Topological vertex and quantum mirror curves

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Based on

1. K. T., arXiv:1301.4548 [math-ph]
2. K. T. and T. Nakatsu, arXiv:1507.07053

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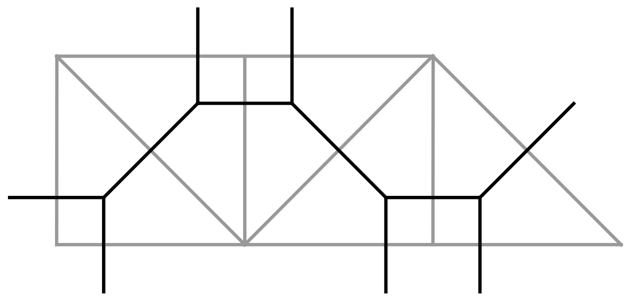
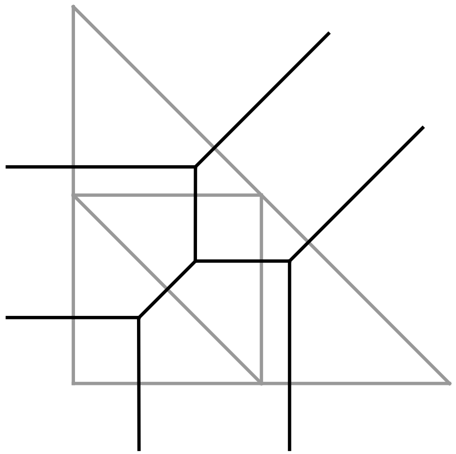
Web diagrams of non-compact toric Calabi-Yau threefolds



On-strip geometry



Closed topological vertex

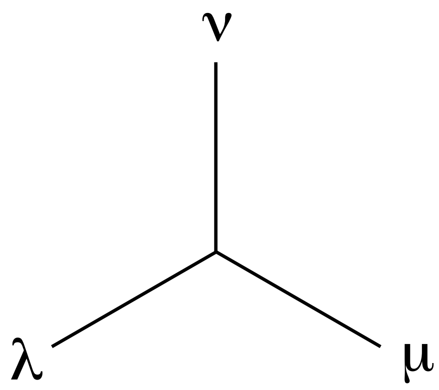
Topological vertex

"Topological vertex" (Aganagic, Klemm, Mariño and Vafa 2003) is a diagrammatic method to construct the partition functions (or amplitudes) of topological string theory on non-compact toric Calabi-Yau threefolds.

Vertex weight

$$C_{\lambda\mu\nu} = q^{\kappa(\mu)/2} s_{{}^{\mathrm{t}}\nu}(q^{-\rho}) \sum_{\eta \in \mathcal{P}} s_{{}^{\mathrm{t}}\lambda/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-{}^{\mathrm{t}}\nu-\rho})$$

- $\lambda = (\lambda_i)_{i=1}^{\infty}$, $\mu = (\mu_i)_{i=1}^{\infty}$, $\nu = (\nu_i)_{i=1}^{\infty}$ are partitions representing Young diagrams of arbitrary shapes. ${}^{\mathrm{t}}\nu$ denotes the conjugate partition of ν.



- $\kappa(\mu)$ is the second Casimir invariant

$$\kappa(\mu) = \sum_{i=1}^{\infty} \mu_i(\mu_i - 2i + 1) = 2 \sum_{(i,j) \in \mu} (j - i)$$

Vertex weight (cont'd)

$$C_{\lambda\mu\nu} = q^{\kappa(\mu)/2} s_{{}^{\mathrm{t}}\nu}(q^{-\rho}) \sum_{\eta \in \mathcal{P}} s_{{}^{\mathrm{t}}\lambda/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-{}^{\mathrm{t}}\nu-\rho})$$

- $s_{{}^{\mathrm{t}}\nu}(q^{-\rho})$, $s_{{}^{\mathrm{t}}\lambda/\eta}(q^{-\nu-\rho})$ and $s_{\mu/\eta}(q^{-{}^{\mathrm{t}}\nu-\rho})$ are special values of the Schur/skew Schur functions $s_{{}^{\mathrm{t}}\nu}(\boldsymbol{x})$, $s_{{}^{\mathrm{t}}\lambda/\eta}(\boldsymbol{x})$, $s_{\mu/\eta}(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, \cdots)$, at

