Logarithmic Lax operators in Toda and lattice KP hierarchies

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Abstract The Lax forms of the 2D Toda hierarchy and the lattice KP hierarchy are constructed by difference operators on a line. The logarithms of the Lax operators can be defined with the help of dressing operators. These logarithmic Lax operators are used to consider exotic reductions of the 2D Toda hierarchy and the lattice KP hierarchy. We present some examples that are related to the 1D/bigrade Toda hierarchy and the lattice Gelfand-Dickey hierarchy.

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References

arXiv:2103.10666, arXiv:2203.06621, arXiv:2211.11353

1. 2D Toda hierarchy

Lax equations of 2D Toda hierarchy

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$
$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}]$$

with the two sets of time variables $t_k, \bar{t}_k$ $(k = 1, 2, \ldots)$ and the difference operators

$$L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n},$$

$$\bar{L}^{-1} = \sum_{n=0}^{\infty} \bar{u}_n \Lambda^{n-1}, \quad \bar{u}_0 \neq 0,$$
$$\Lambda = e^{\partial_s}, \quad \partial_s = \partial/\partial s, \quad \Lambda^n f(s) = f(s+n).$$

Lax equations of 2D Toda hierarchy (cont'd)

B_k's and $\bar{B}_k$'s are constructed from L and $\bar{L}$ as

$$B_k = (L^k)_{\geq 0} = \Lambda^k + b_{k1}\Lambda^{k-1} + \cdots + b_{kk},$$
$$\bar{B}_k = (\bar{L}^{-k})_{<0} = \bar{b}_{k0}\Lambda^{-k} + \cdots + \bar{b}_{kk-1}\Lambda^{-1}.$$

$(\ )_{\geq 0}$ and $(\ )_{<0}$ stand for the projection to the part of non-negative/negative powers of Λ:

$$\left(\sum_{n\in\mathbb{Z}} a_n\Lambda^n\right)_{\geq 0} = \sum_{n\geq 0} a_n\Lambda^n,$$
$$\left(\sum_{n\in\mathbb{Z}} a_n\Lambda^n\right)_{<0} = \sum_{n<0} a_n\Lambda^n.$$

Lax equations of 2D Toda hierarchy (cont'd)

B_k's and $\bar{B}_k$'s satisfy the zero-curvature (Zakharov-Shabat) equations. The lowest equation

$$\frac{\partial B_1}{\partial \bar{t}_1} - \frac{\partial \bar{B}_1}{\partial t_1} + [B_1, \bar{B}_1] = 0$$

for $B_1 = \Lambda + u_1$ and $\bar{B}_1 = \bar{u}_0 \Lambda^{-1}$ becomes the 2D Toda field equation

$$\frac{\partial \phi(s)}{\partial t_1 \partial \bar{t}_1} + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0.$$

Reduction to 1D Toda hierarchy

The 2D Toda hierarchy can be reduced to the 1D Toda hierarchy by setting the constraint

$$L = \bar{L}^{-1}.$$

The flows in the diagonal direction of $(t_k, \bar{t}_k)$ are trivialized:

$$\partial_{t_k} L + \partial_{\bar{t}_k} L = 0, \quad \partial_{t_k} \bar{L} + \partial_{\bar{t}_k} \bar{L} = 0.$$

The reduced Lax operator $\mathfrak{L} = L = \bar{L}^{-1}$ becomes a finite difference operator of the form

$$\mathfrak{L} = \Lambda + u + v\Lambda^{-1} \quad u = u_1, \quad v = \bar{u}_0$$

and satisfies the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial t_k} = [A_k, \mathfrak{L}], \quad A_k = \frac{1}{2}\left((\mathfrak{L}^k)_{\geq 0} - (\mathfrak{L}^k)_{<0}\right).$$

Equivariant Toda hierarchy

The equivariant Toda hierarchy (Getzler) is a deformation of the 1D Toda hierarchy originating in the equivariant Gromov-Witten theory of $\mathbb{CP}^1$ (Getzler, Okounkov & Pandharipande, Milanov).

This integrable hierarchy can be derived from the 1D Toda hierarchy by the constraint

$$L - \nu \log L = \bar{L}^{-1} - \nu \log \bar{L} - \log Q$$

where ν and Q are constant parameters. The 1D Toda hierarchy may be thought of as the non-equivariant limit as $\nu \to 0$ (and $\log Q$ disappears).

The logarithm $\log L, \log \bar{L}$ of $L, \bar{L}$ can be constructed with the help of dressing operators.

