Toda tau functions with quantum torus symmetries

Kanehisa Takasaki (Kyoto University)

Prague, June 18, 2010 (corrected in February, 2012)

References

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1. Quantum torus Lie algebra

1.1 Matrix realization

$\mathbf{Z} \times \mathbf{Z}$ matrices

$$\Lambda = \sum_{i \in \mathbf{Z}} E_{i,i+1} = (\delta_{i+1,j}), \quad \Delta = \sum_{i \in \mathbf{Z}} iE_{ii} = (i\delta_{ij}),$$

$$v_m^{(k)} = q^{-km/2} \Lambda^m q^{k\Delta} \quad (k, m \in \mathbf{Z},\ |q| < 1)$$

Relations

$$[v_m^{(k)},\, v_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2}) v_{m+n}^{(k+l)}$$

Remark: Classical torus Lie algebra (Poisson algebra on 2D torus)

$$\{v_m^{(k)},\, v_m^{(l)}\} = (lm - kn) v_{m+n}^{(k+l)}$$

1.2 Fermionic realization

Fermion creation/annihilation operators ψ_i, ψ_i^* $(i \in \mathbf{Z})$:

anti-commutation relations

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i+j,0}, \quad \psi_i \psi_j + \psi_j \psi_i = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

vacuum states $\langle 0|$, $|0\rangle$ (somewhat unusual definition)

$$\psi_i|0\rangle = 0 \ (i \geq 0), \quad \psi_i^*|0\rangle = 0 \ (i \geq 1),$$
$$\langle 0|\psi_i = 0 \ (i \leq -1), \quad \langle 0|\psi_i^* = 0 \ (i \leq 0),$$

2D fields (somewhat unusual definition)

$$\psi(z) = \sum_{i \in \mathbf{Z}} \psi_i z^{-i-1}, \quad \psi^*(z) = \sum_{i \in \mathbf{Z}} \psi_i^* z^{-i}$$

1.2 Fermionic realization (cont'd)

Fermion bilinears ($E_{ij} \leftrightarrow :\psi_{-i}\psi_j^*: \in \mathrm{gl}(\infty)$)

$$V_m^{(k)} = q^{k/2} \oint \frac{dz}{2\pi i} z^m :\psi(q^{k/2}z)\psi^*(q^{-k/2}z): = q^{-km/2} \sum_{n\in\mathbf{Z}} q^{kn} :\psi_{m-n}\psi_n^*:$$

Relations

$$[V_m^{(k)}, V_n^{(l)}]$$

$$= (q^{(lm-kn)/2} - q^{(kn-lm)/2}) \left(V_{m+n}^{(k+l)} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}} \right)$$

$$= (q^{(lm-kn)/2} - q^{(kn-lm)/2}) V_{m+n}^{(k+l)} - \frac{q^{(k+l)m/2} - q^{-(k+l)m/2}}{1-q^{k+l}} \delta_{m+n,0} q^{k+l}$$

1.2 Fermionic realization (cont'd)

$$
\begin{aligned}
&[V_m^{(k)}, V_n^{(l)}] \\
&= (q^{(lm-kn)/2} - q^{(kn-lm)/2})V_{m+n}^{(k+l)} - \frac{q^{(k+l)m/2} - q^{-(k+l)m/2}}{1-q^{k+l}}\delta_{m+n,0}q^{k+l} \\
&= (q^{-k(m+n)/2} - q^{k(m+n)/2})V_{m+n}^{(0)} + m\delta_{m+n,0} \quad \text{if } k+l=0
\end{aligned}
$$

$V_m^{(0)}$'s span a $\widehat{U(1)}$ (or Heisenberg) subalgebra:

$$
V_m^{(0)} = J_m = \sum_{n\in\mathbf{Z}} {:}\psi_{m-n}\psi_n^*{:}, \quad [J_m, J_n] = m\delta_{m+n,0}
$$

2. Shift symmetries

2.1 Generating operators

$$G_\pm = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{\pm k}\right)$$

generate a linear combination of orthonormal states labelled by partitions λ with weight $s_\lambda(q^\rho)$, $\rho = (q^{1/2}, q^{3/2}, \ldots, q^{k+1/2}, \cdots)$:

$$\langle 0|G_+ = \sum_\lambda s_\lambda(q^\rho)\langle\lambda|, \quad G_-|0\rangle = \sum_\lambda s_\lambda(q^\rho)|\lambda\rangle.$$

The scalar product gives a generating function of all plane partitions (3D Young diagrams):

$$\langle 0|G_+G_-|0\rangle = \sum_\lambda s_\lambda(q^\rho)^2 = \prod_{n=1}^{\infty}(1-q^n)^{-n} = \sum_\pi q^{|\pi|}.$$

