Integrable structure in
melting crystal model of 5D gauge theory
joint work with Toshio Nakatsu

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1. Melting crystal model

melting crystal corner = random plane partition

Okounkov, Reshetikhin & Vafa, “Quantum Calabi-Yau and classical crystal”, hep-th/0309208
ordinary partition = Young diagram

\[ \lambda = (\lambda_1, \lambda_2, \ldots), \lambda_i \geq \lambda_{i+1}, \lambda_i \in \mathbb{Z}_{\geq 0} \text{ (length of } i\text{-th row).} \]

|\lambda| = \sum_i \lambda_i \text{ (area).}

plane partition = 3D Young diagram

\[ \pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix}
\pi_{11} & \pi_{12} & \cdots \\
\pi_{21} & \pi_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}, \quad \pi_{ij} \geq \pi_{i,j+1}, \pi_{ij} \geq \pi_{i+1,j}, \pi_{ij} \in \mathbb{Z}_{\geq 0} \text{ (height of } (i,j)\text{-th stack).} \]

|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij} \text{ (volume).}

partition function of random plane partition

\[ Z = \sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} (1 - q^n)^{-n} \text{ (McMahon function), } 0 < q < 1 \]
diagonal slices of plane partition (Okounkov & Reshetikhin)

The diagonal slices $\{\pi(m)\}_{m=-\infty}^{\infty}$ of the plane partition $\pi$ is a sequence of Young diagrams that satisfy “interlacing relations”

$\cdots \preceq \pi(-2) \preceq \pi(-1) \preceq \pi(0) \succeq \pi(1) \succeq \pi(2) \succeq \cdots$.

interlacing relation:

$\lambda = (\lambda_1, \lambda_2, \ldots) \succeq \mu = (\mu_1, \mu_2, \ldots) \iff \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$
plane partition $\pi \mapsto$ pair $(T, T')$ of semi-standard tableaux

The plane partition $\pi$ determines a pair $(T, T')$ of semi-standard tableaux of shape $\lambda = \pi(0)$ by putting "$m + 1$" in boxes of the skew diagram $\pi(\pm m)/\pi(\pm (m + 1))$.

$T$: $\lambda = \pi(0) \succeq \pi(-1) \succeq \pi(-2) \succeq \cdots$
$T'$: $\lambda = \pi(0) \succeq \pi(1) \succeq \pi(2) \succeq \cdots$
partition function as sum over semi-standard tableaux

By the mapping $\pi \mapsto (T, T')$, the partition function $Z = \sum_{\pi} q^{||\pi||}$ can be converted to a sum over $T, T'$ and their shape $\lambda$:

$$Z = \sum_{\lambda} \sum_{T, T': \text{shape } \lambda} q^T q^{T'}$$

The weights are determined by entries of the tableaux:

$$q^T = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(-m)/\pi(-m-1)|},$$

$$q^{T'} = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(m)/\pi(m+1)|}$$
partition function in terms of Schur functions

The partial sums over the semi-standard tableaux $T, T'$ give a special value of the Shur function:

$$\sum_{T: \text{shape } \lambda} q^T = \sum_{T': \text{shape } \lambda} q^{T'} = s_\lambda(q^\rho), \quad \rho = \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right)$$

The partition function can be thus rewritten as

$$Z = \sum_\lambda s_\lambda(q^\rho)^2$$

Remark: Hook formula for $s_\lambda(q^\rho)$

$$s_\lambda(q^\rho) = q^{n(\lambda)+|\lambda|/2} \prod_{(i,j) \in \lambda} (1 - q^{h(i,j)})^{-1}, \quad n(\lambda) = \sum_{i=1}^{\infty} (i - 1)\lambda_i$$
deformation by potential $\Phi(t, \lambda, p)$

We consider a deformed model

$$Z_p(t) = \sum_{\lambda} s_{\lambda}(q^\rho)^2 e^{\Phi(t, \lambda, p)}, \quad \Phi(t, \lambda, p) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, p)$$

with potentials

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} q^k(p + \lambda_i - i + 1) - \sum_{i=1}^{\infty} q^k(-i + 1)$$

The right hand side of this definition of $\Phi_k(\lambda, p)$ is understood to be a finite sum (hence a rational function of $q$) by cancellation of terms between the two sums:

