# Elliptic soliton equations
# and
# Sato Grassmannian

- Background
- Nonlinear Schrödinger hierarchy
- Factorization problem
- Embedding to Sato Grassmannian

rational

- Landau-Lifshitz hierarchy
- Factorization problem
- Embedding to Sato Grassmannian

elliptic

( • Other examples )

Based on : nlin.SI/0312016 (review)
etc ...

# 1. Background

$\boxed{\begin{array}{l}\text{Sato's thesis (??)} \quad \text{('81 ~ )} \\ \text{Any soliton equation can be described as a simple dynamical} \\ \text{system on an } \infty\text{-dim. Grassmann manifold (universal Grassmannian).}\end{array}}$

This has been confirmed for many soliton equations. However, almost all of them are equations with a rational spectral parameter.

$\boxed{\text{Question}}$: What about equations with an elliptic spectral parameter, or equations some how formulated with elliptic functions?

- "Elliptic analogue" of soliton equations

1) Landau-Lifshitz equation (LL)

$$S_t = S \times S_{xx} + S \times JS$$

$$[\partial_x - A(z), \partial_t - B(z)] = 0$$

$A(z), B(z)$ : matrices of elliptic functions

2) Krichever-Novikov equation

$$c_t = \frac{1}{4} c_{xxx} + \frac{3}{8} \frac{1 - c_{xx}^2}{c_x} - \frac{3}{2} c_x^2 \wp(2c)$$

3) 1+1-dim. analogue of elliptic Calogero-Moser system

(Levin-Olshanetsky-Zotov, Krichever)

4) 1+1-dim. system formulated by Tyurin parameters of vector bundles on a Riemann surface of genus $\geqq 1$ (Krichever, Novikov)

- case study in this talk

$$\mathrm{LL} \xrightarrow[J_1=J_2=J_3]{} S_t = S \times S_{xx}$$

$\wr\wr$ gauge

$$\begin{cases} u_t = \frac{1}{2} u_{xx} - u^2 v \\ v_t = -\frac{1}{2} v_{xx} + u v^2 \end{cases}$$

nonlinear Schrödinger (NLS)

$\downarrow$ well known

dynamical system in Sato Grassmannian

What about LL ?

- cf. Carey, Hannabuss, Mason & Singer, Commun. Math. Phys. 154 (1993), 25-47

deals with a similar issue

(but not satisfactory ...)

## 2. Nonlinear Schrödinger hierarchy

$$A(\lambda) = U_0 \lambda + U_1$$

$$U_0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

- time evolutions $t_2, t_3, \cdots$ $(t_1 = x)$

$$[\partial_x - A(\lambda), \partial_{t_n} - A_n(\lambda)] = 0$$

where $A_n(\lambda) = U_0 \lambda^n + U_1 \lambda^{n-1} + \cdots + U_n$

$$= (U(\lambda)\lambda^n)_+ \quad \text{(polynomial part)}$$

- $U(\lambda)$ is uniquelly determined by

i) $[\partial_x - A(\lambda), U(\lambda)] = 0$

ii) $U(\lambda)^2 = I$

Matrix elements of $U_n$'s are differential polynomials of $u$ and $v$:

$$U_2 = \begin{pmatrix} -\frac{1}{2}uv & \frac{1}{2}u_x \\ -\frac{1}{2}v_x & \frac{1}{2}uv \end{pmatrix}, \text{ etc}$$

- $U(\lambda)$ can also be written

$$U(\lambda) = \phi(\lambda)\sigma_3\phi(\lambda)^{-1}, \quad \phi(\lambda) = I + \sum_{n=1}^{\infty} \phi_n \lambda^{-n}$$

$$\partial_x \phi(\lambda) = A(\lambda)\phi(\lambda) - \phi(\lambda)\sigma_3\lambda$$

# 3. Factorization problem (rational case)

$$\mathfrak{g} = \{ X(\lambda) \mid X(\lambda) = \sum_{n=-\infty}^{\infty} X_n \lambda^n, \ X_n \in sl(2,\mathbb{C}) \}$$