Dressing operators

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^{n}, \quad \bar{w}_0 \neq 0,$$
$$L = W \Lambda W^{-1}, \quad \bar{L} = \bar{W} \Lambda \bar{W}^{-1}.$$

The Lax equations can be converted to the following Sato equations for $W, \bar{W}$:

$$\frac{\partial W}{\partial t_k} = -\left(W \Lambda^k W^{-1}\right)_{<0} W, \quad \frac{\partial W}{\partial \bar{t}_k} = \left(\bar{W} \Lambda^k \bar{W}\right)_{<0} W,$$
$$\frac{\partial \bar{W}}{\partial t_k} = \left(W \Lambda^{-k} W^{-1}\right)_{\geq 0} \bar{W}, \quad \frac{\partial \bar{W}}{\partial \bar{t}_k} = -\left(\bar{W} \Lambda^{-k} \bar{W}\right)_{\geq 0} \bar{W}.$$

Logarithmic Lax operators

The logarithm of $L, \bar{L}$ can be defined as

$$\log L = W \log \Lambda W^{-1} = \partial_s - \frac{\partial W}{\partial s} W^{-1},$$
$$\log \bar{L} = \bar{W} \log \Lambda \bar{W}^{-1} = \log \bar{L} = \partial_s - \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}.$$

The reduced Lax operator

$$\mathfrak{L} = L - \nu \log L = \bar{L}^{-1} - \nu \log \bar{L} - \log Q$$

of the equivariant Toda hierarchy becomes a difference-differential operator:

$$\mathfrak{L} = \Lambda + u + v\Lambda^{-1} - \nu \partial_s, \qquad u = u_1, \quad v = \bar{u}_0$$

Logarithmic flows

The logarithm of the Lax operators are also used to construct an extension of the 1D Toda hierarchy by logarithmic flows (Carlet, Dubrovin & Zhang). Those flows (let x_k, $k = 1, 2, \ldots$, be the time variables) are defined by the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial x_k} = [C_k, \mathfrak{L}], \quad C_k = \frac{1}{2}\left((L^k \log L)_{\geq 0} + (\bar{L}^{-k} \log \bar{L})_{<0}\right).$$

$(\quad)_{\geq 0}$ and $(\quad)_{<0}$ are understood as

$$(A\partial_s + B)_{\geq 0} = (A)_{\geq 0}\partial_s + (B)_{\geq 0},$$
$$(A\partial_s + B)_{<0} = (A)_{<0}\partial_s + (B)_{<0}$$

for genuine difference operators A, B having ∂_s only in $\Lambda = e^{\partial_s}$. Thereby the commutators $[C_k, \mathfrak{L}]$ become difference operators.

Extended bigraded Toda hierarchy

The bigraded Toda hierarchy of type (a, b) (a and b are positive integers) can be derived from the 2D Toda hierarchy by the constraint

$$L^a = \bar{L}^{-b}.$$

The reduced Lax operator is a difference operator of the form

$$\mathfrak{L} = B_a + \bar{B}_b = \Lambda^a + v_1 \Lambda^{a-1} + \cdots + v_{a+b} \Lambda^{-b}.$$

The extended bigraded Toda hierarchy (Carlet) is obtained by adding the logarithmic flows

$$\frac{\partial \mathfrak{L}}{\partial x_k} = [C_k, \mathfrak{L}], \quad C_k = \frac{1}{2} \left((L^{ka} \log L)_{\geq 0} + (\bar{L}^{-kb} \log \bar{L})_{<0} \right)$$

to the bigraded Toda hierarchy.

Equivariant bigraded Toda hierarchy

The equivariant bigraded Toda hierarchy of type (a, b) can be derived from the 2D Toda hierarchy by the constraint

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \log Q.$$

The reduced Lax operator becomes the difference-differential operator

$$\mathfrak{L} = B_a + \bar{B}_b - \nu \partial_s.$$

Remark The bigraded generalizations of the 1D and equivariant Toda hierarchies are related to the stationary and equivariant Gromov-Witten theories of $\mathbb{CP}^1$ with orbifold points of order a at ∞ and of order b at 0 (Milanov & Tseng).

3. Reductions of lattice KP hierarchy

Lattice KP hierarchy

The lattice KP hierarchy is a closed subsystem of the 2D Toda hierarchy with the single set of time variables t_k $(k = 1, 2, \ldots)$ and the Lax operator L:

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad B_k = (L^k)_{\geq 0}.$$

The (pseudo-)differential operators

$$L_{\mathrm{KP}} = \partial_x + \sum_{n=2}^{\infty} u_n \partial_x^{1-n},$$
$$B_k^{\mathrm{KP}} = (L_{\mathrm{KP}}^k)_{\geq 0} = \partial_x^k + b_{k2}\partial_x^{k-2} + \cdots + b_{kk}$$

in the KP hierarchy are replaced by the foregoing difference operators L, B_k.

