Modified melting crystal model and Ablowitz-Ladik hierarchy

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Reference

• K.T., Integrable structure of modified melting crystal model, arXiv:1208.4497 [math-ph]

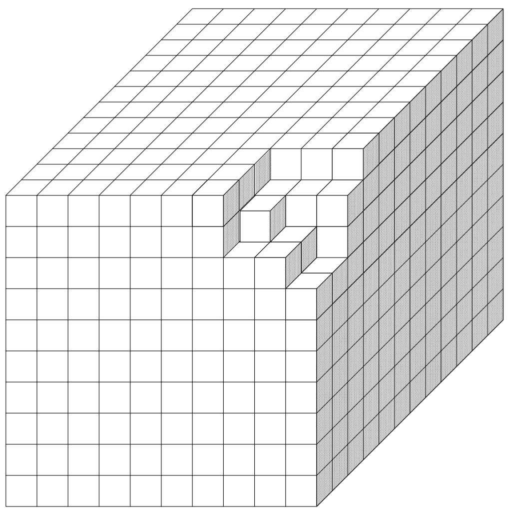
• K.T., Modified melting crystal model and Ablowitz-Ladik hierarchy, arXiv:1302.6129 [math-ph]

Contents

1. Melting crystal model

Crystal corner and 3D Young diagram

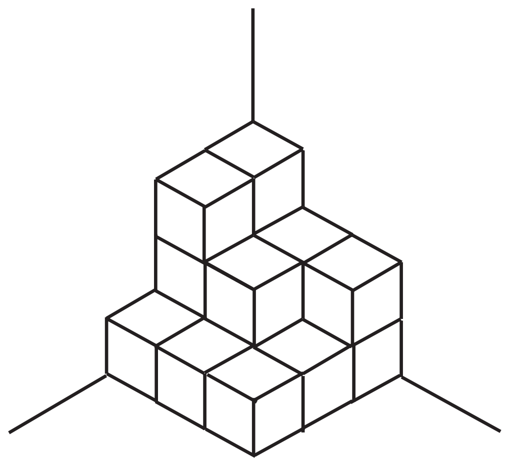
The melting crystal model is a statistical model of a crystal corner in the first octant of the xyz space. The crystal consists of unit cubes, and the complement in the octant is identified with a 3D Young diagram.



Melting crystal corner

complement

$\longleftrightarrow$



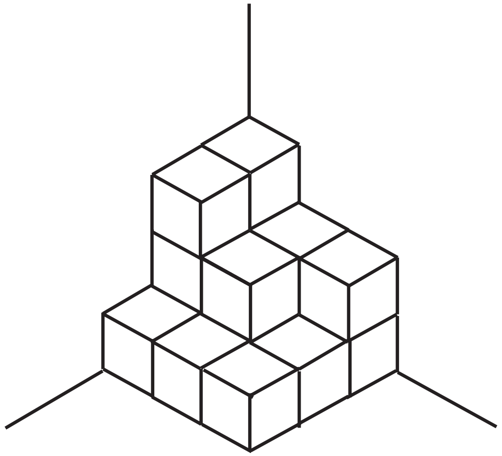
3D Young diagram

Plane partitions and 3D Young diagrams

• Plane partition = decreasing 2D array of non-negative integers

$$\pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{matrix} \pi_{ij} & \geq & \pi_{i,j+1} \\ \text{\rotatebox{90}{$\leq$}} & & \\ \pi_{i+1,j} & & \end{matrix}$$

• 3D Young diagrams can be labelled by plane partitions. π_{ij} is the height of the stack of cubes on the xy plane.



$$\pi = \begin{pmatrix} 3 & 2 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Partition function

The Partition function of this model is the sum

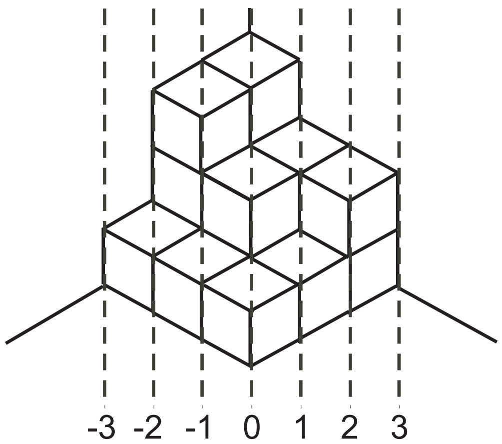
$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|}, \quad |\pi| = \sum_{i,j=1}^{\infty} \pi_{ij}$$

of the Boltzmann weight $q^{|\pi|}$ $(0 < q < 1)$ over the set $\mathcal{PP}$ of all plane partitions. $|\pi|$ is the volume of the 3D Young diagram.