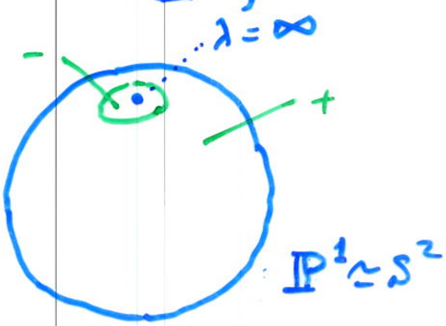
& convergence condition

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$

$$\mathfrak{g}_+ = \{ X(\lambda) \in \mathfrak{g} \mid X_n = 0 \text{ for } n < 0 \}$$

$$\mathfrak{g}_- = \{ X(\lambda) \in \mathfrak{g} \mid X_n = 0 \text{ for } n \geq 0 \}$$

$$(\ )_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$$




- loop group factorization

$$\boxed{G \approx G_+ \cdot G_-}$$

(in a neighborhood of 1)

i.e. any element $g(\lambda)$ of $G$ can be factorized as

$$g(\lambda) = g_+(\lambda)^{-1} g_-(\lambda),$$

$$g_\pm(\lambda) \in G_\pm$$

(if $g(\lambda)$ is close to 1).

## • Solution of NLS hierarchy from factorization

"projection method"

Given initial value $\phi(\lambda) = \phi(0,\lambda)$

$\uparrow$ $x = t_2 = t_3 = \cdots = 0$

do factorization as

$$\phi(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n \sigma_3 \lambda^n\right) = \chi(t,\lambda)^{-1} \phi(t,\lambda)$$

$$\chi(t,\lambda) \in G_+, \quad \phi(t,\lambda) \in G_-$$

Then $\phi(t,\lambda)$ satisfies

$$\boxed{\partial_{t_n} \phi(t,\lambda) = -\left(\phi(t,\lambda)\sigma_3 \lambda^n \phi(t,\lambda)^{-1}\right)_- \phi(t,\lambda)}$$

from which

$$A_n(t,\lambda) = \left(\phi(t,\lambda)\sigma_3 \lambda^n \phi(t,\lambda)\right)_+,$$

$$A(t,\lambda) = A_1(t,\lambda), \text{ and}$$

$$U(t,\lambda) = \phi(t,\lambda)\sigma_3 \phi(t,\lambda)$$

turn out to give a solution of NLS hierarchy.

- What does it mean?

"linear" dynamical system on $G$

$$g(\lambda) \to g(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n \sigma_3 \lambda^n\right)$$

$\downarrow$ projection

coset

"nonlinear" dynamical system on $G_- \approx G_+ \backslash G$

$$G_+ \phi(\lambda) \to G_+ \phi(t,\lambda) = G_+ \phi(\lambda) \exp\left(-\sum_{n=1}^{\infty} t_n \sigma_3 \lambda^n\right)$$

$$G_+ \chi(t,\lambda) = G_+$$

$\chi(t,\lambda)$ is absorbed by $G_+$

- Link with Grassmann manifold

$\exists$ embedding $G_+ \backslash G \hookrightarrow Gr$

# 4. Embedding to Sato Grassmannian

$$V = \{ X(\lambda) \mid X(\lambda) = \sum_{n=-\infty}^{\infty} x_n \lambda^n ,\ x_n \in \underline{\underline{gl}}(2, \mathbb{C}) \}$$

& convergence condition

$$\cup$$

$$V_- = \{ X(\lambda) \mid X_n = 0 \ \text{ for } n \geq 0 \}$$

- $Gr = \{ W \subset V \mid$ closed subspace with

$$\dim \operatorname{Ker}(W \to V/V_-) = \dim \operatorname{Coker}(W \to V/V_-) < \infty \}$$

where $W \to V/V_-$ is the composition of $W \hookrightarrow V$ and $V \to V/V_-$.