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} (q^k(p + \lambda_i - i + 1) - q^k(p-i+1)) + q^k \frac{1 - q^{pk}}{1 - q^k}$$
melting crystal model and 5D SUSY gauge theory

Melting crystal model with external potential:

\[ Z_p(t) = \sum_{\pi} q^{\pi} e^{\Phi(t,\pi(0),p)} = \sum_{\lambda} s_{\lambda}(q^{\rho})^{2} q^{\Phi(t,\lambda,p)} \]

5D \( \mathcal{N} = 1 \) SUSY U(1) gauge theory:

\[ Z_p(t) = \sum_{\pi} q^{\pi} Q^{\pi(0)} e^{\Phi(t,\pi(0),p)} = \sum_{\lambda} s_{\lambda}(q^{\rho})^{2} Q^{\lambda} q^{\Phi(t,\lambda,p)}, \]

\[ q = e^{-R \hat{h}}, \quad Q = (R \Lambda)^2 \]

(5D analogue of Nekrasov’s 4D instanton sum)

Goal: Show that **1D Toda hierarchy** is a common integrable structure in these models.
2. Fermionic representation of partition function

complex fermion system

\[ \psi(z) = \sum_{m=-\infty}^{\infty} \psi_m z^{-m-1}, \quad \psi^*(z) = \sum_{m=-\infty}^{\infty} \psi^*_m z^{-m} \]

with anti-commutation relations

\[ \{\psi_m, \psi^*_n\} = \delta_{m+n,0}, \quad \{\psi_m, \psi_n\} = \{\psi^*_m, \psi^*_n\} = 0 \]

Ground state (Fermi sea) \(|p\rangle\) in charge \(p\) sector

\[ \psi_m|p\rangle = 0 \quad \text{for} \quad m \geq -p, \quad \psi^*_m|p\rangle = 0 \quad \text{for} \quad m \geq p + 1 \]

Fock space spanned by states labelled by partitions (or Young diagrams)

\[ F = \bigoplus_{p=-\infty}^{\infty} F_p, \quad F_p = \bigoplus_{\lambda} \mathbb{C} |\lambda; p\rangle \]
States labelled by Young diagrams (charge 0 sector)

\[
\emptyset = (0, 0, \ldots), \text{ charge } 0 \leftrightarrow |\emptyset; 0\rangle
\]

\[
\lambda = (\lambda_1, \lambda_2, \ldots), \text{ charge } 0 \leftrightarrow |\lambda; 0\rangle
\]

\[
\lambda = (\lambda_1, \lambda_2, \ldots) \leftrightarrow \{\lambda_i - i\}_{i=1}^{\infty} \subset \mathbb{Z} \text{ (Maya diagram)}
\]
States labelled by Young diagrams (charge $p$ sector)

\[
\lambda = (\lambda_1, \lambda_2, \ldots), \text{ charge } p \mapsto |\lambda; p\rangle
\]

\[
(\lambda, p) \mapsto \{p + \lambda_i - i\}_{i=1}^{\infty} \subset \mathbb{Z} \text{ (Maya diagram of charge } p)\]

If $\lambda = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots)$,

\[
|\lambda; p\rangle = \psi_-(p+\lambda_1-1) - 1 \cdots \psi_-(p+\lambda_n-n) - 1 \psi_-(p-n) + 1 \cdots \psi_-(p-1) + 1 |p\rangle
\]
U(1) current and fermionic representation of tau function

\[ J(z) = :\psi(z)\psi^*(z): = \sum_{k=-\infty}^{\infty} J_m z^{-m-1}, \quad J_m = \sum_{n=-\infty}^{\infty} :\psi_{m-n}\psi_n^*: \]

with commutation relations

\[ [J_m, J_n] = m\delta_{m+n,0} \quad \text{(Heisenberg algebra)} \]

