題目:

同変戸田階層とOkounkov–Pandharipande dressing operators

(Equivariant Toda hierarchy and Okounkov–Pandharipande dressing operators)

要旨:

同変戸田階層はGetzlerによってリーマン球面の同変Gromov–Witten不変量における可積分構造として導入されたもので，2次元戸田階層から特殊な簡約条件によって導くことができる．2000年代半ばにOkounkovとPandharipandeはリーマン球面の同変Gromov–Witten不変量をフェルミオン表示することによって，それらの母函数が同変戸田階層のτ函数になることを示した．その証明の中でフォック空間上のdressing opetatorと呼ぶ作用素を用いたが，この作用素の正体はよくわからないままに残った．またLax形式における説明はなされなかった．講演者はこの作用素を2次元戸田階層のLax作用素と同様の意味での差分作用素として再構成し，同変戸田階層が現れる仕組みをLax形式において明らかにすることができた．この講演ではこの結果を紹介し，Hurwitz数の高スピン版に関連すると思われる拡張の試みにも触れる．

（文献：arXiv:2103.10666, 2211.1135~~4~~)
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Plan

- Gromov-Witten partition functions of $\mathbb{CP}^1$
- Equivariant Gromov-Witten invariants of $\mathbb{CP}^1$
- Okounkov-Pandharipande dressing operators
- Equivariant Toda hierarchy
- Reconstruction of OP dressing operators
- Outlook

Gromov-Witten partition functions of $\mathbb{CP}^1$

- GW invariants (connected part)

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle^{\circ}_{g,d} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\omega)$$

where

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d) \to \mathbb{CP}^1, \ (f, C, p_1, \dots, p_n) \mapsto f(p_i),$$

$$\psi_i = c_1(L_i), \ (L_i)_{(f,C,p_1,\cdots,p_n)} = T^*_{p_i} C$$

$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle^{\bullet}_{g,d}$: all curves (connected and disconnected)

$\overline{\mathcal{M}}_{gn}(\mathbb{CP}^1, d)$: moduli space of stable maps $f: C \to \mathbb{CP}^1$
of degree d from stable curve C of genus g
with n marked points $p_1, \cdots, p_n$

$\omega \in H^2(\mathbb{CP}^1)$: Kähler class $c_1(\mathcal{O}(1))$

$\tau_k(\omega)$, $k = 0, 1, \cdots$: gravitational descendents

cf. $\left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_g^0 = \int_{[\overline{\mathcal{M}}_{gn}]} \prod_{i=1}^{n} \psi_{\cancel{k_i}\, i}^{k_i}$: GW invariants of single point

$\to$ Witten-Kontsevich partition function = KdV tau fn

- Generating function (partition function)

$$\left\langle \exp\left(\sum_{k=0}^{\infty} \tau_k(\omega) t_k\right)\right\rangle^{\bullet} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots,k_n=0}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega)\right\rangle^{\bullet} \prod_{i=1}^{n} t_{k_i}$$

$$= \sum_{\lambda\in\mathcal{P}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \exp\left(\sum_{k=0}^{\infty} \frac{p_{k+1}(\lambda)}{(k+1)!} t_k\right)$$

1. KP tau function
2. $\left\langle \exp\left(s\tau_0(1) + \sum_{k=0}^{\infty} \tau_k(\omega) t_k \right)\right\rangle$, $1 \in H^0(\mathbb{CP}^1)$,

   becomes 1D Toda tau function.

- Toda conjecture

types of GW theory	integrable structure	
absolute stationary	extended 1D Toda	(Getzler (conj.) Dubrovin-Zhang)
relative stationary	2D Toda	(OP)
equivariant	equivariant Toda	(Getzler (conj.) OP, Milanov)
orbifold $\mathbb{CP}^1_{a,b}$	extended/equivariant bigraded Toda	(Milanov-Tseng)

Equivariant Gromov-Witten invariants of $\mathbb{C}\mathbb{P}^1$

(Okounkov-Pandharipande arxiv:0207233)

- torus action

$$\mathbb{C}^* \curvearrowright \mathbb{C}\mathbb{P}^1 \qquad [z_0 : z_1] \mapsto [z_0 : \lambda z_1]$$ fixed points: $0, \infty$

- equivariant cohomology

$$H^\bullet_{\mathbb{C}^*}(\mathbb{C}\mathbb{P}^1) \; : \text{ module over } H^\bullet_{\mathbb{C}^*}(pt) = \mathbb{C}[\nu]$$

ν : equivariant parameter

$\nu \to 0$: non-equivariant limit to ordinary cohomology

- equivariant point classes $0, \infty \in H^2_{\mathbb{C}^*}(\mathbb{CP}^1)$

- generating functions of equivariant GW invariants

$$F = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} \hbar^{2g-2} q^d \left\langle \exp\left(\sum_{k=0}^{\infty} x_k \tau_k(0) + \sum_{k=0}^{\infty} x_k^* \tau_k(\infty) \right) \right\rangle_{g,d}^{0}$$

$$Z = e^F$$

- fermionic formula in terms of $\mathcal{A}$-operators

$\mathcal{A}(\nu z, \hbar w)$ and $\mathcal{A}(-\nu z, \hbar w)^*$.