- "big cell"

$$Gr^0 = \{ W \subset V \mid W \xrightarrow{\sim} V/V_- \}$$

For $W \in Gr^0$,

$$\boxed{\begin{array}{l} \exists \text{ basis } \ W_{n,ij}(\lambda) \quad n \geq 0,\ i,j = 1,2 \\ \qquad W_{n,ij}(\lambda) = E_{ij} \lambda^n + O(\lambda^{-1}) \end{array}}$$

— inverse image of $E_{ij}\lambda^n$ by $W \simeq V/V_-$

$$\left( \quad E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ etc} \ldots \right)$$

- embedding of $G_- \approx G_+ \backslash G$ into $Gr$

i) base point ("vacuum") $W_0 \in Gr^0$

$$W_0 = \{ X(\lambda) \mid X_n = 0 \text{ for } n < 0 \}$$

ii) mapping $G_- \to Gr$

$$\phi(\lambda) \mapsto W = W_0 \phi(\lambda)$$

This is an embedding

[ ii') also mapping $G \to Gr$

$$g(\lambda) \to W_0 g(\lambda)$$

which can be reduced to $G_+ \backslash G \to Gr$

$$W_0 \underbrace{h(\lambda)}_{\text{absorbed}} g(\lambda) = W_0 g(\lambda) \quad \text{if } h(\lambda) \in G_+$$ ]

iii) time evolutions of NLS hierarchy is thereby mapped to the exponential flows

$$\boxed{W(0) = W \to W(t) = W \exp\left(-\sum_{n=1}^{\infty} t_n \sigma_3 \lambda^n\right)}$$

on the space $\mathcal{M}$ of "dressed vacua"

$$\boxed{\mathcal{M} = \{ W_0 \phi(\lambda) \mid \phi(\lambda) \in G_- \} \subset Gr}$$

# 5. Landau-Lifshitz hierarchy

- Landau-Lifshitz equation

$$\mathbf{S} = {}^t(S_1\ S_2\ S_3)$$

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S}$$

$$J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}$$

$J_1, J_2, J_3$ distinct (totally anisotropic)

$\Downarrow$

Lax representation (Sklyanin)

$$[\partial_x - A(z), \partial_t - B(z)] = 0$$

with spectral parameter $z$ on a torus $\Sigma = \mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z})$

- weight function (classical limit of Boltzmann weights)

$$w_1(z) = \frac{\alpha\,cn(\alpha z)}{sn(\alpha z)},\quad w_2(z) = \frac{\alpha\,dn(\alpha z)}{sn(\alpha z)},\quad w_3(z) = \frac{\alpha}{sn(\alpha z)},$$

$$\alpha = \sqrt{e_1 - e_3},\quad e_\alpha = \wp(w_\alpha)$$

$$A(z) = \sum_{\alpha=1,2,3} w_\alpha(z) S_\alpha \sigma_\alpha,\quad B(z) = \cdots$$

$$A(z+2w_\alpha) = \sigma_\alpha A(z) \sigma_\alpha \quad (\alpha = 1,2,3)$$
$$B(z+2w_\alpha) = \sigma_\alpha B(z) \sigma_\alpha$$

- higher flows $t_2, t_3, \cdots$ $(t_1 = x)$

$$[\partial_x - A(z),\ \partial_{t_n} - A_n(z)] = 0$$

i) $$A_n(z) = \frac{U_0}{z^n} + \cdots + U_n + O(z) \quad (z \to 0)$$

$U_0 \left(= \sum_\alpha S_\alpha \sigma_\alpha\right), U_1, U_2, \cdots$ determined to be differential polynomials of $S_\alpha$'s

ii) $$A_n(z+2w_\alpha) = \sigma_\alpha A_n(z) \sigma_\alpha \quad (\alpha = 1,2,3)$$

i) & ii) determine $A_n(z)$ uniquely:

$$A_n(z) = \sum_{m=0}^{n} U_{n-m} \cdot \frac{(-1)^m \partial_z^m w_\alpha(z)}{m!} \sigma_\alpha$$

# 6. Factorization problem (elliptic case)

- $\mathfrak{g} = \{ X(z) \mid X(z) = \sum_{n=-\infty}^{\infty} X_n z^n, \quad X_n \in sl(2, \mathbb{C}) \}$

  & convergence condition

$$\mathfrak{g} = \mathfrak{g}_{in} \oplus \mathfrak{g}_{out} \qquad \text{(Reyman \& Semenov-Tian-Shansky)}$$

