Modified melting crystal model and Ablowitz-Ladik hierarchy

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Gallipoli, Italy, June 24, 2013

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1. Melting crystal model

Crystal corner and 3D Young diagram

The melting crystal model is a statistical model of a crystal corner in the first octant of the $xyz$ space. The crystal consists of unit cubes, and the complement in the octant is identified with a 3D Young diagram.
Plane partitions and 3D Young diagrams

- Plane partition = decreasing 2D array of non-negative integers

\[
\pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix}
\pi_{11} & \pi_{12} & \cdots \\
\pi_{21} & \pi_{22} & \cdots \\
\vdots & \vdots & \ddots \\
\end{pmatrix}, \quad \pi_{ij} \geq \pi_{i,j+1}
\]

- 3D Young diagrams can be labelled by plane partitions. \( \pi_{ij} \) is the height of the stack of cubes on the \( xy \) plane.

\[
\pi = \begin{pmatrix}
3 & 2 & 2 \\
3 & 2 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]
1. Melting crystal model

Partition function

The Partition function of this model is the sum

\[ Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|}, \quad |\pi| = \sum_{i,j=1}^{\infty} \pi_{ij} \]

of the Boltzmann weight \( q^{|\pi|} \) \((0 < q < 1)\) over the set \( \mathcal{PP} \) of all plane partitions. \(|\pi|\) is the volume of the 3D Young diagram.
1. Melting crystal model

Diagonal slicing


\[ m\text{-th diagonal slice } \pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0, \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0 \end{cases} \]
There is a one-to-one correspondence between plane partitions $\pi \in \mathcal{PP}$ and triples $(\lambda, T, T')$ of the principal slice $\lambda = \pi(0)$ and two semi-standard tableaux $T, T'$ of shape $\lambda$. 

Mapping $\pi \mapsto (\lambda, T, T')$
Converting $Z$ to sum over $(\lambda, T, T')$

The Boltzmann weight $q^{\pi}$ is factorized as
\[
q^{\pi} = q^T q^{T'},
\]
\[
q^T = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(-m)/\pi(-m-1)|},
\]
\[
q^{T'} = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(m)/\pi(m+1)|}.
\]

The partition function can be thereby decomposed to sums over $\lambda = \pi(0) \in \mathcal{P}$ and $T, T' \in \text{SSTab}(\lambda)$
\[
Z = \sum_{\lambda \in \mathcal{P}} \left( \sum_{T \in \text{SSTab}(\lambda)} q^T \right) \left( \sum_{T' \in \text{SSTab}(\lambda)} q^{T'} \right).
\]
Partial sums over semi-standard tableaux

Partial sums over $T$ and $T'$ turn into the special values

$$\sum_{T \in \text{SSTab}(\lambda)} q^T = \sum_{T' \in \text{SSTab}(\lambda)} q^{T'} = s_\lambda(q^{-\rho})$$

of the Schur functions $s_\lambda(x_1, x_2, \cdots)$ of infinite variables at

$$q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{n+1/2}, \ldots).$$
Final answer

By the Cauchy identity

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(x_1, x_2, \ldots)s_\lambda(y_1, y_2, \ldots) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1},$$

the partition function can be cast into the final form

$$Z = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1})^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$ 

This is the so called MacMahon function.
2. Integrable structure in deformed models

Undeformed models

\[ Z = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{-\rho})^2 Q^{\mid \lambda \mid} = \prod_{n=1}^{\infty} (1 - Q q^n)^{-n}, \]

\[ Z' = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{-\rho}) s_{\lambda}(q^{-\rho}) Q^{\mid \lambda \mid} = \prod_{n=1}^{\infty} (1 + Q q^n)^n \]

1. \( Z \) is a slight modification of the foregoing melting crystal model.
2. \( Z' \) is a modification replacing \( s_{\lambda}(q^{-\rho})^2 \rightarrow s_{\lambda}(q^{-\rho}) s_{\lambda}(q^{-\rho}) \). \( ^t \lambda \) denotes the conjugate partition of \( \lambda \). This is no more a statistical model of 3D Young diagrams. (3D Young diagrams are cut in half and glued together in a twisted way.)
Deformation by external potentials

\[ \Phi(\lambda, s, t) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s), \]

\[ \Phi(\lambda, s, t, \tilde{t}) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s) + \sum_{k=1}^{\infty} \tilde{t}_k \Phi_{-k}(\lambda, s) \]

