Integrable structures of cubic Hodge integrals

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Section 1. Cubic Hodge integrals

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- Defition of two-partition cubic Hodge integrals
- Generating function of cubic Hoge integrals
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Two-partition cubic Hodge integrals

Definition (Liu-Liu-Zhou 0310272, Zhou 0310282)

$$G_{\mu\bar{\mu}}(\tau) = a_{\mu\bar{\mu}}(\tau) \sum_{g=0}^{\infty} \hbar^{2g-2+l(\mu)+l(\bar{\mu})}$$

$$\times \int_{\overline{\mathcal{M}}_{g,l(\mu)+l(\bar{\mu})}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu)} \frac{1}{\mu_i}\left(\frac{1}{\mu_i} - \psi_i\right) \prod_{i=1}^{l(\bar{\mu})} \frac{\tau}{\bar{\mu}_i}\left(\frac{\tau}{\bar{\mu}_i} - \psi_{l(\mu)+i}\right)}$$

- τ is a parameter. $a_{\mu\bar{\mu}}(\tau)$ is a numerical factor depending on τ and the integer partitions $\mu = (\mu_i)_{i\geq 1}$, $\bar{\mu} = (\bar{\mu}_i)_{i\geq 1}$.
- $\overline{\mathcal{M}}_{g,n}$is the compactified moduli space of complex curves of genus g with n marked points. ψ_i is the ψ-class, $\psi_i = c_1(L_i)$, corresponding to the i-th marked point.

Two-partition cubic Hodge integrals (cont'd)

Definition (Liu-Liu-Zhou 0310272, Zhou 0310282)

$$G_{\mu\bar{\mu}}(\tau) = a_{\mu\bar{\mu}}(\tau) \sum_{g=0}^{\infty} \hbar^{2g-2+l(\mu)+l(\bar{\mu})}$$

$$\times \int_{\overline{\mathcal{M}}_{g,l(\mu)+l(\bar{\mu})}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu)} \frac{1}{\mu_i}(\frac{1}{\mu_i} - \psi_i) \prod_{i=1}^{l(\bar{\mu})} \frac{\tau}{\bar{\mu}_i}(\frac{\tau}{\bar{\mu}_i} - \psi_{l(\mu)+i})}$$

- $\Lambda_g^\vee(u)$ is the special linear combination

$$\Lambda_g^\vee(u) = u^g - u^{g-1}\lambda_1 + \cdots + (-1)^g \lambda_g$$

of the Hodge classes $\lambda_k = c_k(E_g)$.

Generating functions of cubic Hodge integrals

Introduce two sets of variables $\mathbf{p} = (p_k)_{k\geq 1}$, $\bar{\mathbf{p}} = (\bar{p}_k)_{k\geq 1}$, and make generating functions of the cubic Hodge integrals:

Definition (Liu-Liu-Zhou 0310272, Zhou 0310282)

$$G(\tau, \mathbf{p}, \bar{\mathbf{p}}) = \sum_{\mu,\bar{\mu}\in\mathcal{P}} G_{\mu,\bar{\mu}}(\tau) p_\mu \bar{p}_{\bar{\mu}},$$

$$p_\mu = \prod_{i\geq 1} p_{\mu_i}, \quad \bar{p}_{\bar{\mu}} = \prod_{i\geq 1} \bar{p}_{\bar{\mu}_i},$$

$$G^\bullet(\tau, \mathbf{p}, \bar{\mathbf{p}}) = \exp G(\tau, \mathbf{p}, \bar{\mathbf{p}}).$$