\( J_m \)'s play the role of "Hamiltonians" in the usual fermionic formula of tau functions of the KP and (2D) Toda hierarchies:

\[ \tau_p(t, \bar{t}) = \langle p | \exp(\sum_{m=1}^{\infty} t_m J_m) g \exp(-\sum_{m=1}^{\infty} \bar{t}_m J_{-m}) | p \rangle, \ g \in \text{GL}(\infty) \]
Hamiltonians for fermionic representation of $Z_p(t)$

$$H_k = \sum_{n=-\infty}^{\infty} q^{kn} :\psi_n \psi_n^* :$$

The states $|\lambda; p\rangle$ are eigenvectors of these “Hamiltonians” and the potential functions $\Phi_k(\lambda, p)$ are their eigenvalues:

$$H_k |\lambda; p\rangle = \Phi_k(\lambda, p) |\lambda; p\rangle$$

ferminonic representation of $Z_p(t)$

$$Z_p(t) = \langle p| G_+ e^{H(t)} G_- |p\rangle$$

where

$$H(t) = \sum_{k=1}^{\infty} t_k H_k, \quad G_{\pm} = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1 - q^k)} J_{\pm k}\right)$$
\( G_\pm \) generate random plane partition (Okounkov & Reshetikhin)

\( G_\pm \) are a product of vertex operators \( \Gamma_\pm(m) \):

\[
G_+ = \prod_{m=-\infty}^{-1} \Gamma_+(m), \quad G_- = \prod_{m=0}^{\infty} \Gamma_-(m),
\]

\[
\Gamma_\pm(m) = \exp\left( \sum_{k=1}^{\infty} \frac{1}{k} q^{\mp k(m+1/2)} J_{\pm k} \right)
\]

They generate a “half” of random plane partition \( \pi \):

\[
\langle p | G_+ \rangle = \sum_{\lambda} \sum_{T: \text{shape } \lambda} q^T \langle \lambda; p \rangle = \sum_{\lambda} s_\lambda(q^0) \langle \lambda; p \rangle,
\]

\[
G_- | p \rangle = \sum_{\lambda} \sum_{T: \text{shape } \lambda} q^T | \lambda; p \rangle = \sum_{\lambda} s_\lambda(q^0) | \lambda; p \rangle
\]

Consequently, \( \langle p | G_+ e^{H(t)} G_- | p \rangle = \sum_{\lambda} s_\lambda(q^0)^2 e^{\Phi(t, \lambda, p)} = Z_p(t) \).
3. Quantum torus Lie algebra

basis $V_m^{(k)}$ ($k = 0, 1, \ldots, m \in \mathbb{Z}$)

$$V_m^{(k)} = q^{-km/2} \sum_{n=-\infty}^{\infty} q^{kn} \psi_{m-n} \psi_n^*;$$

$$= q^{k/2} \oint \frac{dz}{2\pi i} z^m \psi(q^{k/2}z) \psi^*(q^{-k/2}z);$$

Remark: $J_m = V_m^{(0)}$, $H_k = V_0^{(k)}$. $V_m^{(k)}$ coincides with Okounkov and Pandharipande’s operator $\mathcal{E}_m(z)$ specialized to $z = q^k$.

commutation relations

$$[V_m^{(k)}, V_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2})(V_{m+n}^{(k+l)} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}})$$

Remark: This is a (central extension of) $q$-deformation of the Poisson algebra of functions on a 2-torus.
adjoint action by $G_{\pm}$ (1)

Fermion fields $\psi(z), \psi^*(z)$ transform as

$$
G_+ \psi(z) G_+^{-1} = (q^{1/2}z; q)_\infty^{-1} \psi(z),
$$

$$
G_+ \psi^*(z) G_+^{-1} = (q^{1/2}z; q)_\infty \psi^*(z),
$$

$$
G_- \psi(z) G_-^{-1} = (q^{1/2}z^{-1}; q)_\infty \psi(z),
$$

$$
G_- \psi^*(z) G_-^{-1} = (q^{1/2}z^{-1}; q)_\infty^{-1} \psi^*(z)
$$

where $(z; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n)$. 
adjoint action by $G_\pm$ (2)

The foregoing formulae for fermion fields imply that the fermion bilinear $\psi^* (q^{-k/2} z) \psi (q^{k/2} z)$ transforms as

$$ G_+ \psi^* (q^{-k/2} z) \psi (q^{k/2} z) G_+^{-1} $$

$$ = \frac{(q^{1/2} \cdot q^{-k/2} z; q)_\infty}{(q^{1/2} \cdot q^{k/2} z; q)_\infty} \psi^* (q^{-k/2} z) \psi (q^{k/2} z) $$

$$ = \prod_{m=1}^{k} (1 - z q^{(k+1)/2-m}) \psi^* (q^{-k/2} z) \psi (q^{k/2} z) $$