Diagonal slicing

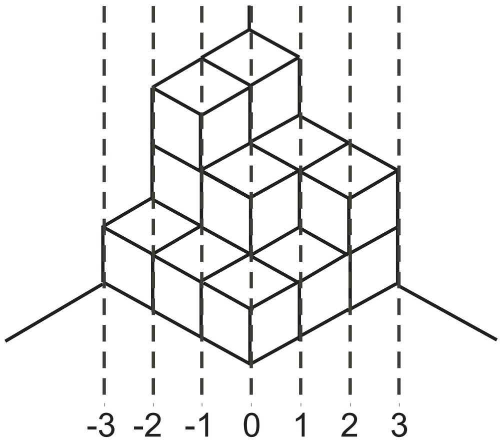
The partition function can be calculated by the method of diagonal slicing (A. Okounkov and N. Reshetikhin, J. Amer. Math. Soc. **16**, (2003), 581–603, arXiv:math.CO/0107056).

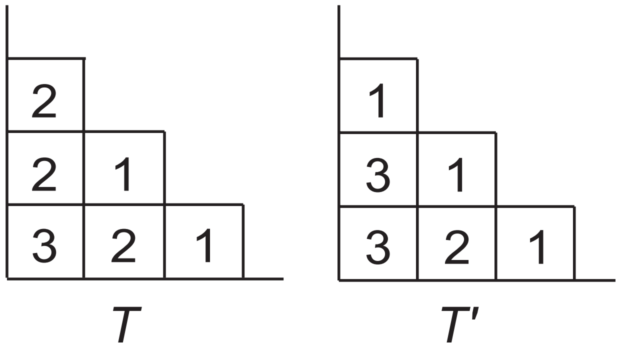
$$m\text{-th diagonal slice } \pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0, \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0 \end{cases}$$



Mapping $\pi \mapsto (\lambda, T, T')$

There is a one-to-one correspondence between plane partitions $\pi \in \mathcal{PP}$ and triples (λ, T, T') of the principal slice $\lambda = \pi(0)$ and two semi-standard tableaux T, T' of shape λ.





$\lambda = (3, 2, 1)$

Converting Z to sum over (λ, T, T')

The Boltzmann weight $q^{|\pi|}$ is factorized as

$$q^{|\pi|} = q^T q^{T'},$$

$$q^T = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(-m)/\pi(-m-1)|},$$

$$q^{T'} = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(m)/\pi(m+1)|}$$

The partition function can be thereby decomposed to sums over $\lambda = \pi(0) \in \mathcal{P}$ and $T, T' \in \mathrm{SSTab}(\lambda)$

$$Z = \sum_{\lambda \in \mathcal{P}} \left(\sum_{T \in \mathrm{SSTab}(\lambda)} q^T \right) \left(\sum_{T' \in \mathrm{SSTab}(\lambda)} q^{T'} \right).$$

Partial sums over semi-standard tableaux

Partial sums over T and T' turn into the special values

$$\sum_{T \in \mathrm{SSTab}(\lambda)} q^{T} = \sum_{T' \in \mathrm{SSTab}(\lambda)} q^{T'} = s_\lambda(q^{-\rho})$$

of the Schur functions $s_\lambda(x_1, x_2, \cdots)$ of infinite variables at

$$q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{n+1/2}, \ldots).$$

Final answer

By the Cauchy identity

$$\sum_{\lambda\in\mathcal{P}} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1},$$

the partition function can be cast into the final form

$$Z = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1})^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$

This is the so called MacMahon function.

2. Integrable structure in deformed models

Undeformed models

$$Z = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} = \prod_{n=1}^{\infty} (1 - Qq^n)^{-n},$$

$$Z' = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho}) s_{{}^{\mathrm{t}}\lambda}(q^{-\rho}) Q^{|\lambda|} = \prod_{n=1}^{\infty} (1 + Qq^n)^n$$

1. Z is a slight modification of the foregoing melting crystal model.
2. Z' is a modification replacing $s_\lambda(q^{-\rho})^2 \longrightarrow s_\lambda(q^{-\rho}) s_{{}^{\mathrm{t}}\lambda}(q^{-\rho})$. ${}^{\mathrm{t}}\lambda$ denotes the conjugate partion of λ. This is no more a statistical model of 3D Young diagrams. (3D Young diagrams are cut in half and glued together in a twisted way.)