Schur functions

- Let $s_\nu(\mathbf{x})$ and $s_{\bar{\nu}}(\bar{\mathbf{x}})$ denote the Schur functions of $\mathbf{x} = (x_i)_{i\geq 1}$ and $\bar{\mathbf{x}} = (\bar{x}_i)_{i\geq 1}$ in the sense of Macdonald's book.
- There are polynomials $S_\nu(\mathbf{p})$ and $S_{\bar{\nu}}(\bar{\mathbf{p}})$ of $\mathbf{p}$ and $\bar{\mathbf{p}}$ from which $s_\nu(\mathbf{x})$ and $s_{\bar{\mu}}(\bar{\mathbf{x}})$ are obtained by substituting the power sums

$$p_k = \sum_{i\geq 1} x_i^k, \quad \bar{p}_k = \sum_{i\geq 1} \bar{x}_i^k.$$

Schur functions (cont'd)

- These power sum variables are related to the time variables $\mathbf{t} = (t_k)_{k\geq 1}$, $\bar{\mathbf{t}} = (\bar{t}_k)_{k\geq 1}$ of the 2D Toda hierarchy as

$$p_k = kt_k, \quad \bar{p}_k = k\bar{t}_k.$$

  Let $\mathrm{S}_\nu(\mathbf{t})$ and $\mathrm{S}_{\bar{\nu}}(\bar{\mathbf{t}})$ denote $S_\nu(\mathbf{p})$ and $S_{\bar{\nu}}(\bar{\mathbf{p}})$ as the polynomials in $\mathbf{t}$ and $\bar{\mathbf{t}}$.

- $\mathrm{S}_\nu(\mathbf{t})$ and $\mathrm{S}_{\bar{\nu}}(\bar{\mathbf{t}})$ can be directly defined as

$$\mathrm{S}_\nu(\mathbf{t}) = \det(\mathrm{S}_{\nu_i - i + j}(\mathbf{t}))_{i,j\geq 1},$$

$$\sum_{m=0}^{\infty} \mathrm{S}_m(\mathbf{t}) z^m = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right).$$

Combinatorial expression of cubic Hodge integrals

Theorem (Liu-Liu-Zhou 0310272, Zhou 0310282)

$$G^\bullet(\tau, \mathbf{p}, \bar{\mathbf{p}}) = R^\bullet(\tau, \mathbf{p}, \bar{\mathbf{p}})$$
$$= \sum_{\nu, \bar{\nu} \in \mathcal{P}} q^{(\kappa(\nu)\tau + \kappa(\bar{\nu})\tau^{-1}))/2} \mathcal{W}_{\nu\bar{\nu}}(q) S_\nu(\mathbf{p}) S_{\bar{\nu}}(\bar{\mathbf{p}}),$$

where

$$\mathcal{W}_{\nu\bar{\nu}}(q) = s_\nu(q^\rho) s_{\bar{\nu}}(q^{\nu+\rho}), \quad q = e^{\sqrt{-1}\hbar},$$
$$q^\rho = (q^{-i+1/2})_{i\geq 1}, \quad q^{\nu+\rho} = (q^{\nu_i - i+1/2})_{i\geq 1},$$
$$\kappa(\nu) = \sum_{i\geq 1} \nu_i(\nu_i - 2i + 1), \ \kappa(\bar{\nu}) = \sum_{i\geq 1} \bar{\nu}_i(\bar{\nu}_i - 2i + 1).$$

Section 2: Lift to tau function

Contents

- Two-leg topological vertex
- Fermionic expression of generating function
- Lift to tau function

Two-leg topological vertex $\mathcal{W}_{\nu\bar{\nu}}(q)$

- $\mathcal{W}_{\nu\bar{\nu}}(q)$ is a rational function of $q^{1/2}$, and satisfies the identities

$$\mathcal{W}_{\nu\bar{\nu}}(q) = \mathcal{W}_{\bar{\nu}\nu}(q) = (-1)^{|\nu|+|\bar{\nu}|}\mathcal{W}_{{}^{\mathrm{t}}\nu\,{}^{\mathrm{t}}\bar{\nu}}(q^{-1}).$$

- Fermionic formula for $|q| > 1$

$$\mathcal{W}_{\nu\bar{\nu}}(q) = \langle {}^{\mathrm{t}}\nu|q^{-K/2}\Gamma_{-}(q^{\rho})\Gamma_{+}(q^{\rho})q^{-K/2}|{}^{\mathrm{t}}\bar{\nu}\rangle.$$