A similar transformation law holds for the adjoint action by $G_-$ as well.
shift symmetry among $V^{(k)}_m$'s

From the foregoing formulae, one can deduce the following symmetry among the basis of the quantum torus Lie algebra:

$$G_- G_+ \left(V^{(k)}_m - \delta_{m,0} \frac{q^k}{1 - q^k}\right) (G_- G_+)^{-1} = (-1)^k \left(V^{(k)}_{m+k} - \delta_{m+k,0} \frac{q^k}{1 - q^k}\right)$$

In particular,

$$G_- G_+ \left(V^{(k)}_0 - \frac{q^k}{1 - q^k}\right) (G_- G_+)^{-1} = (-1)^k V^{(k)}_k,$$

$$(G_- G_+)^{-1} \left(V^{(k)}_0 - \frac{q^k}{1 - q^k}\right) G_- G_+ = (-1)^k V^{(k)}_{-k}$$

This is a key to identification of the integrable structure.
4. Integrable structure

rewriting partition function of melting crystal model (1)

\[ Z_p(t) = \langle p| G_+ e^{H(t)} G_- |p \rangle \]

Split \( G_+ e^{H(t)} G_- \) into several pieces as

\[
G_+ e^{H(t)} G_- = G_+ e^{H(t)/2} e^{H(t)/2} G_- \\
= G_+ e^{H(t)/2} G_-^{-1} \cdot G_+ G_- \cdot G_-^{-1} e^{H(t)/2} G_- 
\]

and use the formulae (a special case of shift symmetry)

\[
G_- G_+ \left( H_k - \frac{q^k}{1 - q^k} \right) (G_- G_+)^{-1} = (-1)^k V^{(k)}_k,
\]

\[
(G_- G_+)^{-1} \left( H_k - \frac{q^k}{1 - q^k} \right) G_- G_+ = (-1)^k V^{(-k)}_k
\]
The foregoing formulae imply that

\[ G_+ \left( H_k - \frac{q^k}{1 - q^k} \right) G_+^{-1} = (-1)^k G_-^{-1} V_k^{(k)} G_- , \]

\[ G_-^{-1} \left( H_k - \frac{q^k}{1 - q^k} \right) G_- = (-1)^k G_+ V_{-k}^{(k)} G_+^{-1} \]

\[ V_{\pm k}^{(k)} \] on the right hand side can be transformed to \( J_{\pm k} \) as

\[ q^{W/2} V_k^{(k)} q^{-W/2} = V_k^{(0)} = J_k , \quad q^{-W/2} V_{-k}^{(k)} q^{W/2} = V_{-k}^{(0)} = J_{-k} \]

where \( W \) is a special element of \( W_\infty \) algebra:

\[ W = W_0^{(3)} = \sum_{n=-\infty}^{\infty} n^2 : \psi_n \psi_n^* : \]
Thus we have the relation

\[ G_+ \left( H_k - \frac{q^k}{1 - q^k} \right) G_+^{-1} = (-1)^k G_-^{-1} q^{-W/2} J_k q^{W/2} G_-, \]

\[ G_-^{-1} \left( H_k - \frac{q^k}{1 - q^k} \right) G_- = (-1)^k G_+ q^{W/2} J_k q^{-W/2} G_+^{-1} \]

hence

\[ G_+ e^{H(t)/2} G_+^{-1} = \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{2(1 - q^k)} \right) G_-^{-1} q^{-W/2} \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k J_k}{2} q^{W/2} G_- \right), \]

and a similar expression for \( G_-^{-1} e^{H(t)/2} G_- \).
rewriting partition function of melting crystal model (4)

We can thus eventually rewrite $G_+ e^{H(t)} G_-$ as

$$G_+ e^{H(t)} G_- = \exp \left( \sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) G_-^{-1} q^{-W/2} \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k J_k}{2} \right) \times$$

$$\times g \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k J_{-k}}{2} \right) q^{-W/2} G_+^{-1}$$

where

$$g = q^{W/2} (G_- G_+)^2 q^{W/2} \in \text{GL}(\infty)$$
rewriting partition function of melting crystal model (5)