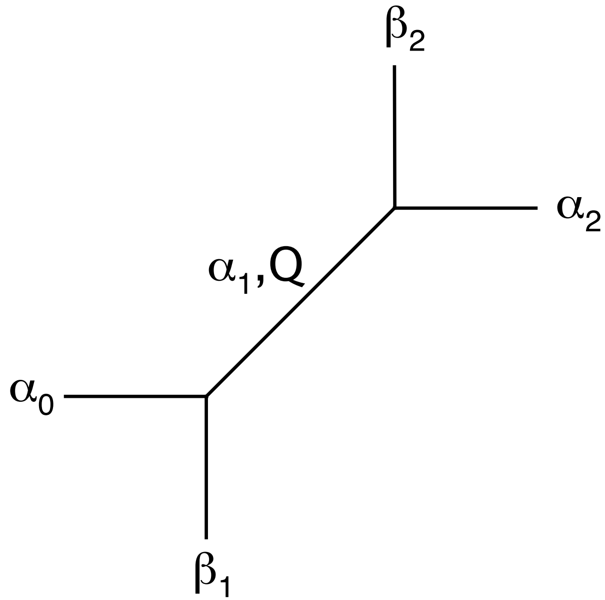
$$q^{-\rho} = (q^{i-1/2})_{i=1}^{\infty}, \quad q^{-\nu-\rho} = (q^{-\nu_i+i-1/2})_{i=1}^{\infty},$$
$$q^{-{}^{\mathrm{t}}\nu-\rho} = (q^{-{}^{\mathrm{t}}\nu_i+i-1/2})_{i=1}^{\infty}$$

- Vertex weights are glued together along the internal lines. Edge weights are assigned to those lines.

Examples (local $\mathbb{P}^1$ geometry)

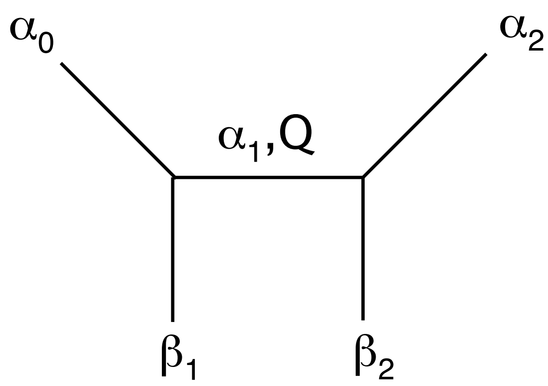
1. Open string amplitudes of $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ (resolved conifold):

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = \sum_{\alpha_1 \in \mathcal{P}} C_{\alpha_1\alpha_0\beta_1}(-Q)^{|\alpha_1|} C_{{}^t\alpha_1\alpha_2\beta_2}$$



2. Open string amplitudes of $X = \mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{CP}^1$:

$$Z^{\alpha_0\alpha_2}_{\beta_1\beta_2} = \sum_{\alpha_1 \in \mathcal{P}} C_{\alpha_1\alpha_0\beta_1}(-Q)^{|\alpha_1|} \times (-1)^{|\alpha_1|} q^{-\kappa(\alpha_1)/2} C_{\alpha_2{}^t\alpha_1\beta_2}$$



(A framing factor is inserted in this case.)

Building blocks of operator formalism

• Charge-zero sector of fermionic Fock space spanned by the ground states $\langle 0|$, $|0\rangle$ and the excited states

$$\langle\lambda| = \langle -\infty| \cdots \psi^*_{\lambda_i - i + 1} \cdots \psi^*_{\lambda_2 - 1}\psi^*_{\lambda_1},$$

$$|\lambda\rangle = \psi_{-\lambda_1}\psi_{-\lambda_2+1} \cdots \psi_{-\lambda_i+i-1} \cdots |-\infty\rangle$$

• Fermion bilinears

$$L_0 = \sum_{n\in\mathbb{Z}} n{:}\psi_{-n}\psi^*_n{:}, \quad K = \sum_{n\in\mathbb{Z}} (n-1/2)^2{:}\psi_{-n}\psi^*_n{:},$$

$$J_m = \sum_{n\in\mathbb{Z}} {:}\psi_{m-n}\psi^*_n{:}, \quad m \in \mathbb{Z},$$

$$V^{(k)}_m = q^{-km/2}\sum_{n\in\mathbb{Z}} q^{kn}{:}\psi_{m-n}\psi^*_n{:}, \quad k, m \in \mathbb{Z}.$$