Lattice GD hierarchy

The lattice GD (Gelfand-Dickey) hierarchy is a reduction derived by the constraint

$$(L^a)_{<0} = 0.$$

The reduced Lax operator $\mathfrak{L} = L^a = B_a$ is a finite difference operator of the form

$$\mathfrak{L} = \Lambda^a + v_1\Lambda^{a-1} + \cdots + v_a$$

and satisfies the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial t_k} = [B_k, \mathfrak{L}], \quad B_k = (\mathfrak{L}^{k/a})_{\geq 0}.$$

Remarks

- Every a-th flow is trivial: $\frac{\partial \mathfrak{L}}{\partial t_{ka}} = 0$, $k \geq 1$.
- The ordinary GD hierarchy is hidden behind. This system may be thought of as an infinite chain of ordinary GD hierarchies connected by Darboux transformations.
- Frenkel considered this system in terms of q-difference operators ($\Lambda = q^{x\partial_x}$).
- Buryak and Rossi (arxiv: 1806.09285) proposed the $a = 2$ case (the lattice KdV hierarchy) as an integrable structure of an exotic cohomological field theory. They introduced therein a set of extra flows, which turn out to be logarithmic flows shown below.

3. Reductions of lattice KP hierarchy

Extension by logarithmic flows

The logarithmic flows of the lattice GD hierarchy are defined by the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial x_k} = [C_k, \mathfrak{L}],$$

$$C_k = (L^{ka} \log L)_{\geq 0} = \frac{1}{a}(\mathfrak{L}^k \log \mathfrak{L})_{\geq 0}.$$

Cf. The case of the 1D and bigraded Toda hierarchies:

$$C_k = \frac{1}{2}\left((L^{ka} \log L)_{\geq 0} + (\bar{L}^{-kb} \log \bar{L})_{<0}\right).$$

Generalized ILW hierarchy

A generalization of the ILW (intermediate long wave) hierarchy can be derived from the lattice KP hierarchy by the constraint

$$(L^a - \nu \log L)_{<0} = 0.$$

The reduced Lax operator $\mathfrak{L} = L^a - \nu \log L$ is a difference-differential operator of finite order:

$$\mathfrak{L} = B_a - \nu\partial_s = \Lambda^a + v_1\Lambda^{a-1} + \cdots + v_a - \nu\partial_s.$$

Cf. The case of the bigraded equivariant Toda hierarchy:

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \log Q,$$
$$\mathfrak{L} = B_a + \bar{B}_b - \nu\partial_s = \Lambda^a + \cdots + v_{a+b}\Lambda^{-b} - \nu\partial_s.$$

Remarks

- The ordinary ILW hierarchy amounts to the case where $a = 1$. The reduced Lax operator becomes the difference-differential operator $\mathfrak{L} = \Lambda + u - \nu\partial_s$, $u = u_1$, and u satisfies the ILW equation

$$\frac{\partial u}{\partial t_2} = \partial_s \left(u^2 + \nu(1+\Lambda)(1-\Lambda)^{-1}\frac{\partial u}{\partial s} \right).$$

  $(1+\Lambda)(1-\Lambda)^{-1}$ is an algebraic expression of the well-known integral operator (Kodama et al., Lebedev & Radul).

- Buryak and Rossi (arxiv:1809.00271) proposed to use the operator $\mathfrak{L}$ to formulate an integrable structure of the single Hurwitz numbers of $\mathbb{CP}^1$. Liu, Wang and Zhang (arxiv:2110.03317) considered a reduction of the 2D Toda hierarchy for the ILW equation.

The Lax forms of the 2D Toda and lattice KP hierarchies are formulated by difference operators on a line. Unlike the KP hierarchy, this enables us to consider the logarithm of the Lax operators without technical difficulty.

These logarithmic Lax operators plays a role in the description of some exotic reduced systems such as the equivariant Toda hierarchy and the ILW hierarchy. They are also used in the construction of logarithmic flows in the 1D/bigraded Toda hierarchy and the lattice GD hierarchy.

Another usage of the logarithmic Lax operators can be found in a class of special solutions, e.g., the solution obtained from the double Hurwitz numbers of $\mathbb{CP}^1$ can be characerized by algebraic relations for the logarithmic Lax operators and their canonical conjugates (Orlov-Schulman operators).