, this implies that the difference Fay identities are equivalent to the Toda hierarchy itself. (Details remain to be checked.)

6. Conclusion

• The dispersionless Hirota equations of the dKP/dToda hierarchies appear as the quasi-classical ($\hbar \to 0$) limit of the differential/difference Fay identities.

• These special Fay-type identities carry full information of the KP and Toda hierarchies.

• Systematic $\hbar$-expansion of these Fay-type identities may have application to, say, light-cone string field theory ? (Bonora et al. 05)

Hirota formalism		Lax formalism
DJKM bilinear eqn	$\Longleftrightarrow$	auxiliary linear eqns (usual form)
$\updownarrow$		$\updownarrow$
Fay-type identity	$\Longleftrightarrow$	auxiliary linear eqns (generating form)
$\downarrow \hbar \to 0$		$\downarrow \hbar \to 0$
dispersionless Hirota eqn	$\Longleftrightarrow$	Hamilton-Jacobi eqns

• Similar results have been obtained for

◦ multi-component (charged) KP hierarchies (T. & Takebe 06)
◦ 1- and 2-component BKP hierarchies (Bogdanov & Konopelchenko 05; T. 06; Chen & Tu 06)
◦ DKP hierarchy (also called "Pfaff lattice" or "coupled KP hierarchy") (T., work in progress)

• An even more interesting case will be the "elliptic" DKP hierarchy and the Landau-Lifshitz hierarchy (studied by DJKM in early 80's using free fermions on an elliptic curve).

• Any relation to limit shape of 3D Young diagrams, Schur process, dimer models, etc. ??