Since \( \langle p | G_+^{-1} q^{-W/2} = q^{-p(p+1)(2p+1)/12} \langle p | \) and \( q^{-W/2} G_+^{-1} | p \rangle = q^{-p(p+1)(2p+1)/12} | p \rangle \), the partition function \( Z_p(t) \) can be expressed as

\[
Z_p(t) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k}\right) q^{-p(p+1)(2p+1)/6} \times \\
\times \langle p \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k t_k J_k}{2}\right) g \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k t_k J_{-k}}{2}\right) | p \rangle
\]

The last piece \( \langle p | \cdots | p \rangle \) may be interpreted as a special value of the tau function

\[
\tau_p(t, \bar{t}) = \langle p | \exp(\sum_{k=1}^{\infty} t_k J_k) g \exp(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}) | p \rangle
\]

of 2D Toda hierarchy. However, this is not the end of the story.
identities of expectation values

Actually, we can start from different splitting of $G_+ e^H(t) G_-$ as well:

$$G_+ e^H(t) G_- = G_+ e^H(t) G_+^{-1} \cdot G_+ G_- = G_+ G_- \cdot G_-^{-1} e^H(t) G_-$$

This leads to apparently different expressions of $Z_p(t)$, which imply that the following identities hold:

$$\langle p | \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k J_k}{2} \right) g \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k t_k J_{-k}}{2} \right) | p \rangle$$

$$= \langle p | \exp \left( \sum_{k=1}^{\infty} (-1)^k t_k J_k \right) g | p \rangle$$

$$= \langle p | g \exp \left( \sum_{k=1}^{\infty} (-1)^k t_k J_{-k} \right) | p \rangle$$

What do they mean?
\[ g = q^{W/2}(G_-G_+)^2q^{W/2} \] determines solution of 1D Toda hierarchy

The foregoing identities can be directly derived from the relations

\[ J_kg = gJ_{-k}, \quad k = 1, 2, 3, \ldots \]

(a consequence of the shift symmetry of \( V_{m}^{(k)} \)'s). From these relations one can derive the identities

\[ \tau_p(t, \bar{t}) = \tau_p(t - \bar{t}, 0) = \tau_p(0, \bar{t} - t) \]

for the tau function \( \tau_p(t, \bar{t}) \) of 2D Toda hierarchy, which thereby reduces to a tau function of 1D Toda hierarchy. Thus 1D Toda hierarchy turns out to be an underlying integrable structure of the partition function \( Z_p(t) \) of the melting crystal model.
integrable structure in 5D SUSY U(1) gauge theory

$Z_p(t)$ has a fermionic representation of the form

$$Z_p(t) = \langle p| G + Q^{L_0} e^{H(t)} G^- |p \rangle$$

where $L_0 = \sum_{n=-\infty}^{\infty} n : \psi_n \psi_n^*: \quad \text{(element of Virasoro algebra)}$. The foregoing calculations can be repeated for this case as well and lead to a similar conclusion. The counterpart of $g$ is given by

$$g = q^{W/2} G^- G^+ Q^{L_0} G^- G^+ q^{W/2}$$

and satisfies the relation

$$J_k g = g J_{-k}, \; k = 1, 2, 3, \ldots$$

Thus a relevant integrable structure is again 1D Toda hierarchy.
Concluding remarks

4D limit ($R \to 0$) (cf. Marshakov and Nekrasov's work on 4D case)
Not straightforward

relation to topological strings
1. Another interpretation of $\langle p | G + Q^L_0 e^{H(t)} G_- | p \rangle$ ($q = e^{-g_{st}}$, $Q = e^{-a}$) as A-model amplitude on $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{CP}^1$
2. Generating function of $W_{\lambda \mu} \sim c_{\lambda \mu}$ as solution of 2D Toda hierarchy with $g = q^{W/2} G_+ G_- q^{W/2}$ (Zhou)

thermodynamic limit (rescaling $t_k$'s and letting $\hbar \to 0$ in $q = e^{-Rh}$)
Dispersionless Toda hierarchy? (work in progress)

more relations satisfied by $g$ Constraints with quantum/classical torus algebraic structure? (work in progress)