- Vertex operators

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right), \quad \Gamma'_\pm(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k}\right)$$

and the multi-variable extensions

$$\Gamma_\pm(\boldsymbol{x}) = \prod_{i\geq 1} \Gamma_\pm(x_i), \quad \Gamma'_\pm(\boldsymbol{x}) = \prod_{i\geq 1} \Gamma'_\pm(x_i)$$

Fermionic expression of vertex weight

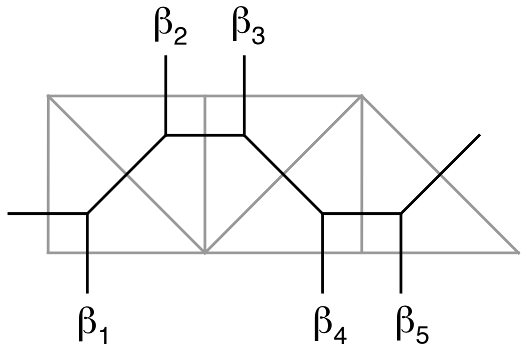
$$\begin{aligned} C_{\lambda\mu\nu} &= s_{{}^{\mathrm{t}}\nu}(q^{-\rho})\langle {}^{\mathrm{t}}\lambda|\Gamma_-(q^{-\nu-\rho})\Gamma_+(q^{-{}^{\mathrm{t}}\nu-\rho})q^{K/2}|\mu\rangle \\ &= s_{{}^{\mathrm{t}}\nu}(q^{-\rho})\langle \lambda|\Gamma'_-(q^{-\nu-\rho})\Gamma'_+(q^{-{}^{\mathrm{t}}\nu-\rho})q^{-K/2}|{}^{\mathrm{t}}\mu\rangle \end{aligned}$$

(Okounkov, Reshetikhin and Vafa 2003)

Cases where amplitudes are computed explicitly

1. On-strip geometry

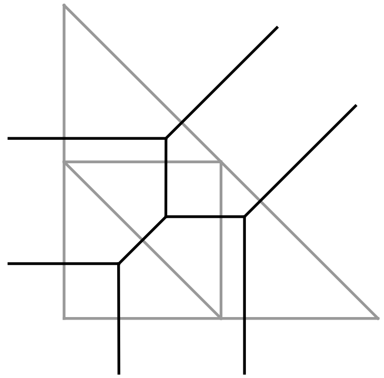
Iqbal and Kashani-Poor 2004 (by topological vertex)



2. Closed topological vertex

Bryan and Karp 2003, Karp, Liu and Mariño 2005 (by algebraic geometry and topological vertex)

Sułkowski 2006 (by crystal model and topological vertex)



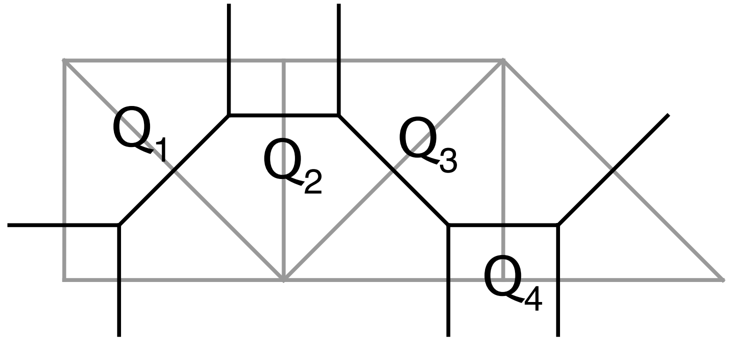
Our goal: To derive a q-difference equation that may be interpreted as quantization of the mirror curve.

2. On-strip geometry

General setup

(a) The toric diagram is a triangulation of a strip (trapezoid of height 1).

(b) The web diagram has N vertices, $N-1$ internal lines and $N+2$ external lines (or "legs").