- $$\mathcal{A}(z,w) = \left(\frac{\varsigma(w)}{w}\right)^z \sum_{k\in\mathbb{Z}} \frac{\varsigma(w)^k}{(1+z)_k}\,\mathcal{E}_k(z)$$

$$\varsigma(w) = e^{w/2} - e^{-w/2}, \quad (1+z)_k = \Gamma(1+k+z)/\Gamma(1+z)$$

$$\mathcal{E}_k(z) = \sum_{n\in\mathbb{Z}} e^{z(n+1/2-k/2)} :\psi_{n-k}\psi_n^*:$$

$$[\psi_m, \psi_n^*]_+ = \delta_{mn}, \quad [\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0$$

Remark: $\quad :\psi_m\psi_n^*: \longleftrightarrow E_{mn} \in \mathfrak{gl}(\infty)$

$$(:\psi_m\psi_n^*:)^* = :\psi_n\psi_m^*: \quad \longleftrightarrow \text{transpose}$$

Okounkov-Pandharipande dressing operators (OP arxiv 0207233)

- fermionic formula of partition function

$$Z = \langle 0| \exp\left(\sum_{k=0}^{\infty} x_k A_k\right) e^{\alpha_1} \left(\frac{q}{\hbar^2}\right)^H e^{\alpha_{-1}} \exp\left(\sum_{k=0}^{\infty} x_k^* A_k^*\right)|0\rangle$$

$$\alpha_n,\ n \in \mathbb{Z},\quad [\alpha_m, \alpha_n] = m\,\delta_{m+n,0}$$

$$H = L_0,\qquad A_k, A_k^* \notin \operatorname{Span}\{\alpha_n \mid n \in \mathbb{Z}\}$$

$$A(z) = \frac{1}{\hbar}\mathcal{A}(\nu z, \hbar z) = \sum_{k\in\mathbb{Z}} A_k z^{k+1}$$

$$A^*(z) = \frac{1}{\hbar}\mathcal{A}^*(-\nu z, \hbar z) = \sum_{n\in\mathbb{Z}} A_k^* z^{k+1}$$

- dressing operators V, V^* (W and W^* in OP's notations)

$$\langle 0| V = \langle 0|, \quad V^* |0\rangle = |0\rangle$$

$$V^{-1} A_k V = \widetilde{A}_k \in \mathrm{Span}\{\alpha_n \,|\, n \geq 1\}$$

$$V^* A_k^* V^{*-1} = \widetilde{A}_k^* \in \mathrm{Span}\{\alpha_{-n} \,|\, n \geq 1\}$$

- $$Z = \langle 0| \exp\Big(\underbrace{\sum_{k=0}^{\infty} x_k \widetilde{A}_k}_{= \sum_{k=1}^{\infty} t_k \alpha_k}\Big) V^{-1} e^{\alpha_1} \left(\frac{q}{\hbar^2}\right)^H e^{\alpha_{-1}} V^{*-1} \exp\Big(\underbrace{\sum_{k=0}^{\infty} x_k^* \widetilde{A}_k^*}_{\text{“} - \sum_{k=1}^{\infty} \bar{t}_k \alpha_{-k}\text{”}}\Big) |0\rangle$$

$\rightarrow$ 2-KP tau function !

Equivariant Toda hierarchy

$(Q = q/\hbar^2)$

- $\tau(s, t, \bar{t}) = Q^{s^2/2} Z(t_1 + \frac{s}{\nu}, t_2, \cdots, \bar{t}_1 + \frac{s}{\nu}, \bar{t}_2, \cdots)$

  is a tau function of the 2D Toda hierarchy that satisfies the reduction condition

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial \bar{t}_1} - \nu \partial_s\right)\left(\tau Q^{-s^2/2}\right) = 0$$

to the equivariant Toda hierarchy.

Remark: Okounkov and Pandharipande's reasoning is not very clear.

- Lowest equation (equivariant Toda equation)

$$\frac{\partial^2 \log Z(t_1,\bar{t}_1)}{\partial t_1 \partial \bar{t}_1} + Q\,\frac{Z(t_1+\frac{1}{\nu},\bar{t}_1+\frac{1}{\nu})\,Z(t_1-\frac{1}{\nu},\bar{t}_1-\frac{1}{\nu})}{Z(t_1,\bar{t}_1)^2} = 0$$

from $Q^{S^2/2}$

- Question : What about Lax formalism ?

cf. Milanov and Tseng, 0707.3172, Appendix

good review of the equiv. Toda hierarchy.

