Melting crystal model and its 4D limit

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Contents

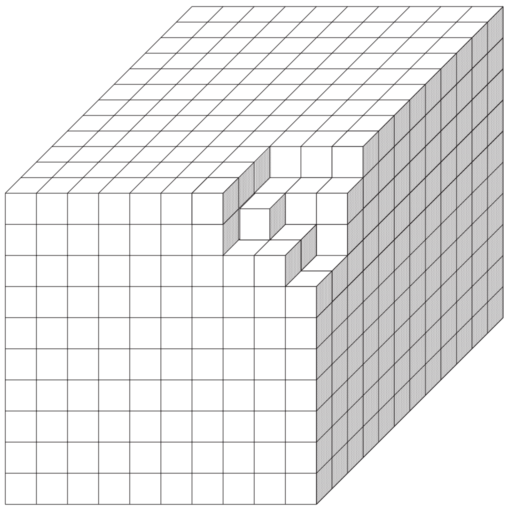
Based on

K.T., Quantum curve and 4D limit of melting crystal model, arXiv:1704.02750 [math-ph]

1. Melting crystal model

Statistical model (Okounkov, Reshetikhin & Vafa)

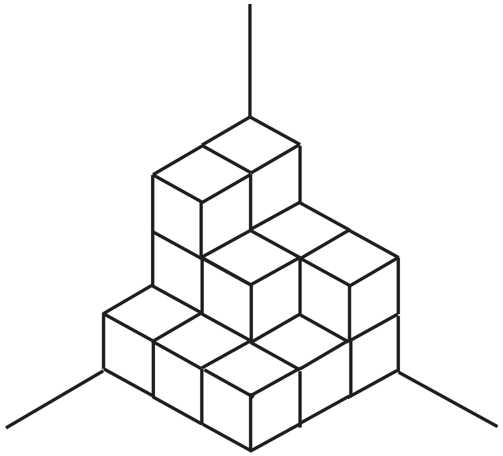
The melting crystal model is a statistical model of the crystal corner in the first octant of the xyz space. The complement of the crystal is a 3D Young diagram and represented by the plane partition $\pi = (\pi_{ij})_{i,j=1}^{\infty}$ (π_{ij} = height of (i, j)-th column).



crystal corner

complement

$\longleftrightarrow$



3D Young diagram

Undeformed partition function

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|} \qquad \left(|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij} = \text{volume of 3DYD}\right)$$

$$= \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 \quad \left(\text{sum over ordinary partitions } \lambda = (\lambda_i)_{i=1}^{\infty}\right)$$

- q is a parameter in the range $0 < q < 1$.
- $s_\lambda(q^{-\rho})$ is the special value of the infinite-variate Schur function $s_\lambda(\boldsymbol{x})$ at

$$\boldsymbol{x} = q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{i-1/2}, \ldots).$$

Hook-length formula

$$s_\lambda(q^{-\rho}) = \frac{\dim_q \lambda}{|\lambda|!} = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda}(q^{-h(i,j)/2}) - q^{h(i,j)/2})}$$

- $h(i,j)$ denotes the length of the (i,j)-th hook.
- $\kappa(\lambda) = 2\sum_{(i,j)\in\lambda}(j-i) = \sum_{i=1}^{\infty}\lambda_i(\lambda_i - 2i + 1)$.

This is a q-deformation of the building block

$$\frac{\dim \lambda}{|\lambda|!} = \frac{1}{\prod_{(i,j)\in\lambda} h(i,j)}$$

of the Poissonized Plancherel measure on $\mathcal{P}$.

Deformed partition function

$$Z(\boldsymbol{t}) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\boldsymbol{t},\lambda)}$$

Q is a new parameter, and $\boldsymbol{t} = (t_1, t_2, \ldots)$ are coupling constants of a set $\{\phi_k(\lambda)\}_{k=1}^{\infty}$ of external potentials:

$$\phi(\boldsymbol{t},\lambda) = \sum_{k=1}^{\infty} t_k \phi_k(\lambda), \quad \phi_k(\lambda) = \sum_{i=1}^{\infty} \left(q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

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> $Z(\boldsymbol{t})$ is a tau function of the KP hierarchy (Nakatsu & T. 2007).

Tools for proof: fermions and quantum torus algebra.