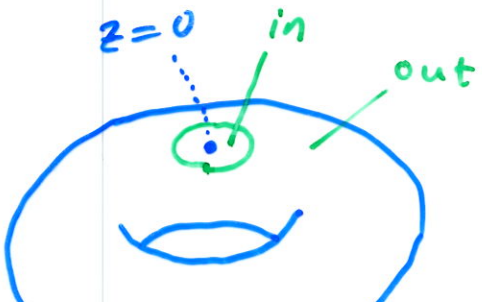
where

$\mathfrak{g}_{in} = \{ X(z) \mid X_n = 0 \text{ for } n < 0 \}$

$\quad = \{ X(z) \mid$ holomorphic in a neighborhood of $z = 0 \}$

$\mathfrak{g}_{out} = \{ X(z) \mid$ holomorphic in a neighborhood of $z = 0$ except $z = 0$, singular at $z = 0$, and can be extended to a holomorphic function on $\Sigma - (2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z})$ with quasi-periodicity

$$X(z + 2\omega_\alpha) = \sigma_\alpha X(z) \sigma_\alpha \quad (\alpha = 1, 2, 3) \}$$




$\Sigma$

- i) $\mathfrak{g}_{out}$ is spanned by $\{ \partial_z^m w_\alpha(z) \sigma_\alpha \mid m = 0, 1, \cdots, \alpha = 1, 2, 3 \}$

  ii) projection $(\cdot)_{out} : \mathfrak{g} \to \mathfrak{g}_{out}$

$$\boxed{(z^{-m-1}\sigma_\alpha)_{out} = \frac{(-1)^m}{m!} \partial_z^m w_\alpha(z) \sigma_\alpha}$$

  iii) $A_n(z) = (U(z) z^{-n})_{out}, \quad U(z) = \sum_{n=0}^{\infty} U_n z^{-n}$

- loop group factorization

$$G \approx G_{out} \cdot G_{in}$$

- solution of LL hierarchy by factorization

  initial value $\phi(z) = \phi(0, z) \in G_{in}$

$$\phi(z) \exp\left(-\sum_{n=1}^{\infty} t_n \sigma_3 z^{-n}\right) = \chi(t,z)^{-1} \phi(t,z),$$

$$\chi(t,z) \in \underline{G_{out}}, \quad \phi(t,z) \in \underline{G_{in}}$$

  i) $\phi(t,z)$ satisfies

$$\partial_{t_n} \phi(t,z) = -\left(\phi(t,z) \sigma_3 z^{-n} \phi(t,z^{-1})\right)_{\underline{out}} \phi(t,z)$$

  ii) $A_n(t,z) = (\phi(t,z) \sigma_3 z^{-n} \phi(t,z))_{\underline{in}}$ satisfies the Lax equations.

- LL hierarchy is a dynamical system on $G_{in}$ derived from the "linear" dynamics on $G$

⋮

Everything is parallel to NLS hierarchy.

$$G_{in} \approx G_{out} \backslash G \hookrightarrow Gr \ ?$$

$$V = \{ X(z) \mid X(z) = \sum_{n=-\infty}^{\infty} X_n z^n , \ X_n \in gl(2,\mathbb{C}) \}$$

& convergence condition

$$\cup$$

$$V_+ = \{ X(z) \mid X_n = 0 \ \text{ for } n \leq 0 \}$$

basis : $z^{n+1}\sigma_\alpha$, $z^{n+1}1$ $(n \geq 0)$ $(\alpha = 1,2,3)$

$$z \leftrightarrow \lambda^{-1}$$

# 7. Embedding into Sato Grassmannian



- Due to speciality of LL, Gr and $Gr^\circ$ have to be replaced by

$$Gr_{-4} = \{ W \subset V \mid \text{closed subspace with}$$
$$\dim \mathrm{Ker}(W \to V/V_+) = \dim \mathrm{Coker}(W \to V/V_+) - 4 < \infty \}$$

index $-4$

and

$$Gr^\circ_{-4} = \{ W \subset V \mid W \simeq V/\overbrace{(V_+ \oplus sl(2,\mathbb{C}) \oplus \mathbb{C}z^{-1}1)}^{4 \text{ gaps}} \}$$

$$\boxed{\begin{array}{l} \text{basis of } V/(V_+ \oplus sl(2,\mathbb{C}) \oplus \mathbb{C}z^{-1}1): \\ z^{-n-1}\sigma_\alpha, \quad z^{-n-2}1, \quad 1 \qquad (n \geq 0,\ \alpha = 1,2,3) \end{array}}$$