Deformation by external potentials

$$\Phi(\lambda, s, \boldsymbol{t}) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s),$$

$$\Phi(\lambda, s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s) + \sum_{k=1}^{\infty} \bar{t}_k \Phi_{-k}(\lambda, s)$$

$$Z(s, \boldsymbol{t}) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda| + s(s+1)/2} e^{\Phi(\lambda, s, \boldsymbol{t})},$$

$$Z'(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho}) s_{{}^{\mathrm{t}}\lambda}(q^{-\rho}) Q^{|\lambda| + s(s+1)/2} e^{\Phi(\lambda, s, \boldsymbol{t}, \bar{\boldsymbol{t}})}$$

$\boldsymbol{t} = (t_1, t_2, \ldots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$ play the role of "time variables", s a lattice coordinate in an integrable lattice system.

External potentials

Heuristic definition

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} q^{k(\lambda_i + s - i + 1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}$$

True definition by recombination of terms

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} (q^{k(\lambda_i + s - i + 1)} - q^{k(s-i+1)}) + \frac{1 - q^{ks}}{1 - q^k} q^k$$

This is related to “normal ordering” of operators in a 2D free fermion system.

Previous result: summary

(K.T. and T. Nakatsu, Commun. Math. Phys. **285** (2009), 445–468, arXiv:0710.5339 [hep-th])

$Z(s, \boldsymbol{t})$ is related to a tau function $\tau(s, \boldsymbol{t})$ of the 1D Toda hierarchy as

$$Z(s, \boldsymbol{t}) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k}\right) q^{-s(s+1)(2s+1)/6} \tau(s, \iota(\boldsymbol{t})),$$

where $\iota(\boldsymbol{t})$ denotes the alternating inversion

$$\iota(\boldsymbol{t}) = (-t_1, t_2, -t_3, \ldots, (-1)^k t_k, \ldots)$$

of $\boldsymbol{t} = (t_1, t_2, \ldots)$. $\tau(s, \boldsymbol{t})$ has a fermionic representation.

New result: summary

(K.T., arXiv:1208.4497 [math-ph], arXiv:1302.6129 [math-ph])

$Z'(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ is related to a tau function $\tau'(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ of the 2D Toda hierarchy. $\tau'(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ has a fermionic representation. Moreover, this solution of the 2D Toda hierarchy is actually a solution of the Ablowitz-Ladik (or relativistic Toda) hierarchy.

Remark: $Z'(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ coincides with the amplitude of topological string theory (equivalently, a generating function of Gromov-Witten invariants) on the resolved conifold. Brini conjectured that this generating function is related to the Ablowitz-Ladik hierarchy, and confirmed the conjecture for genus ≤ 1 of the genus expansion (A. Brini, Commun. Math. Phys. **313** (2012), 571–605, arXiv:1002.0582 [math-ph]). Our result is an answer to Brini's conjecture from a different approach.

3. Partition functions in fermionic formalism

Fermions

- 2D fermion fields

$$\psi(z) = \sum_{n\in\mathbf{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n\in\mathbf{Z}} \psi_n^* z^{-n}.$$

- Creation-annihilation operators

$$\psi_m\psi_n^* + \psi_n^*\psi_n = \delta_{m+n,0}, \quad \psi_m\psi_n + \psi_n\psi_m = \psi_m^*\psi_n^* + \psi_n^*\psi_m^* = 0$$

- Ground states of charge s in the Fock space

$$\langle s| = \langle -\infty| \cdots \psi_{s-1}^*\psi_s^*, \quad |s\rangle = \psi_{-s}\psi_{-s+1}\cdots|-\infty\rangle$$

- Excited states are labelled by partitions $\lambda \in \mathcal{P}$ as $\langle\lambda, s|$ and $|\lambda, s\rangle$.

Fermionic representation of partition functions

$$Z(s,\boldsymbol{t}) = \langle s|\Gamma_+(q^{-\rho})Q^{L_0}e^{H(\boldsymbol{t})}\Gamma_-(q^{-\rho})|s\rangle,$$
$$Z'(s,\boldsymbol{t},\bar{\boldsymbol{t}}) = \langle s|\Gamma_+(q^{-\rho})Q^{L_0}e^{H(\boldsymbol{t},\bar{\boldsymbol{t}})}\Gamma'_-(q^{-\rho})|s\rangle$$

- L_0 and H_k are fermion bilinears:

$$L_0 = \sum_{n\in\mathbf{Z}} n{:}\psi_{-n}\psi_n^*{:}, \quad H_k = \sum_{n\in\mathbf{Z}} q^{kn}{:}\psi_{-n}\psi_n^*{:}.$$