4D partition function

$$Z_{4\mathrm{D}}(\boldsymbol{t}) = \sum_{\lambda\in\mathcal{P}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \left(\frac{\Lambda}{\hbar}\right)^{|\lambda|} e^{\phi_{4\mathrm{D}}(\boldsymbol{t},\lambda)},$$

$$\phi_{4\mathrm{D}}(\boldsymbol{t},\lambda) = \sum_{k=1}^{\infty} t_k \phi_k^{4\mathrm{D}}(\lambda),$$

$$\phi_k^{4\mathrm{D}}(\lambda) = \sum_{i=1}^{\infty} \Big((\lambda_i - i + 1)^k - (-i+1)^k\Big)$$

$Z(\boldsymbol{t})$ and $Z_{4\mathrm{D}}(\boldsymbol{t})$ are Nekrasov's instanton partition functions.

$Z(\boldsymbol{t})$ — 5D $\mathcal{N}=1$ supersymmetric $U(1)$ Yang-Mills theory

$Z_{4\mathrm{D}}(\boldsymbol{t})$ — 4D $\mathcal{N}=2$ supersymmetric $U(1)$ Yang-Mills theory

Relation to Gromov-Witten theory

$Z_{4\mathrm{D}}(\boldsymbol{t})$ is also related to Gromov-Witten theory of $\mathbb{CP}^1$.

• A generating function of the Gromov-Witten invariants in a fermionic expression (Okounkov & Pandharipande) coincides with the 4D Nekrasov partition function (Losev, Marshakov & Nekrasov).

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Both $Z(\boldsymbol{t})$ and $Z_{4D}(\boldsymbol{t})$ can be extended to tau functions $Z(\boldsymbol{t}, s)$, $Z_{4D}(\boldsymbol{t}, s)$ of the 1D Toda hierarchy. The results of Getzler, Dubrovin-Zhang and Milanov are formulated in that form.

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1) The radius parameter R that tends to 0 in the 4D limit.

2) R-dependent parametrization $q = q(R)$, $Q = Q(R)$, $\boldsymbol{t} = \boldsymbol{t}(R, \boldsymbol{T})$ of q, Q and $\boldsymbol{t}$. $\boldsymbol{T} = (T_1, T_2, \ldots)$ are identified with the coupling constants of $\phi_k^{\mathrm{4D}}(\lambda)$'s.

3) Derivation of the quantum spectral curve of $\mathbb{CP}^1$ Gromov-Witten theory from the melting crystal model.

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3) Derivation of the quantum spectral curve of $\mathbb{CP}^1$ Gromov-Witten theory from the melting crystal model.

- To give yet another (more direct) proof of the fact that $Z_{4\mathrm{D}}(\boldsymbol{t})$ is a KP tau function. Bilinear equations play a role here.

2. Formulation of 4D limit

What is 4D limit?

Nekrasov's instanton partition functions of 5D gauge theories are derived for theories on $\mathbb{R}^4 \times S^1$. The partition functions are expected to turn into those of 4D gauge theories on $\mathbb{R}^4$ as the radius R of S^1 tends to 0.

R-dependent parametrization of q, Q

The 4D limit $Z(\mathbf{0}) \to Z_{4\mathrm{D}}(\mathbf{0})$ of the undeformed partition function can be achieved in the following well-known manner:

- q and Q are parametrized as

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2.$$

- As $R \to 0$, the Boltzmann weights behave nicely:

$$\lim_{R\to 0} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} = \left(\frac{\dim\lambda}{|\lambda|}\right)^2 \left(\frac{\Lambda}{\hbar}\right)^{2|\lambda|}.$$

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We want to extend this prescription to the deformed partition function $Z(\boldsymbol{t})$.

Prescription for external potentials

5D external potentials

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left(q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

4D external potentials

$$\phi_k^{4\mathrm{D}}(\lambda) = \sum_{i=1}^{\infty} \left((\lambda_i - i + 1)^k - (-i + 1)^k \right)$$

Question: How $\phi_k^{4\mathrm{D}}(\lambda)$ can be derived from $\phi_k(\lambda)$ ($q = e^{-R\hbar}$) in the limit as $R \to 0$?

Hint: Take linear combinations of $\phi_j(\lambda)$'s to derive $\phi_k^{4\mathrm{D}}(\lambda)$.

Prescription for external potentials

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \phi_k^{\mathrm{4D}}(\lambda)(-R\hbar)^k + O(R^{k+1})$$

Proof: Substitute $q = e^{-R\hbar}$ in

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left(q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

and use the identity

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} (q^{ju} - q^{jv}) = (q^u - 1)^k - (q^v - 1)^k.$$

R-dependent parametrization of $\boldsymbol{t}$

The foregoing relation implies the identity

$$\lim_{R\to 0}\sum_{k=1}^{\infty}\frac{T_k}{(-R\hbar)^k}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}\phi_j(\lambda)=\sum_{k=1}^{\infty}T_k\phi_k^{\mathrm{4D}}(\lambda).$$