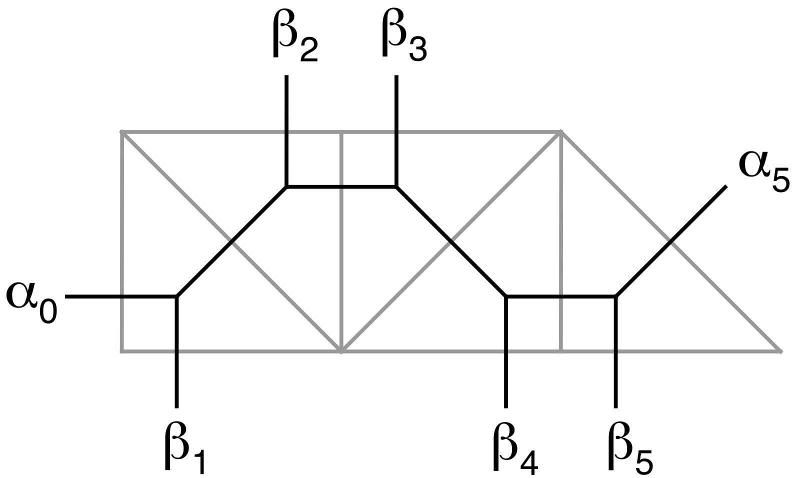
Example of on-strip geometry ($N = 5$)

(c) The internal lines are assigned with Kähler parameters $Q_1, \ldots, Q_{N-1}$.

General setup (cont'd)

(d) The external lines are assigned with partitions $\alpha_0, \beta_1, \ldots, \beta_N, \alpha_N$.

(e) The sign (or type) σ_n of the n-th vertical leg are defined as



$$\sigma_n = \begin{cases} +1 & \text{if the leg points} \uparrow \\ -1 & \text{if the leg points} \downarrow \end{cases}$$

$$\sigma_2 = \sigma_3 = +1,$$
$$\sigma_1 = \sigma_4 = \sigma_5 = -1$$

Let $Z^{\alpha_0 \alpha_N}_{\beta_1 \ldots \beta_N}$ denote the open string amplitude constructed by topological vertex.

Infinite-product formula of amplitude

If $\alpha_0 = \alpha_N = \emptyset$, the amplitude can be computed explicitly with the aid of the Cauchy identities for skew Schur functions (Iqbal and Kashani-Poor 2004):

$$Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N} = s_{{}^{\mathrm{t}}\beta_1}(q^{-\rho}) \cdots s_{{}^{\mathrm{t}}\beta_N}(q^{-\rho}) \times \prod_{1\le m<n\le N} \prod_{i,j=1}^{\infty} (1 - Q_{mn} q^{-{}^{\mathrm{t}}\beta_i^{(m)} - \beta_j^{(n)} + i + j - 1})^{-\sigma_m \sigma_n}$$

where

$$\beta^{(n)} = \begin{cases} \beta_n & \text{if } \sigma_n = +1, \\ {}^{\mathrm{t}}\beta_n & \text{if } \sigma_n = -1, \end{cases} \qquad Q_{mn} = Q_m Q_{m+1} \cdots Q_{n-1}$$

Fermionic expression of amplitude

The infinite-product formula holds only for $\alpha_0 = \alpha_N = \emptyset$. For general cases, the following fermionic expression is available:

$$
\begin{aligned}
Z^{\alpha_0\alpha_N}_{\beta_1\cdots\beta_N} &= q^{(1-\sigma_1)\kappa(\alpha_0)/4} q^{(1+\sigma_N)\kappa(\alpha_N)/4} s_{{}^{\mathrm{t}}\beta_1}(q^{-\rho}) \cdots s_{{}^{\mathrm{t}}\beta_N}(q^{-\rho}) \\
&\quad \times \langle {}^{\mathrm{t}}\alpha_0 | \Gamma^{\sigma_1}_{-}(q^{-\beta^{(1)}-\rho}) \Gamma^{\sigma_1}_{+}(q^{-{}^{\mathrm{t}}\beta^{(1)}-\rho}) (\sigma_1 Q_1 \sigma_2)^{L_0} \cdots \\
&\quad \times \Gamma^{\sigma_{N-1}}_{-}(q^{-\beta^{(N-1)}-\rho}) \Gamma^{\sigma_{N-1}}_{+}(q^{-{}^{\mathrm{t}}\beta^{(N-1)}-\rho}) (\sigma_{N-1} Q_{N-1} \sigma_N)^{L_0} \\
&\quad \times \Gamma^{\sigma_N}_{-}(q^{-\beta^{(N)}-\rho}) \Gamma^{\sigma_N}_{+}(q^{-{}^{\mathrm{t}}\beta^{(N)}-\rho}) | \alpha_N \rangle
\end{aligned}
$$

where $\Gamma^{\sigma}_{\pm}$ denote $\Gamma_{\pm}$ if $\sigma = +1$ and $\Gamma'_{\pm}$ if $\sigma = -1$ (Eguchi and Kanno 2003, Bryan and Young 2008 for special cases; Nagao 2009 and Sułkowski 2009 for general cases)