- Base point (vacuum) $W_0 \in Gr^\circ_{-4}$ spanned by

$$\begin{cases} \dfrac{(-1)^n}{n!} \partial_z^n w_\alpha(z)\sigma_\alpha = \underline{z^{-n-1}\sigma_\alpha} + O(1) & \begin{pmatrix} n \geq 0 \\ \alpha = 1,2,3 \end{pmatrix} \\ \dfrac{(-1)^n}{(n+1)!} \partial_z^n \wp(z) 1 = \underline{z^{-n-2} 1} + O(1) & (n \geq 0) \\ 1 & \end{cases}$$

i.e.

$W_0 = \{ X(z) \mid$ holomorphic in a neighborhood of $z=0$ except $z=0$, and can be extended to a holomorphic function on $\Sigma - (2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z})$ with quasi-periodicity

$X(z) $ can be holomorphic at $z=0$

$$X(z+2\omega_\alpha) = \sigma_\alpha X(z) \sigma_\alpha, \quad \alpha = 1, 2, 3 \}$$

- Embedding of $G_{in} \simeq G_{out} \backslash G$ into $Gr_{-4}$

$$\boxed{\begin{array}{rcl} G_{out}\backslash G \simeq G_{in} & \longrightarrow & Gr_{-4} \\ \phi(z) & \longmapsto & W = W_0 \phi(z) \end{array}}$$

$$\left[ \; h(\lambda) \in G_{out} \Rightarrow W_0 \underbrace{h(\lambda)}_{\text{absorbed}} g(\lambda) = W_0 g(\lambda) \; \right]$$

- LL hierarchy as exponential dynamics

$$W(0) = W \rightarrow W(t) = W \exp\left(-\sum_{n=1}^{\infty} t_n \sigma_3 z^{-n}\right)$$

on the space of dressed vacua

$$\mathcal{M} = \{ W_0 \phi(z) \mid \phi(z) \in G_{in} \} \subset Gr_{-4}$$

- Geometric meaning :

$$\mathcal{M} \ni W = W_0 \, \phi(z)$$

$\Gamma(\Sigma - \{z=0\}, E)$

local trivialization of $E$ at $z=0$

$E$ : holomorphic $sl(2,\mathbb{C})$ bundle (rigid) over $\Sigma$ characterizing quasi-periodicity

$$\chi(z+2\omega_\alpha) = \sigma_\alpha \chi(z) \sigma_\alpha$$

$\mathcal{M}$ can be identified with a moduli space of $(E, \phi)$

(Note : local coordinate $z$ is fixed)

- Cf. NLS hierarchy : $E$ is a trivial bundle. Only the local trivialization is deformed.

- cf. 1+1-dim systems formulated by Tyurin parameters

E has moduli $(\underline{\gamma}, \underline{\alpha})$ ← dynamical variables

$\parallel$

$E(\underline{\gamma}, \underline{\alpha})$

$W_0$ also depends on these moduli

$\parallel$

$W_0(\underline{\gamma}, \underline{\alpha})$

$$\mathcal{M} = \{\, W_0(\underline{\gamma}, \underline{\alpha})\phi \mid \underline{\gamma}, \underline{\alpha}, \phi \cdots \}$$

Time evolutions of the system is mapped to exponential flows on $\mathcal{M}$.

## Conclusion

For elliptic (and probably higher genus) soliton equations, an underlying bundle structure is essential. Interpretation in the language of Sato Grassmannian, too, respects this structure. The bundle $E$ determines a base point (vacuum) of the phase space $\mathcal{M}$ embedded in the Grassmann manifold. The "vacuum" is dressed by a loop group element $\phi$, which corresponds to the data of local trivialization of $E$.