Since

$$\sum_{k=1}^{\infty}\frac{T_k}{(-R\hbar)^k}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}\phi_j(\lambda)=\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j}\frac{(-1)^{k-j}T_k}{(-R\hbar)^k}\phi_j(\lambda),$$

the 4D limit $\phi(\boldsymbol{t},\lambda)\to\phi^{\mathrm{4D}}(\boldsymbol{T},\lambda)$ is achieved if t_k's are defined as

$$t_j=\sum_{k=j}^{\infty}\binom{k}{j}\frac{(-1)^{k-j}T_k}{(-R\hbar)^k}.$$

Summary

As $R \to 0$ under the R-dependent parametrization

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2,$$

$$t_j = \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k},$$

the 5D partition function converges to the 4D partition function:

$$\lim_{R \to 0} Z(\boldsymbol{t}) = Z_{4\mathrm{D}}(\boldsymbol{T}).$$

3. Quantum spectral curve

Single-variate specialization of $Z(\boldsymbol{t})$

Substituting

$$t_k = -\frac{q^{-k/2} x^k}{k}, \ k = 1, 2, \ldots,$$

in $e^{\phi(\boldsymbol{t},\lambda)}$ $(\phi_k(\lambda) = \sum_{i=1}^{\infty} \left(q^{k(\lambda_i - i + 1)} - q^{k(-i+1)}\right))$ gives

$$e^{\phi(\boldsymbol{t},\lambda)} = \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i+1/2} x},$$

hence

$$Z(x) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i+1/2} x}.$$

Remark: This is a Baker-Akhiezer function at $\boldsymbol{t} = \boldsymbol{0}$.

Quantum spectral curve of melting crystal model

$Z(x)$ satisfies the q-difference equation

$$A(x, q^D)Z(x) = Z(x),$$

where $D = x\frac{d}{dx}$ and

$$\begin{aligned} A(x, q^D) = &\left(1 + q^{1/2}xq^{-D}(1 - q^{1/2}x)^{-1}\right) \\ &\times \left(1 + Qq^{1/2}xq^{-D}(1 - q^{1/2}x)^{-1}\right) \\ &\times (1 - q^{1/2}x)q^D. \end{aligned}$$

Classical limit ($q \to 1$, $q^D \to y$): $A_{\mathrm{cl}}(x, y) = 1$.

4D limit of $Z(x)$

As $R \to 0$ under the R-dependent parametrization

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2, \quad x = e^{R(X-\hbar/2)},$$

$Z(x)$ converges to

$$Z_{\mathrm{4D}}(X) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{i=1}^{\infty} \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i + 1)\hbar}.$$

Proof: As $R \to 0$,

$$\frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i+1/2} x} = \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i+1)\hbar} (1 + O(R)).$$

4D limit of quantum spectral curve

$Z_{4\mathrm{D}}(X)$ satisfies the difference equation

$$\left((X-\hbar)(e^{-\hbar d/dX}-1)+\frac{\Lambda^2}{X}e^{\hbar d/dX}\right)Z_{4\mathrm{D}}(X)=0.$$

Remark (Dunin-Barkowski et al.):

$$\Psi(X)=\exp\left(B\left(-\hbar\frac{d}{dX}\right)\frac{X-X\log X}{\hbar}\right)Z_{4\mathrm{D}}(X+\hbar)$$

$(B(z)=z/(e^z-1))$ satisfies the equation

$$\left(e^{-\hbar d/dX}+\Lambda^2 e^{\hbar d/dX}-X\right)\Psi(X)=0$$

of the quantum spectral curve of $\mathbb{CP}^1$ Gromov-Witten theory.

4. Bilinear equations of Fay type

Bilinear equations for $Z(\boldsymbol{t})$

Let $Z(\boldsymbol{t}, x_1, \ldots, x_N)$ denote $Z(\boldsymbol{t})$ with t_k's shifted as follows:

$$Z(\boldsymbol{t}, x_1, \ldots, x_N) = Z\Big(\ldots, t_k - \sum_{j=1}^{N} \frac{q^{-k/2} x_j^k}{k}, \ldots\Big).$$

Being a KP tau function, $Z(\boldsymbol{t})$ satisfies the following and other (infinitely many) bilinear equations of the Fay type:

$$\begin{aligned}
&(x_1 - x_2)(x_3 - x_4) Z(\boldsymbol{t}, x_1, x_2) Z(\boldsymbol{t}, x_3, x_4) \\
&- (x_1 - x_3)(x_2 - x_4) Z(\boldsymbol{t}, x_1, x_3) Z(\boldsymbol{t}, x_2, x_4) \\
&+ (x_1 - x_4)(x_2 - x_3) Z(\boldsymbol{t}, x_1, x_4) Z(\boldsymbol{t}, x_2, x_3) = 0
\end{aligned}$$