Wave functions

Wave functions are defined as

$$\Psi_n(x) = \sum_{k=0}^{\infty} \frac{Z_{n,(1^k)}}{Z_{n,\emptyset}} x^k, \quad \tilde{\Psi}_n(x) = \sum_{k=0}^{\infty} \frac{Z_{n,(k)}}{Z_{n,\emptyset}} x^k$$

for $n = 0, 1, \ldots, N, N+1$, where

$$Z_{0,\lambda} = Z^{\lambda\emptyset}_{\emptyset\cdots\emptyset},\ Z_{n,\lambda} = Z^{\emptyset\emptyset}_{\cdots\emptyset\lambda\emptyset\cdots}\ (1 \le n \le N),\ Z_{N+1,\lambda} = Z^{\emptyset\lambda}_{\emptyset\cdots\emptyset}$$

Remark: $Z_{n,\lambda}$'s are the coefficients of Schur function expansion of a KP tau function τ_n. In this sense, these wave functions are Baker-Akhiezer functions (at the initial time $\boldsymbol{t} = \boldsymbol{0}$) that correspond to free fermion fields $\psi(-x), \psi^*(x)$.

Ψ_n and $\tilde{\Psi}_n$ for $n = 1, \ldots, N$ are q-hypergeometric series

$$\Psi_n(x) = 1 + \sum_{k=1}^{\infty} \frac{C_n(1)C_n(q)\cdots C_n(q^{k-1})}{B_n(1)B_n(q)\cdots B_n(q^{k-1})(1-q)\cdots(1-q^k)} q^{k/2} x^k$$

where $B_n(y)$ and $C_n(y)$ are Laurent polynomials in y:

$$B_n(y) = \prod_{m<n,\, \sigma_m\sigma_n>0} (1 - Q_{mn} y^{\sigma_n}) \times \prod_{m>n,\, \sigma_m\sigma_n>0} (1 - Q_{nm} y^{-\sigma_n}),$$

$$C_n(y) = \prod_{m<n,\, \sigma_m\sigma_n<0} (1 - Q_{mn} y^{\sigma_n}) \times \prod_{m>n,\, \sigma_m\sigma_n<0} (1 - Q_{nm} y^{-\sigma_n})$$

Remark: When $N = 1$, Ψ reduces to a quantum dilog:

$$\Psi(x) = 1 + \sum_{k=1}^{\infty} \frac{q^{k/2} x^k}{(1-q)\cdots(1-q^k)} = \prod_{i=1}^{\infty} (1 - q^{i-1/2} x)^{-1}$$

$\Psi_1, \ldots, \Psi_N$ satisfy q-difference equation

$$\Psi_n(x) - \Psi_n(qx) = q^{1/2} x \frac{C_n(q^{x\partial_x})}{B_n(q^{x\partial_x})} \Psi_n(x)$$

or, equivalently,

$$B_n(q^{-1} q^{x\partial_x})(1 - q^{x\partial_x})\Psi_n(x) = q^{1/2} x C_n(q^{x\partial_x})\Psi_n(x)$$

(Kashani-Poor 2006, Hyun and Yi 2006, Gukov and Sułkowski 2011 for the resolved conifold).

Remark: These equations are also studied in the context of the AGT correspondence (Kozçaz, Pasquetti and Wyllard 2010, Taki 2010) and the vortex partition function (Bonelli, Tanzini and Zhao 2011).

Classical limit

As $q \to 1$ ($\hat{y} = q^{x\partial_x} \to y$), the q-difference equation

$$B_n(q^{-1}q^{x\partial_x})(1 - q^{x\partial_x})\Psi_n(x) = q^{1/2}xC_n(q^{x\partial_x})\Psi_n(x)$$

reduces to the algebraic equation

$$B_n(y)(1 - y) = xC_n(y)$$

of the the mirror curve. In this sense, the q-difference equation may be thought of as quantization of the mirror curve.

Remark: The Newton polygon of $B_n(y)(1 - y) - xC_n(y)$ can be mapped to the outline of the toric diagram by an $SL(2, \mathbb{Z})$ transformation.