- Fermionic formula for $|q| < 1$

$$\mathcal{W}_{\nu\bar{\nu}}(q) = (-1)^{|\nu|+|\bar{\nu}|}\langle\nu|q^{K/2}\Gamma_{-}(q^{-\rho})\Gamma_{+}(q^{-\rho})q^{K/2}|\bar{\nu}\rangle.$$

Operators on fermionic Fock space

- K is diagonal:
$$\langle \mu | K | \nu \rangle = \delta_{\mu\nu} \kappa(\mu).$$
- $\Gamma_\pm(q^{\pm\rho})$'s are specializations of the vertex operators

$$\Gamma_\pm(\mathbf{x}) = \prod_{i \geq 1} \Gamma_\pm(x_i), \quad \Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right),$$

e.g.,

$$\Gamma_\pm(q^{-\rho}) = \exp\left(\sum_{k,i=1}^{\infty} \frac{q^{(i-1/2)k}}{k} J_{\pm k} \right) = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{\pm k} \right).$$

Fermionic expression of $R^{\bullet}(\tau, \mathbf{p}, \bar{\mathbf{p}})$

$$
\begin{aligned}
R^{\bullet}(\tau, \mathbf{p}, \bar{\mathbf{p}}) &= \sum_{\nu,\bar{\nu}\in\mathcal{P}} q^{(\kappa(\nu)\tau+\kappa(\bar{\nu})\tau^{-1}))/2}\mathcal{W}_{\nu\bar{\nu}}(q)S_{\nu}(\mathbf{p})S_{\bar{\nu}}(\bar{\mathbf{p}}) \\
&= \langle 0|\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k p_k}{k}J_k\right) \\
&\quad \times q^{(\tau+1)K/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{(\tau^{-1}+1)K/2} \\
&\quad \times \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k \bar{p}_k}{k}J_{-k}\right)|0\rangle
\end{aligned}
$$

Lift to tau function

A tau function of the 2D Toda hierarchy can be obtained by replacing

$$\frac{(-1)^k p_k}{k} \to t_k, \quad \frac{(-1)^p \bar{p}_k}{k} \to -\bar{t}_k,$$

$$\langle 0| \to \langle s|, \quad |0\rangle \to |s\rangle, \quad s \in \mathbb{Z}$$

in $R^{\bullet}(\tau, \mathbf{p}, \bar{\mathbf{p}})$:

$$\mathcal{T}(s, \mathbf{t}, \bar{\mathbf{t}}) = \langle s| \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}\right) |s\rangle,$$

$$g = q^{(\tau+1)K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{(\tau^{-1}+1)K/2}.$$

s-dependence of tau function

$$\mathcal{T}(s,\mathbf{t},\bar{\mathbf{t}}) = \sum_{\nu,\bar{\nu}\in\mathcal{P}} q^{(\tau+1)(\kappa(\nu)/2+s|\nu|+(4s^2-1)s/24)} \times q^{(\tau^{-1}+1)(\kappa(\bar{\nu})/2+s|\bar{\nu}|+(4s^2-1)s/24)} \times \langle\nu|\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})|\bar{\nu}\rangle \mathrm{S}_\nu(\mathbf{t})\mathrm{S}_{\bar{\nu}}(\bar{\mathbf{t}})$$

Consequences:

- s may be thought of as a continuous variable: $s \in \mathbb{R}$.
- For any $c \in \mathbb{R}$, $\mathcal{T}(s+c,\mathbf{t},\bar{\mathbf{t}})$ $(s \in \mathbb{Z})$ persists to be a tau function of the 2D Toda hierarchy (with $K/2$ in g shifted to $K/2 + cL_0$).