4D limit of $Z(\boldsymbol{t}, x_1, \ldots, x_N)$

$$Z(\boldsymbol{t}, x_1, \ldots, x_N) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\boldsymbol{t},\lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x_j}{1 - q^{-i+1/2} x_j}.$$

Its 4D limit

$$Z_{\mathrm{4D}}(\boldsymbol{T}, X_1, \ldots, X_N) = \lim_{R \to 0} Z(\boldsymbol{t}, x_1, \ldots, x_N)$$

can be obtained by letting $R \to 0$ under the same R-dependent parametrization of $q, Q, \boldsymbol{t}$ as for $Z(\boldsymbol{t})$ and the substitution

$$x_j = e^{R(X_j - \hbar/2)}$$

of the same form as used for $Z(x)$.

4D limit of $Z(\boldsymbol{t}, x_1, \ldots, x_N)$

$$
\begin{aligned}
&Z_{\mathrm{4D}}(\boldsymbol{T}, X_1, \ldots, X_N) \\
&= \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{\mathrm{4D}}(\boldsymbol{T}, \lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{X_j - (\lambda_i - i + 1)\hbar}{X_j - (-i + 1)\hbar} \\
&= Z_{\mathrm{4D}}\Big(\ldots,\ T_k - \sum_{j=1}^{N} \frac{h^k}{k X_j^k},\ \ldots \Big).
\end{aligned}
$$

Remark: Compared with the definition of $Z(\boldsymbol{t}, x_1, \ldots, x_N)$, T_k's in this expression are shifted in a slightly different way.

4D limit of $Z(\boldsymbol{t}, x_1, \ldots, x_N)$

$$Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, \ldots, X_N)$$

$$= \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{4\mathrm{D}}(\boldsymbol{T}, \lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{X_j - (\lambda_i - i + 1)\hbar}{X_j - (-i + 1)\hbar}$$

$$= Z_{4\mathrm{D}}\Big(\ldots,\ T_k - \sum_{j=1}^{N} \frac{h^k}{k X_j^k},\ \ldots \Big).$$

$$Z(\boldsymbol{t}, x_1, \ldots, x_N) = Z\Big(\ldots,\ t_k - \sum_{j=1}^{N} \frac{q^{-k/2} x_j^k}{k},\ \ldots \Big)$$

$$= \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(\boldsymbol{t}, \lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x_j}{1 - q^{-i + 1/2} x_j}.$$

4D limit of three-term bilinear equation

$$
\begin{aligned}
&(X_1 - X_2)(X_3 - X_4)Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, X_2)Z_{4\mathrm{D}}(\boldsymbol{T}, X_3, X_4) \\
-\,&(X_1 - X_3)(X_2 - X_4)Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, X_3)Z_{4\mathrm{D}}(\boldsymbol{T}, X_2, X_4) \\
+\,&(X_1 - X_4)(X_2 - X_3)Z_{4\mathrm{D}}(\boldsymbol{T}, X_1, X_4)Z_{4\mathrm{D}}(\boldsymbol{T}, X_2, X_3) = 0
\end{aligned}
$$

Proof: As $R \to 0$, $Z(\boldsymbol{t}, x_i, x_j)$ converges to $Z_{4\mathrm{D}}(\boldsymbol{T}, X_i, X_j)$ and

$$x_i - x_j = R(X_i - X_j) + O(R).$$

Corollary:

$Z_{4\mathrm{D}}(\boldsymbol{T})$ is a tau function of the KP hierarchy.

Conclusion

• The 4D limit $Z(\boldsymbol{t}) \to Z_{\mathrm{4D}}(\boldsymbol{T})$ $(R \to 0)$ can be formulated by R-dependent transformations $q = q(R)$, $Q = Q(R)$, $\boldsymbol{t} = \boldsymbol{t}(R, \boldsymbol{T})$ of the parameters and the coupling constants.

• The quantum spectral curve of $\mathbb{CP}^1$ Gromov-Witten theory can be derived from that of the melting crystal model. The 4D limit $Z(x) \to Z_{\mathrm{4D}}(X)$ of the wave function is achieved by an R-dependent transformation $x = x(R, X)$ of the variable.

• Bilinear equations of the Fay type turns out to survive the 4D limit. $Z_{\mathrm{4D}}(\boldsymbol{T})$ thereby satisfies the three-term bilinear equation. This leads to yet another proof of the fact that $Z_{\mathrm{4D}}(\boldsymbol{T})$ is a tau function of the KP hierarchy.