How the equations for different n's are related

$$x = (1-y)\frac{B_n(y)}{C_n(y)} \xrightarrow{(\heartsuit)} \tilde{x} = (1-\tilde{y})\frac{B_{n+1}(\tilde{y})}{C_{n+1}(\tilde{y})}$$

$$(\heartsuit) \quad y = Q_n^{\sigma_n} \tilde{y}^{\sigma_n \sigma_{n+1}}, \; x = g(y,\tilde{y})\tilde{x}^{\sigma_n \sigma_{n+1}}$$

where

$$g(y,\tilde{y}) = \begin{cases} -\tilde{y}^{-1} & \text{if } \sigma_n = +1,\, \sigma_{n+1} = +1, \\ 1 & \text{if } \sigma_n = +1,\, \sigma_{n+1} = -1, \\ y\tilde{y} & \text{if } \sigma_n = -1,\, \sigma_{n+1} = +1, \\ -y & \text{if } \sigma_n = -1,\, \sigma_{n+1} = -1 \end{cases}$$

Birational map $(\heartsuit)$ preserves the symplectic structure:

$$d\log x \wedge d\log y = d\log \tilde{x} \wedge d\log \tilde{y}$$

What about Ψ_0 and Ψ_{N+1} ?

They are related to quantum dilogs. E.g., if $\sigma_1 = +1$, $\Psi_0(x)$ is a product of quantum dilogs and satisfies the equation

$$\Psi_0(qx) = (1 - q^{1/2}x)^{-1} \prod_{n=2}^{N} (1 - Q_{n-1}q^{1/2}x)^{-\sigma_n} \Psi_0(x).$$

Its classical limit is the algebraic equation

$$y = (1 - x)^{-1} \prod_{n=2}^{N} (1 - Q_{n-1}x)^{-\sigma_n}.$$

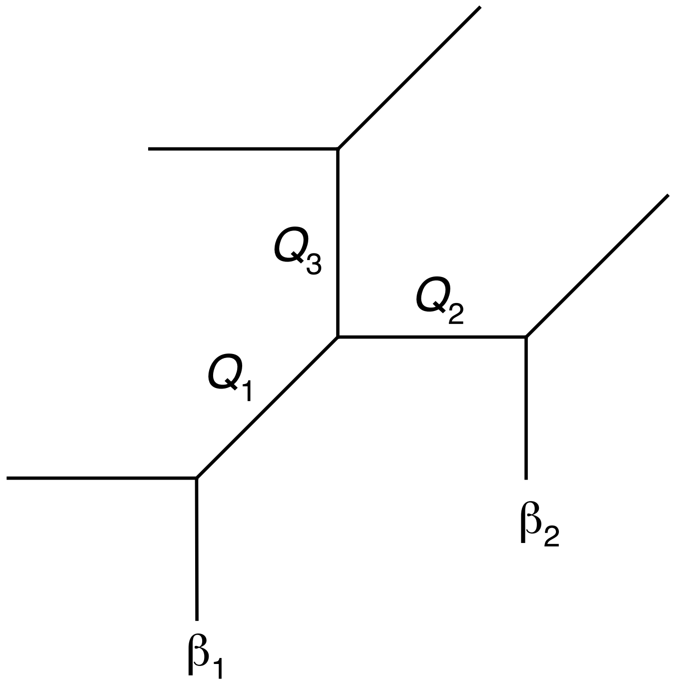
This equation can be transformed to the equation $\tilde{x} = (1 - \tilde{y})B_1(\tilde{y})/C_1(\tilde{y})$ by a birational symplectic map $(x, y) \mapsto (\tilde{x}, \tilde{y})$: $d\log y \wedge d\log x = d\log\tilde{x} \wedge d\log\tilde{y}$.

3. Closed topological vertex

Setup

(a) Three internal lines are assigned with Kähler parameters Q_1, Q_2, Q_3

(b) Two vertical external lines are assigned with partitions β_1, β_2. All other external lines are given $\emptyset$.