2.2 Two types of shift symmetries

First symmetry

$$G_-G_+\left(V_m^{(k)}-\delta_{m,0}\frac{q^k}{1-q^k}\right)(G_-G_+)^{-1}=(-1)^k\left(V_{m+k}^{(k)}-\delta_{m+k,0}\frac{q^k}{1-q^k}\right)$$

Second symmetry

$$q^{W_0/2}V_m^{(k)}q^{-W_0/2}=V_m^{(k-m)}$$

where

$$W_0=\sum_{n\in\mathbf{Z}}n^2{:}\psi_{-n}\psi_n^*{:}\in W^{(3)}$$

Remark: W_0 is a fermionic realization of the cut-and-join operator.

2.3 Shift symmetries in matrix realization

Key identities

$$\exp\left(\sum_{m=1}^{\infty} t_m \Lambda^m\right) q^{k\Delta} \exp\left(-\sum_{m=1}^{\infty} t_m \Lambda^m\right) = \exp\left(\sum_{m=1}^{\infty} t_m (1 - q^{-km}) \Lambda^m\right) q^{k\Delta},$$

$$\exp\left(\sum_{m=1}^{\infty} t_m \Lambda^{-m}\right) q^{k\Delta} \exp\left(-\sum_{m=1}^{\infty} t_m \Lambda^{-m}\right) = \exp\left(\sum_{m=1}^{\infty} t_m (1 - q^{km}) \Lambda^{-m}\right) q^{k\Delta}.$$

Proof: Compare the action of both hand side on the infinite column vector $(z^i)_{i \in \mathbf{Z}}$.

2.3 Shift symmetries in matrix realization (cont'd)

By specializing $t_k = q^{k/2}/k(1-q^k)$, the exponential factors on the left hand side become matrix analogues of $G_\pm$,

$$g_\pm = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)}\Lambda^{\pm k}\right),$$

and the exponential factors on the right hand side simplify to

$$\frac{(q^{-k+1/2}\Lambda;q)_\infty}{(q^{1/2}\Lambda;q)_\infty} = (1-q^{-1/2}\Lambda)(1-q^{-3/2}\Lambda)\cdots(1-q^{-k+1/2}\Lambda),$$

$$\frac{(q^{k+1/2}\Lambda^{-1};q)_\infty}{(q^{1/2}\Lambda^{-1};q)_\infty} = (1-q^{1/2}\Lambda^{-1})(1-q^{3/2}\Lambda^{-1})\cdots(1-q^{k-1/2}\Lambda^{-1}).$$

2.3 Shift symmetries in matrix realization (cont'd)

Consequently, we have the identities

$$g_+ \Lambda^m q^{k\Delta} {g_+}^{-1} = (1 - q^{-1/2}\Lambda)(1 - q^{-3/2}\Lambda) \cdots (1 - q^{-k+1/2}\Lambda)\Lambda^m q^{k\Delta},$$

$${g_-}^{-1} \Lambda^{m+k} q^{k\Delta} g_- = (-1)^k q^{k^2/2}(1 - q^{-1/2}\Lambda) \cdots (1 - q^{-k+1/2}\Lambda)\Lambda^m q^{k\Delta},$$

which imply the matrix analogue

$$g_- g_+ v_m^{(k)} (g_- g_+)^{-1} = (-1)^k v_{m+k}^{(k)}$$

of the first symmetry.

The matrix analogue

$$q^{\Delta^2/2} v_m^{(k)} q^{-\Delta^2/2} = v_m^{(k-m)}$$

of the second symmetry can be derived by straightforward calculations (in much the same way as in fermionic calculations).

3. Toda tau functions with quantum torus symmetries

3.1 Fermionic formula of tau function of 2D Toda hierarchy

$$\tau(\boldsymbol{T}, \bar{\boldsymbol{T}}, s) = \langle s| \exp\left(\sum_{k=1}^{\infty} T_k J_k\right) g \exp\left(-\sum_{k=1}^{\infty} \bar{T}_k J_{-k}\right) |s\rangle,$$

where $\langle s|$ and $|s\rangle$ are the ground states in the charge-s sector of the Fock space,

$$|s\rangle = \psi_{-s}\psi_{-s+1}\cdots|-\infty\rangle, \quad \langle s| = \langle -\infty|\cdots\psi^*_{s-1}\psi^*_s,$$

and g is an element of $\mathrm{GL}(\infty) = \exp\big(\mathrm{gl}(\infty)\big)$.