- 2D Toda hierarchy in Lax formalism $(\hbar = 1)$

$$L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n}, \quad \bar{L}^{-1} = \sum_{n=0}^{\infty} \bar{u}_n \Lambda^{n-1} \qquad \Lambda = e^{\partial_s}$$

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$
$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}],$$

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{<0}.$$

- Reduction condition to orbifold generalization of equivariant Toda hierarchy :

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \nu \log Q$$

($a = b = 1$ for ordinary equivariant Toda hierarchy)

Cf. Dressing operators $W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}$, $\bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n$:

$$L = W\Lambda W^{-1}, \quad \bar{L} = \bar{W}\Lambda\bar{W}^{-1}$$

$\Lambda = e^{\partial_s}$

$$\log L = W \log \Lambda W^{-1} = \partial_s - \frac{\partial W}{\partial s} W^{-1},$$

$\log \Lambda = \partial_s$

$$\log \bar{L} = \bar{W} \log \Lambda \bar{W}^{-1} = \partial_s - \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}.$$

Reconstruction of OP dressing operators

(T. 2103.10666) ミス発覚 (p16で訂正)

- Redefine V and $\bar{V}$ as difference operators

$$V = 1 + \sum_{n=1}^{\infty} v_n \Lambda^{-n}, \quad \bar{V} = 1 + \sum_{n=1}^{\infty} \bar{v}_n \Lambda^{n},$$
$$v_n = v_n(s), \quad \bar{v}_n = \bar{v}_n(s),$$

$\Lambda = e^{\partial_s}$

$\log \Lambda = \partial_s$

that satisfy the intertwining relations

$$(\Lambda^a + H - \nu \log \Lambda) V = V (\Lambda^a - \nu \log \Lambda),$$
$$\bar{V} (\Lambda^{-b} + H - \nu \log \Lambda) = (\Lambda^{-b} - \nu \log \Lambda) \bar{V}.$$

$H = s + \frac{1}{2} \sim L_0$

Remark: Construction of $\bar{V}$ is parallel ($\bar{V}^* \sim V$)

- Power series expansion $V = \sum_{k=0}^{\infty} \nu^k V_k$ yields

$$[\Lambda^a, V_0] + HV_0 = 0,$$
$$[\Lambda^a, V_0^{-1} V_k] = V_0^{-1}[\log \Lambda, V_{k-1}]. \quad k \geq 1.$$

- We can find a solution (not unique) of the form

$$V_0 = 1 + \sum_{n=\cancel{1}\, a}^{\infty} v_{0n} \Lambda^{-n}, \quad V_k = \sum_{n=ka}^{\infty} v_{kn} \Lambda^{-n}, \quad k \geq 1.$$

where v_{kn}'s are polynomials in s.

1) Equation for the coeff. of $V_0 = 1 + \sum_{n=a}^{\infty} v_{0n}\Lambda^{-n}$:

$$v_{0,n+a}(s+a) - v_{0,n+a}(s) = -H v_{0n}(s) \qquad (*)$$

Choose $v_{0a}(s) = 1$. Then

$$v_{0,2a}(s+a) - v_{0,2a}(s) = -H = -s - \frac{1}{2}$$

We can find a solution :

$$v_{0,2a}(s) = -\frac{1}{2a}(s+a)s - \frac{1}{2a}s$$

$(*)$ can be solved recursively with the aid of the Bernoulli polynomials $B_k(x)$:

$$B_k(x+1) - B_k(x) = k x^{k-1}$$

We can thus find u_{0n}'s such that

$$\begin{cases} u_{0,na}(s) & \text{are polynomials in } s. \\ u_{0,n}(s) = 0 & \text{for } n \not\equiv 0 \mod a. \end{cases}$$

2) Equations for the coeff. of $V_0^{-1} V_k = \sum_{n=ka}^{\infty} v'_{kn} \Lambda^{-n}$:

$$v'_{k,n+a}(s+a) - v'_{k,n+a}(s) = f_{kn}(s) \qquad (**)$$

where $V_0^{-1}[\log \Lambda, V_{k-1}] = \sum_{n=(k-1)a}^{\infty} f_{kn} \Lambda^{-n}$. We can solve

(**) in much the same way.

Reduction condition for Lax operators

- The dressing operators $W, \bar{W}$ can be characterized by the factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1}\bar{W},$$

where

$$U = V^{-1} e^{\Lambda^a/a} Q^H e^{\Lambda^{-b}/b} \bar{V}^{-1} \quad . \qquad \left(H = s + \tfrac{1}{2}\right)$$

$$\tau = \langle s| \exp\left(\sum_{k=1}^{\infty} t_k \alpha_k\right) V^{-1} e^{\alpha_a/a} Q^H e^{\alpha_{-b}/b} \bar{V}^{-1} \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \alpha_{-k}\right) |s\rangle$$

- U satisfies the intertwining relation

$$(\Lambda^a - \nu \log \Lambda)U = U(\Lambda^{-b} - \nu \log \Lambda - \nu \log Q).$$

This implies the reduction condition

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \nu \log Q$$

for the Lax operators.

Outlook

- "Higher spin" analogue r-spin

  $((\Lambda^a + H)^r - \nu \log \lambda) V = V(\Lambda^{ar} - \nu \log \Lambda), \cdots$

- $\nu \to 0$ limit: $L^a = \bar{L}^{-b}$

  Bigraded Toda hierarchy of type (a, b) emerges.

  Additional (logarithmic) flows can be derived.

  (T. 2211.11353)

- Explicit construction of $V_0 = \lim_{\nu \to 0} V$ by Bernoulli polynomials. (cf. Alexandrov's work)