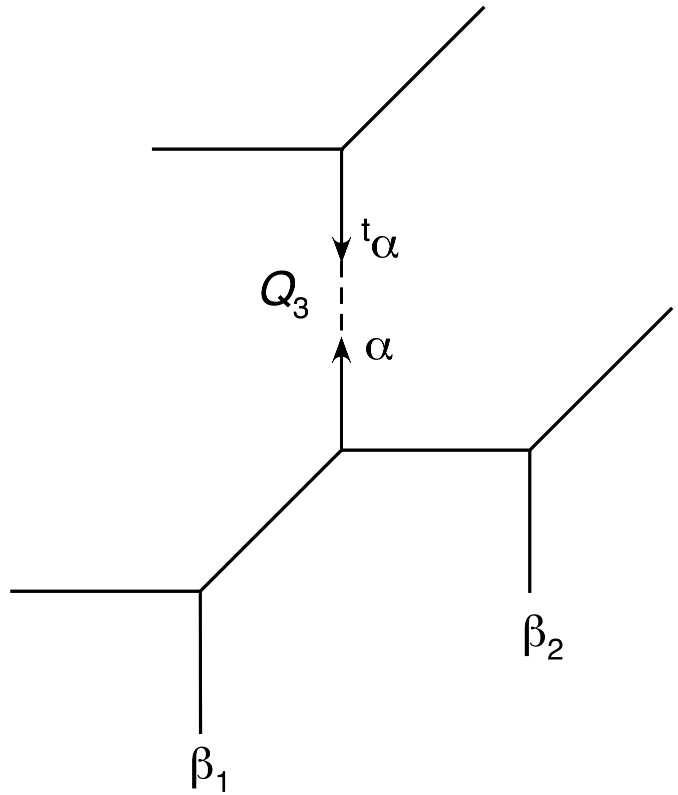


Let $Z^{\text{ctv}}_{\beta_1\beta_2}$ denote the open string amplitude in this setting.

Method of computation of $Z^{\mathrm{ctv}}_{\beta_1\beta_2}$

- Reconstruct the amplitude $Z^{\mathrm{ctv}}_{\beta_1\beta_2}$ by gluing a single vertex $C_{{}^{\mathrm{t}}\alpha\emptyset\emptyset}$ to the amplitude of an on-strip geometry.

- Borrow tools from our previous study on integrable structures of the melting crystal models (5D $U(1)$ instanton sum and its variants) (Nakatsu and K.T, since 2007) to compute the sum over $\alpha \in \mathcal{P}$.



Gluing a vertex (top) to an on-strip geometry (bottom)

Result of computation of $Z^{\mathrm{ctv}}_{\beta_1\beta_2}$

$$
\begin{aligned}
Z^{\mathrm{ctv}}_{\beta_1\beta_2} = q^{\kappa(\beta_2)/2} \prod_{i,j=1}^{\infty} & (1 - Q_1 Q_2 q^{-\beta_{1i} - {}^{\mathrm{t}}\beta_{2j} + i + j - 1})^{-1} \\
& \times \langle {}^{\mathrm{t}}\beta_1 | \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) (-Q_1)^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) (-Q_3)^{L_0} \\
& \times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) (-Q_2)^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) | {}^{\mathrm{t}}\beta_2 \rangle.
\end{aligned}
$$

- The main part $Y_{\beta_1\beta_2} := \langle {}^{\mathrm{t}}\beta_1 | \cdots | {}^{\mathrm{t}}\beta_2 \rangle$ is essentially the open string amplitude of yet another on-strip geometry. This strange coincidence is a key to derive a q-difference equation.

- Letting $\beta_1 = \beta_2 = \emptyset$, this expression reduces to the known result of the closed string amplitudes (Bryan and Karp 2003, Karp, Liu and Mariño 2005).

Wave functions

Wave functions are defined as

$$\Psi(x) = \sum_{k=0}^{\infty} \frac{Z^{\mathrm{ctv}}_{(1^k)\emptyset}}{Z^{\mathrm{ctv}}_{\emptyset\emptyset}} x^k, \quad \tilde{\Psi}(x) = \sum_{k=0}^{\infty} \frac{Z^{\mathrm{ctv}}_{(k)\emptyset}}{Z^{\mathrm{ctv}}_{\emptyset\emptyset}} x^k$$

along with the auxiliary wave functions

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{Y_{(1^k)\emptyset}}{Y_{\emptyset\emptyset}} x^k, \quad \tilde{\Phi}(x) = \sum_{k=0}^{\infty} \frac{Y_{(k)\emptyset}}{Y_{\emptyset\emptyset}} x^k$$

obtained from the main part $Y_{\beta_1\beta_2} = \langle {}^{\mathrm{t}}\beta_1 | \cdots | {}^{\mathrm{t}}\beta_2 \rangle$ of $Z^{\mathrm{ctv}}_{\beta_1\beta_2}$.