\[ Z(s, t) = \sum_{\lambda \in \mathcal{P}} s_\lambda (q^{-\rho})^2 \mathcal{Q}_{[\lambda]} + s(s+1)/2 \epsilon_{s, t}, \]

\[ Z'(s, t, \tilde{t}) = \sum_{\lambda \in \mathcal{P}} s_\lambda (q^{-\rho}) s_{t, \lambda} (q^{-\rho}) \mathcal{Q}_{[\lambda]} + s(s+1)/2 \epsilon_{s, t, \tilde{t}} \]

\[ t = (t_1, t_2, \ldots) \] and \[ \tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots) \] play the role of “time variables”, s a lattice coordinate in an integrable lattice system.
2. Integrable structure in deformed models

External potentials

Heuristic definition

\[ \Phi_k(\lambda, s) = \sum_{i=1}^{\infty} q_k(\lambda_i + s - i + 1) - \sum_{i=1}^{\infty} q_k(-i + 1) \]

True definition by recombination of terms

\[ \Phi_k(\lambda, s) = \sum_{i=1}^{\infty} (q_k(\lambda_i + s - i + 1) - q_k(s - i + 1)) + \frac{1 - q^{ks}}{1 - q^k} q_k \]

This is related to “normal ordering” of operators in a 2D free fermion system.
Previous result: summary


$Z(s; t)$ is related to a tau function $\tau(s, t)$ of the 1D Toda hierarchy as

$$Z(s, t) = \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) q^{-s(s+1)(2s+1)/6} \tau(s, \iota(t)),$$

where $\iota(t)$ denotes the alternating inversion

$$\iota(t) = (-t_1, t_2, -t_3, \ldots, (-1)^k t_k, \ldots)$$

of $t = (t_1, t_2, \ldots)$. $\tau(s, t)$ has a fermionic representation.
New result: summary


\( Z'(s, t, \bar{t}) \) is related to a tau function \( \tau'(s, t, \bar{t}) \) of the 2D Toda hierarchy. \( \tau'(s, t, \bar{t}) \) has a fermionic representation. Moreover, this solution of the 2D Toda hierarchy is actually a solution of the Ablowitz-Ladik (or relativistic Toda) hierarchy.

Remark: \( Z'(s, t, \bar{t}) \) coincides with the amplitude of topological string theory (equivalently, a generating function of Gromov-Witten invariants) on the resolved conifold. Brini conjectured that this generating function is related to the Ablowitz-Ladik hierarchy, and confirmed the conjecture for genus \( \leq 1 \) of the genus expansion (A. Brini, Commun. Math. Phys. 313 (2012), 571–605, arXiv:1002.0582 [math-ph]). Our result is an answer to Brini’s conjecture from a different approach.
3. Partition functions in fermionic formalism

**Fermions**

- 2D fermion fields

\[ \psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}. \]

- Creation-annihilation operators

\[ \psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m+n,0}, \quad \psi_m \psi_n + \psi_n \psi_m = \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0 \]

- Ground states of charge \( s \) in the Fock space

\[ \langle s \rangle = \langle -\infty | \cdots \psi_{s-1}^* \psi_s^* \rangle, \quad |s\rangle = \psi_{-s} \psi_{-s+1} \cdots | -\infty \rangle \]

- Excited states are labelled by partitions \( \lambda \in \mathcal{P} \) as \( \langle \lambda, s \rangle \) and \( |\lambda, s\rangle \).
3. Partition functions in fermionic formalism

Fermionic representation of partition functions

\[ Z(s, t) = \langle s| \Gamma_+ (q^{-\rho}) Q^L_0 e^{H(t)} \Gamma_- (q^{-\rho}) |s \rangle, \]
\[ Z'(s, t, \bar{t}) = \langle s| \Gamma_+ (q^{-\rho}) Q^L_0 e^{H(t, \bar{t})} \Gamma'_- (q^{-\rho}) |s \rangle \]

- \( L_0 \) and \( H_k \) are fermion bilinears:
  \[ L_0 = \sum_{n \in \mathbb{Z}} n: \psi_n \psi^*_n :, \quad H_k = \sum_{n \in \mathbb{Z}} q^{kn}: \psi_n \psi^*_n :. \]

- \( H(t) \) and \( H(t, \bar{t}) \) are linear combinations of \( H_k \)'s:
  \[ H(t) = \sum_{k=1}^{\infty} t_k H_k, \quad H(t, \bar{t}) = \sum_{k=1}^{\infty} t_k H_k + \sum_{k=1}^{\infty} \bar{t}_k H_{-k}, \]
3. Partition functions in fermionic formalism