Section 3: Main result and implications

Contents

- Lax and dressing operators of 2D Toda hierarchy
- Main result
- Reduced systems in special cases

Lax operators

$$L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n}, \quad \bar{L}^{-1} = \sum_{n=0}^{\infty} \bar{u}_n \Lambda^{n-1}, \quad \Lambda = e^{\partial_s}$$

satisfy the Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$
$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}],$$
$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{<0}$$

of the 2D Toda hierarchy.

Dressing operators

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n$$

express the Lax operators in the dressed form

$$L = W \Lambda W^{-1}, \quad \bar{L} = \bar{W} \Lambda \bar{W}^{-1}.$$

The logarithm and the fractional powers of the Lax operators can be thereby defined as

$$\log L = W \log \Lambda W^{-1}, \ \log \bar{L} = \bar{W} \log \Lambda \bar{W}^{-1}, \ \log \Lambda = \partial_s,$$
$$L^\alpha = W \Lambda^\alpha W^{-1}, \ \bar{L}^\alpha = \bar{W} \Lambda^\alpha \bar{W}^{-1}, \ \Lambda^\alpha = e^{\alpha \partial_s}.$$

Main result

If $\tau = -1$, $\mathcal{T}(s, \mathbf{t}, \bar{\mathbf{t}})$ becomes a trivial (exponential) tau function. Let us consider the case where $\tau \neq -1$.

Theorem

The Lax operators obtained from the tau function $\mathcal{T}(s, \mathbf{t}, \bar{\mathbf{t}})$ satisfy the algebraic relation

$$L^{1/(\tau+1)} = -\bar{L}^{-\tau/(\tau+1)}.$$

Corollary

There is a function $u = u(s, \mathbf{t}, \bar{\mathbf{t}})$ such that

$$L^{1/(\tau+1)} = -\bar{L}^{-\tau/(\tau+1)} = (1 - u\Lambda^{-1})\Lambda^{1/(\tau+1)}.$$

What this implies?

- The reduced Lax operator

$$\mathcal{L} = (1 - u\Lambda^{-1})\Lambda^{1/(\tau+1)}$$

satisfies the Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_k} = [(L^k)_{\geq 0}, \mathcal{L}] = -[(L^k)_{<0}, \mathcal{L}],$$

$$\frac{\partial \mathcal{L}}{\partial \bar{t}_k} = [(\bar{L}^{-k})_{<0}, \mathcal{L}] = -[(\bar{L}^{-k})_{\geq 0}, \mathcal{L}].$$

What this implies? (cont'd)

- By the two expressions for each equation, these equations turn into equations of the form

$$\frac{\partial u}{\partial t_k} = f_k, \quad \frac{\partial u}{\partial \bar{t}_k} = \bar{f}_k.$$

  If one can express f_k's and $\bar{f}_k$'s in terms of u appropriately (this is not obvious), these equations become a single field reduction of the 2D Toda hierarchy.
- A number of such reduced systems emerge when τ takes various rational values.

Reduced systems in special cases

1. $\tau = N =$ positive integer

$$\mathcal{L} = \Lambda^{1/(N+1)} - u\Lambda^{-N/(N+1)}$$

- This coincides with the Lax formalism of the hungry Lotka-Volterra (aka Bogoyavlensky-Itoh-Narita) system on the fractional lattice $\frac{1}{N+1}\mathbb{Z}$.
- The $N+1$-st power

$$\mathcal{L}^{N+1} = L = (-1)^{N+1}\bar{L}^{-N} = \Lambda + u_1 + \cdots + u_{N+1}\Lambda^{-N}$$

is the Lax operator of the bi-graded Toda hierarchy of the $(1, N)$ type with time variables $(\mathbf{t}, \bar{\mathbf{t}})$.

Reduced systems in special cases (cont'd)

2. $\tau = -\frac{N}{N+1}$, $N =$ positive integer

$$\mathcal{L} = \Lambda^{N+1} - u\Lambda^{N}$$

- Since the $N+1$-st power of L is a difference operator

$$L^{N+1} = \mathcal{L} = \Lambda^{N+1} - u\Lambda^{N},$$

the **t**-flows become trivial at every $N+1$-st step:

$$\frac{\partial \mathcal{L}}{\partial t_{(N+1)k}} = [L^{(N+1)k}, \mathcal{L}] = 0, \quad k = 1, 2, \ldots$$

(though this is not an $N+1$-periodic reduction).