Wave functions (cont'd)

- The coefficients of $\Psi(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\Phi(x) = \sum_{k=0}^{\infty} b_k x^k$, $a_0 = b_0 = 1$, are related as

$$a_k = b_k \prod_{i=1}^{k} (1 - Q_1 Q_2 q^{i-1})^{-1}$$

- $\Phi(x)$ is build from quantum dilogs, and satisfies the q-difference equation

$$\Phi(qx) = \frac{(1 - q^{1/2}x)(1 - Q_1 Q_3 q^{1/2} x)}{(1 - Q_1 q^{1/2} x)(1 - Q_1 Q_2 Q_3 q^{1/2} x)} \Phi(x).$$

Transforming q-difference equation

The q-difference equation

$$
\begin{aligned}
&(1 - Q_1 q^{1/2} x)(1 - Q_1 Q_2 Q_3 q^{1/2} x)\Phi(qx) \\
&= (1 - q^{1/2} x)(1 - Q_1 Q_3 q^{1/2} x)\Phi(x)
\end{aligned}
$$

for $\Phi(x)$ is transformed to the q-difference equation

$$
\begin{aligned}
&(1 - Q_1 Q_2 q^{-2} q^{x\partial_x} - Q_1 q^{1/2} x)(1 - Q_1 Q_2 q^{-1} q^{x\partial_x} - Q_1 Q_2 Q_3 q^{1/2} x)\Psi(qx) \\
&= (1 - Q_1 Q_2 q^{-2} q^{x\partial_x} - q^{1/2} x)(1 - Q_1 Q_2 q^{-1} q^{x\partial_x} - Q_1 Q_3 q^{1/2} x)\Psi(x)
\end{aligned}
$$

for $\Psi(x)$. Let us rewrite this equation as

$$
H(x, q^{x\partial_x})\Psi(x) = 0
$$

and examine the q-difference operator $H(x, q^{x\partial_x})$.

Reducing q-difference equation to final form

The operator $H(x, q^{x\partial_x})$ can be factorized as

$$H(x, q^{x\partial_x}) = (1 - Q_1 Q_2 q^{-2} q^{x\partial_x}) K(x, q^{x\partial_x})$$

where

$$K(x, q^{x\partial_x}) = (1 - Q_1 Q_2 q^{-1} q^{x\partial_x})(1 - q^{x\partial_x}) - (1 + Q_1 Q_3) q^{1/2} x + Q_1 (1 + Q_2 Q_3) q^{1/2} x q^{x\partial_x} + Q_1 Q_3 q x^2.$$

Since the prefactor $1 - Q_1 Q_2 q^{-2} q^{x\partial_x}$ is invertible on the space of power series of x, the equation $H(x, q^{x\partial_x})\Psi(x) = 0$ reduces to

$$K(x, q^{x\partial_x})\Psi(x) = 0$$

This is the final form of our quantum mirror curve.

Classical limit

As $q \to 1$, the q-difference operator

$$K(x, q^{x\partial_x}) = (1 - Q_1 Q_2 q^{-1} q^{x\partial_x})(1 - q^{x\partial_x}) - (1 + Q_1 Q_3) q^{1/2} x + Q_1 (1 + Q_2 Q_3) q^{1/2} x q^{x\partial_x} + Q_1 Q_3 q x^2$$

turns into the polynomial

$$K_{\mathrm{cl}}(x, y) = (1 - Q_1 Q_2 y)(1 - y) - (1 + Q_1 Q_3) x + Q_1 (1 + Q_2 Q_3) x y + Q_1 Q_3 x^2$$

Its Newton polygon has the same shape as the triangular outline of the toric diagram.

What about wave functions obtained from $\beta_2 = (k), (1^k)$?

The q-difference equations become slightly more complicated because of the framing factor $q^{\kappa(\beta_2)/2}$.

What about putting β_1 and β_2 on other legs ?

The open string amplitude can be expressed in a similar form. However $q^{K/2}$'s remain in the operator product, and they prevent us from doing explicit computation. A similar difficulty takes place when one attempts to prolong the branches of the closed topological vertex.

What about web diagrams with cycle(s) ?

It is an ultimate goal of our project, but we have currently no idea.