- $\Gamma_\pm(q^{-\rho})$ and $\Gamma'_\pm(q^{-\rho})$ are infinite products

$$
\Gamma_\pm(q^{-\rho}) = \prod_{i=1}^{\infty} \Gamma_\pm(q^{i-1/2}), \quad \Gamma'_\pm(q^{-\rho}) = \prod_{i=1}^{\infty} \Gamma'_\pm(q^{i-1/2})
$$

of the vertex operators (Okounkov & Reshetikhin, Bryan & Young)

$$
\Gamma_\pm(z) = \exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right), \quad \Gamma'_\pm(z) = \exp \left( - \sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k} \right)
$$

specialized to $z = q^{i-1/2}$, $i = 1, 2, \ldots$. $J_k$'s are the usual basis of the $U(1)$ current algebra:

$$
J_k = \sum_{n \in \mathbb{Z}} :\psi_{-n} \psi^*_{n+k}:.
$$
The partition function $Z(s, t)$ is related to a tau function $\tau(x, t)$ of the 1D Toda hierarchy as

$$Z(s, t) = \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) q^{-s(s+1)(2s+1)/6} \tau(s, \iota(t)),$$

where

$$\tau(s, t) = \langle s | \exp \left( \sum_{k=1}^{\infty} t_k J_k \right) g | s \rangle = \langle s | g \exp \left( \sum_{k=1}^{\infty} t_k J_{-k} \right) | s \rangle,$$

and

$$g = q^{W_0/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{W_0/2}.$$

Previous result (K.T. & Nakatsu, loc. cit.)
$W_0$ is the fermion bilinear

$$W_0 = \sum_{n \in \mathbb{Z}} n^2 \psi_n \psi_n^*.$$ 

$g$ satisfies the algebraic relations

$$J_k g = g J_{-k}, \quad k = 1, 2, \ldots.$$ 

This implies that the associated tau function

$$\tau(s, t, \bar{t}) = \langle s | \exp \left( \sum_{k=1}^{\infty} t_k J_k \right) g \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) | s \rangle$$

of the 2D Toda hierarchy depends on $t$ and $\bar{t}$ through $t - \bar{t}$:

$$\tau(s, t, \bar{t}) = \tau(s, t - \bar{t}).$$
3. Partition functions in fermionic formalism


The partition function is related to a tau function $\tau'(s, t, \bar{t})$ of the 2D Toda hierarchy as

$$Z'(s, t, \bar{t}) = \exp \left( \sum_{k=1}^{\infty} \frac{q^k t_k - \bar{t}_k}{1 - q^k} \right) \tau'(s, \nu(t), -\bar{t}).$$

The tau function $\tau'(s, t, \bar{t})$ is defined by the fermionic formula

$$\tau'(s, t, \bar{t}) = \langle s | \exp \left( \sum_{k=1}^{\infty} t_k J_k \right) g' \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) | s \rangle,$$

where

$$g' = q^{W_0/2} \Gamma_-(q^{-\rho}) \Gamma_++(q^{-\rho}) Q_{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) q^{-W_0/2}.$$
“Shift symmetries” (K.T. & Nakatsu, loc. cit.) in a quantum torus algebra of the fermion bilinears

\[ V^{(k)}_m = q^{-km/2} \sum_{n \in \mathbb{Z}} q^{kn} \psi_{m-n} \psi^*_{n}; \]

\[ H_k = V^{(k)}_0, \quad J_m = V^{(0)}_m. \]

Shift symmetries imply algebraic relations among \( H_k, J_{\pm k} \):

\[ \Gamma_+ (q^{-\rho}) H_k \Gamma_+ (q^{-\rho})^{-1} = (-1)^k \Gamma_- (q^{-\rho})^{-1} q^{-W_0/2} J_k q^{W_0/2} \Gamma_- (q^{-\rho}) + \frac{q^k}{1 - q^k}, \]

\[ \Gamma'_- (q^{-\rho})^{-1} H_{-k} \Gamma'_- (q^{-\rho}) = \Gamma'_+ (q^{-\rho}) q^{-W_0/2} J_{-k} q^{W_0/2} \Gamma'_+ (q^{-\rho})^{-1} - \frac{1}{1 - q^k}. \]

They are used to convert \( Z(s, t) \) and \( Z'(s, t, \bar{t}) \) to the tau functions.
4. Integrable structure in Lax formalism