3.2 Tau function from deformed melting crystal model

Partition function of "deformed" melting crystal model:

$$Z(Q,\boldsymbol{t},s) = \langle s|G_+ e^{H(\boldsymbol{t})} Q^{L_0} G_-|s\rangle,$$

where

$$H(\boldsymbol{t}) = \sum_{k=1}^{\infty} t_k H_k, \quad H_k = V_0^{(k)}, \quad L_0 = \sum_{n\in\mathbf{Z}} n{:}\psi_{-n}\psi_n^*{:} \in W^{(2)}.$$

This partition function is related to the Toda tau function with the $\mathrm{GL}(\infty)$ element

$$g = q^{W_0/2} G_- G_+ Q^{L_0} G_- G_+ q^{W_0/2}.$$

3.3 Intertwining relations as constraints

By the shift symmetries, one can derive the intertwining relations

$$J_k g = g J_{-k} \quad (k = 1, 2, \ldots).$$

They imply the constraints

$$\left(\frac{\partial}{\partial T_k} + \frac{\partial}{\partial \bar{T}_k}\right) \tau(\boldsymbol{T}, \bar{\boldsymbol{T}}, s) = 0 \quad (k = 1, 2, \ldots)$$

on the tau function. The tau function thereby becomes a function of $\boldsymbol{T} - \bar{\boldsymbol{T}}$:

$$\tau(\boldsymbol{T}, \bar{\boldsymbol{T}}, s) = \tau(\boldsymbol{T} - \bar{\boldsymbol{T}}, s).$$

The reduced tau function $\tau(\boldsymbol{T}, s)$ is a tau function of the 1D Toda hierarchy. Up to a simple factor, $Z(Q, \boldsymbol{t}, s)$ coincides with $\tau(\boldsymbol{T}, s)$ $(T_k = (-1)^k t_k)$.

3.3 Intertwining relations as constraints (cont'd)

Proof of intertwining relations:

$$
\begin{aligned}
J_k g &= V_k^{(0)} q^{W_0/2} G_- G_+ Q^{L_0} G_- G_+ q^{W_0/2} \\
&= q^{W_0/2} V_k^{(k)} G_- G_+ Q^{L_0} G_- G_+ q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ (-1)^k (V_0^{(k)} - \frac{q^k}{1-q^k}) Q^{L_0} G_- G_+ q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ Q^{L_0} (-1)^k (V_0^{(k)} - \frac{q^k}{1-q^k}) G_- G_+ q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ Q^{L_0} G_- G_+ V_{-k}^{(k)} q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ Q^{L_0} G_- G_+ q^{W_0/2} V_{-k}^{(0)} \\
&= g J_{-k}
\end{aligned}
$$

3.3 Intertwining relations as constraints (cont'd)

The constraints $J_k g = gJ_{-k}$ are a special case of more general constraints

$$(V_m^{(k)} - \delta_{m,0}\frac{q^k}{1-q^k})g = Q^{-k}g(V_{-2k-m}^{(-k)} - \delta_{2k+m,0}\frac{q^{-k}}{1-q^{-k}}).$$

The constraints take a simpler form in the language of the Lax and Orlov-Schulman operators $L, M, \bar{L}, \bar{M}$ as

$$L = \bar{L}^{-1}, \quad q^M = q^{-1}Q^{-1}\bar{L}^{-2}q^{-\bar{M}}.$$

Remark: Since L, M and $\bar{L}, \bar{M}$ satisfy the commutation relations $[L, M] = L$ and $[\bar{L}, \bar{M}] = \bar{L}$, $q^{-km/2}L^m q^{kM}$ and $q^{-km/2}\bar{L}^m e^{k\bar{M}}$ give another realization of the quantum torus Lie algebra.

3.4 Other examples

• Generating function of two-leg amplitude $W_{\lambda\mu}$ in topological vertex (J. Zhou):

$$g = q^{W_0/2} G_+ G_- q^{W_0/2}$$

Constraints for Lax and Orlov-Schulman operators:

$$L = -q^{\bar{M}}, \quad \bar{L}^{-1} = -q^{M}$$

• Generating function of double Hurwitz numbers of Riemann sphere (A. Okounkov):

$$g = e^{\beta W_0} Q^{L_0}$$

Constraints for Lax and Orlov-Schulman operators:

$$L = e^{-\beta/2} Q \bar{L} e^{\beta \bar{M}}, \quad \bar{L}^{-1} = e^{\beta/2} Q L^{-1} e^{\beta M}$$

Summary

- Two realization of quantum torus Lie algebra (matrix realization $v_m^{(k)}$ and fermionic realization $V_m^{(k)}$)
- Generating operators $G_\pm$
- Auxiliary operator $W_0 \in W^{(3)}$
- Two types of shift symmetries induced by G_-G_+ and $q^{W_0/2}$
- Matrix realization of shift symmetries
- Toda tau function with quantum torus symmetries
- Constraints on tau function and Lax-Orlov-Schulman operators