- $H(\boldsymbol{t})$ and $H(\boldsymbol{t},\bar{\boldsymbol{t}})$ are linear combinations of H_k's:

$$H(\boldsymbol{t}) = \sum_{k=1}^{\infty} t_k H_k, \quad H(\boldsymbol{t},\bar{\boldsymbol{t}}) = \sum_{k=1}^{\infty} t_k H_k + \sum_{k=1}^{\infty} \bar{t}_k H_{-k},$$

• $\Gamma_\pm(q^{-\rho})$ and $\Gamma'_\pm(q^{-\rho})$ are infinite products

$$\Gamma_\pm(q^{-\rho}) = \prod_{i=1}^{\infty} \Gamma_\pm(q^{i-1/2}), \quad \Gamma'_\pm(q^{-\rho}) = \prod_{i=1}^{\infty} \Gamma'_\pm(q^{i-1/2})$$

of the vertex operators (Okounkov & Reshetikhin, Bryan & Young)

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right), \quad \Gamma'_\pm(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k}\right)$$

specialized to $z = q^{i-1/2}$, $i = 1, 2, \ldots$. J_k's are the usual basis of the $U(1)$ current algebra:

$$J_k = \sum_{n \in \mathbf{Z}} {:}\psi_{-n}\psi^*_{n+k}{:}.$$

Previous result (K.T. & Nakatsu, loc. cit.)

The partition function $Z(s,\boldsymbol{t})$ is related to a tau function $\tau(x,\boldsymbol{t})$ of the 1D Toda hierarchy as

$$Z(s,\boldsymbol{t}) = \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right) q^{-s(s+1)(2s+1)/6}\tau(s,\iota(\boldsymbol{t})).$$

The tau function $\tau(s,\boldsymbol{t})$ is defined by the fermionic formula

$$\tau(s,\boldsymbol{t}) = \langle s|\exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g|s\rangle = \langle s|g\exp\left(\sum_{k=1}^{\infty} t_k J_{-k}\right)|s\rangle,$$

where

$$g = q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{W_0/2}.$$

W_0 is the fermion bilinear

$$W_0 = \sum_{n \in \mathbf{Z}} n^2 {:}\psi_{-n}\psi_n^*{:}.$$

g satisfies the algebraic relations

$$J_k g = g J_{-k}, \quad k = 1, 2, \ldots.$$

This implies that the associated tau function

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \langle s| \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}\right) |s\rangle$$

of the 2D Toda hierarchy depends on $\boldsymbol{t}$ and $\bar{\boldsymbol{t}}$ through $\boldsymbol{t} - \bar{\boldsymbol{t}}$:

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \tau(s, \boldsymbol{t} - \bar{\boldsymbol{t}}).$$

New result (K.T., arXiv:1208.4497 [math-ph])

The partition function is related to a tau function $\tau'(s,\boldsymbol{t},\bar{\boldsymbol{t}})$ of the 2D Toda hierarchy as

$$Z'(s,\boldsymbol{t},\bar{\boldsymbol{t}}) = \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k - \bar{t}_k}{1-q^k}\right) \tau'(s,\iota(\boldsymbol{t}),-\bar{\boldsymbol{t}}).$$

The tau function $\tau'(s,\boldsymbol{t},\bar{\boldsymbol{t}})$ is defined by the fermionic formula

$$\tau'(s,\boldsymbol{t},\bar{\boldsymbol{t}}) = \langle s|\exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g' \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}\right)|s\rangle,$$

where

$$g' = q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2}.$$

Technical clue

"Shift symmetries" (K.T. & Nakatsu, loc. cit.) in a quantum torus algebra of the fermion bilinears

$$V_m^{(k)} = q^{-km/2} \sum_{n\in\mathbf{Z}} q^{kn} {:}\psi_{m-n}\psi_n^*{:},$$

$$H_k = V_0^{(k)}, \quad J_m = V_m^{(0)}.$$

Shift symmetries imply algebraic relations among $H_k, J_{\pm k}$:

$$\Gamma_+(q^{-\rho})H_k\Gamma_+(q^{-\rho})^{-1} = (-1)^k\Gamma_-(q^{-\rho})^{-1}q^{-W_0/2}J_kq^{W_0/2}\Gamma_-(q^{-\rho}) + \frac{q^k}{1-q^k},$$

$$\Gamma'_-(q^{-\rho})^{-1}H_{-k}\Gamma'_-(q^{-\rho}) = \Gamma'_+(q^{-\rho})q^{-W_0/2}J_{-k}q^{W_0/2}\Gamma'_+(q^{-\rho})^{-1} - \frac{1}{1-q^k}.$$

They are used to convert $Z(s,\boldsymbol{t})$ and $Z'(s,\boldsymbol{t},\bar{\boldsymbol{t}})$ to the tau functions.