Lax formalism of 2D Toda hierarchy

- Lax operators of the 2D Toda hierarchy

\[ L = e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + \cdots, \]
\[ \bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \cdots \]

- Lax equations

\[ \frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \]
\[ \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}] \]

where

\[ B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{< 0}. \]
Reduction to 1D Toda hierarchy

Reduction to the 1D Toda hierarchy is achieved by the condition

\[ L = \bar{L}^{-1}. \]

The reduced Lax operator \( \mathcal{L} = L = \bar{L}^{-1} \) has the familiar form

\[ \mathcal{L} = e^{\partial_s} + b + ce^{-\partial_s} \]

and satisfies the Lax equations

\[ \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}] = -[\bar{B}_k, \mathcal{L}] \]

for the single set \( t = (t_1, t_2, \cdots) \) of time variables.
Reduction to Ablowitz-Ladik hierarchy

Reduction to the Ablowitz-Ladik (equivalently, relativistic Toda) hierarchy is achieved by assuming the factorized form

\[ L = BC^{-1}, \quad \bar{L}^{-1} = -CB^{-1}, \]
\[ B = e^{\partial_s} - b, \quad C = 1 + ce^{-\partial_s} \]


Remark: \( L \neq -\bar{L} \). \( C^{-1} \) in \( L \) is an operator of the form \( 1 + c_1 e^{-\partial_s} + \cdots \). Formally, \( \bar{L} = -BC^{-1} \), but \( C^{-1} \) in \( \bar{L} \) is a different operator of the form \( e^{\partial_s} \cdot c + \bar{c}_1 e^{2\partial_s} + \cdots \).
4. Integrable structure in Lax formalism


The Lax operators $L, \tilde{L}$ associated with $\tau'(s, t, \bar{t})$ do have the factorized form of Brini et al., hence give a solution of the Ablowitz-Ladik hierarchy.

Technical clue

1. Correspondence between fermion bilinears and $\mathbb{Z} \times \mathbb{Z}$ matrices

$$X = \sum_{i,j \in \mathbb{Z}} x_{ij} E_{ij} \quad \longleftrightarrow \quad \hat{X} = \sum_{i,j \in \mathbb{Z}} x_{ij}:\psi_i \psi_j^*:$$

2. Matrix factorization problem

$$\exp \left( \sum_{k=1}^{\infty} t_k \Lambda^k \right) U \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k} \right) = W^{-1} \tilde{W}$$
Matrix representations of important fermion bilinears read

\[ L_0 = \Delta, \quad W_0 = \Delta^2, \quad H_k = q^k \Delta, \quad J_k = \Lambda^k, \]

\[ \Gamma_{\pm}(z) = (1 - z\Lambda^{\pm1})^{-1}, \quad \Gamma'_{\pm}(z) = 1 + z\Lambda^{\pm1} \]

where

\[ \Delta = \sum_{i \in \mathbb{Z}} iE_{ii}, \quad \Lambda = \sum_{i \in \mathbb{Z}} E_{i,i+1}. \]

In particular, the vertex operators turn into **matrix-valued quantum dilogarithm**:

\[ \Gamma_{\pm}(q^{-\rho}) = \prod_{i=1}^{\infty} (1 - q^{i-1/2} \Lambda^{\pm1})^{-1}, \quad \Gamma'_{\pm}(q^{-\rho}) = \prod_{i=1}^{\infty} (1 + q^{i-1/2} \Lambda^{\pm1}). \]
4. Integrable structure in Lax formalism

- The factorization problem

\[
\exp \left( \sum_{k=1}^{\infty} t_k \Lambda^k \right) U \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k} \right) = W^{-1} \bar{W}
\]

captures all solutions of the 2D Toda hierarchy. \( W \) is a “monic” lower triangular matrix, and \( \bar{W} \) is an upper triangular matrix with nonzero diagonal elements. The Lax operators are obtained from \( W \) and \( \bar{W} \) as \( L = W \Lambda W^{-1} \) and \( \bar{L} = \bar{W} \Lambda \bar{W}^{-1} \).

- In the case of \( \tau'(s, t, \bar{t}) \), the matrix \( U \) reads

\[
U = q^{\Delta^2/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^\Delta \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) q^{-\Delta^2/2}.
\]

Fortunately, one can solve this problem explicitly at the “initial point” \( t = \bar{t} = 0 \). This is enough to show that \( L \) and \( \bar{L} \) satisfy the factorization ansatz of Brini et al.