4. Integrable structure in Lax formalism

Lax formalism of 2D Toda hierarchy

- Lax operators of the 2D Toda hierarchy

$$L = e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + \cdots ,$$

$$\bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \cdots$$

- Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}],$$

$$\frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}]$$

where

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{<0}.$$

Reduction to 1D Toda hierarchy

Reduction to the 1D Toda hierarchy is achieved by the condition

$$L = \bar{L}^{-1}.$$

The reduced Lax operator $\mathfrak{L} = L = \bar{L}^{-1}$ has the familiar form

$$\mathfrak{L} = e^{\partial_s} + b + ce^{-\partial_s}$$

and satisfies the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial t_k} = [B_k, \mathfrak{L}] = -[\bar{B}_k, \mathfrak{L}]$$

for the single set $\boldsymbol{t} = (t_1, t_2, \cdots)$ of time variables.

Reduction to Ablowitz-Ladik hierarchy

Reduction to the Ablowitz-Ladik (equivalently, relativistic Toda) hierarchy is achieved by assuming the factorized form

$$L = BC^{-1}, \quad \bar{L}^{-1} = -CB^{-1},$$
$$B = e^{\partial_s} - b, \quad C = 1 + ce^{-\partial_s}$$

of the Lax operators. This condition is preserved by time evolutions. (A. Brini, G. Carlet and P. Rossi, Physica **D241** (2012), 2156–2162, arXiv:1002.0582 [math-ph])

Remark: $L \neq -\bar{L}$. C^{-1} in L is an operator of the form $1 + c_1 e^{-\partial_s} + \cdots$. Formally, $\bar{L} = -BC^{-1}$, but C^{-1} in $\bar{L}$ is a different operator of the form $e^{\partial_s} \cdot c + \bar{c}_1 e^{2\partial_s} + \cdots$.

Result (K.T., arXiv:1302.6129 [math-ph])

The Lax operators $L, \bar{L}$ associated with $\tau'(s,\boldsymbol{t},\bar{\boldsymbol{t}})$ do have the factorized form of Brini et al., hence give a solution of the Ablowitz-Ladik hierarchy.

Technical clue

1. Correspondence between fermion bilinears and $\mathbf{Z}\times\mathbf{Z}$ matrices

$$X = \sum_{i,j\in\mathbf{Z}} x_{ij}E_{ij} \longleftrightarrow \hat{X} = \sum_{i,j\in\mathbf{Z}} x_{ij}{:}\psi_{-i}\psi_j^*{:}$$

2. Matrix factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k\Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty}\bar{t}_k\Lambda^{-k}\right) = W^{-1}\bar{W}$$

- Matrix representations of important fermion bilinears read

$$L_0 = \Delta, \quad W_0 = \Delta^2, \quad H_k = q^{k\Delta}, \quad J_k = \Lambda^k,$$
$$\Gamma_\pm(z) = (1 - z\Lambda^{\pm 1})^{-1}, \quad \Gamma'_\pm(z) = 1 + z\Lambda^{\pm 1}$$

where

$$\Delta = \sum_{i\in\mathbf{Z}} iE_{ii}, \quad \Lambda = \sum_{i\in\mathbf{Z}} E_{i,i+1}.$$

In particular, the vertex operators turn into matrix-valued quantum dilogarithm:

$$\Gamma_\pm(q^{-\rho}) = \prod_{i=1}^{\infty}(1 - q^{i-1/2}\Lambda^{\pm 1})^{-1}, \quad \Gamma'_\pm(q^{-\rho}) = \prod_{i=1}^{\infty}(1 + q^{i-1/2}\Lambda^{\pm 1}).$$

- The factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1}\bar{W}$$

captures all solutions of the 2D Toda hierarchy. W is a "monic" lower triangular matrix, and $\bar{W}$ is an upper triangular matrix with nonzero diagonal elements. The Lax operators are obtained from W and $\bar{W}$ as $L = W\Lambda W^{-1}$ and $\bar{L} = \bar{W}\Lambda\bar{W}^{-1}$.

- In the case of $\tau'(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$, the matrix U reads

$$U = q^{\Delta^2/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{\Delta}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-\Delta^2/2}.$$

Fortunately, one can solve this problem explicitly at the "initial point" $\boldsymbol{t} = \bar{\boldsymbol{t}} = \boldsymbol{0}$. This is enough to show that L and $\bar{L}$ satisfy the factorization ansatz of Brini et al.