Reduced systems in special cases (cont'd)

2. $\tau = -\frac{N}{N+1}$, N = positive integer

$$\mathcal{L} = \Lambda^{N+1} - u\Lambda^{N}$$

- The wave function $\Psi(z)$ of the 2D Toda hierarchy turns out to satisfy a linear equation of the form

$$z^{N+1}\Psi(z) = (\partial_{t_1}^{N+1} + c_1\partial_{t_1}^{N} + \cdots + c_{N+1})\Psi(z).$$

  The **t**-flows give isospectral deformations of this spectral problem. The reduced system is thus related to the generalized KdV (i.e., Gelfand-Dickey) hierarchy.

Reduced system in special cases (cont'd)

Other rational values of τ

3. (i) $\tau = \frac{1}{N}$ and (ii) $\tau = -\frac{N+1}{N}$, $N =$ positive integer: These are parallel to the cases 1 and 2 by the duality under the exchange $\tau \leftrightarrow \tau^{-1}$, $\mathbf{t} \leftrightarrow \bar{\mathbf{t}}$.

4. $\tau = \frac{b}{a}$, $a, b =$ positive coprime integers: A generalization of the cases 1 and 3 (i). A further generalized Lotka-Volterra hierarchy (included in Bogoyavlensky's work?) emerges.

5. $\tau = -\frac{b}{a}$, $a, b =$ positive coprime integers: This case is a generalization of the cases 2 and 3 (ii), and again related to the Gelfand-Dickey hierarchy.

Remarks

- Our result in the case of $\tau = N$ explains an origin of the Volterra-type hierarchies in the work of B. Dubrovin, S.-Q. Liu, D. Yang and Y. Zhang, arXiv:1612.02333.
- The relevance of the Gelfand-Dickey hierarchy in the case of $\tau = -(N+1)/N$ is pointed out in our recent preprint, T. Nakatsu and K.T., arXiv:1812.11726, by a different method.
- When $\tau = -b/a$ $(a > b)$, we have the difference operator

$$\mathcal{L}^{a-b} = L^a = (-1)^{a-b}\bar{L}^b = \Lambda^a + v_1\Lambda^{a-1} + \cdots + v_{a-b}\Lambda^b.$$

This is a lattice version of the Gelfdand-Dickey hierarchy (cf, A. Buryak and P. Rossi, arXiv:1806.09825).

Section 4: Outline of proof

Contents

- Factorization problem
- Initial values of W and $\bar{W}$
- Initial values of $L^{1/(\tau+1)}$ and $\bar{L}^{-\tau/(\tau+1)}$

Factorization problem

The dressing operators can be characterized by the factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1}\bar{W}$$

where

$$U = q^{(\tau+1)(s-1/2)^2/2} \cdot \prod_{i=1}^{\infty}(1 - q^{i-1/2}\Lambda^{-1})^{-1}$$
$$\times \prod_{i=1}^{\infty}(1 - q^{i-1/2}\Lambda)^{-1} \cdot q^{(\tau^{-1}+1)(s-1/2)^2/2}.$$

Initial values of W and $\bar{W}$

At the initial time $\mathbf{t} = \bar{\mathbf{t}} = \mathbf{0}$, the factorization problem can be solved explicitly. This leads to the following expression of the initial values $W_0 = W|_{\mathbf{t}=\bar{\mathbf{t}}=\mathbf{0}}$ and $\bar{W}_0 = \bar{W}|_{\mathbf{t}=\bar{\mathbf{t}}=\mathbf{0}}$ of the dressing operators:

$$W_0 = q^{(\tau+1)(s-1/2)^2/2} \cdot \prod_{i=1}^{\infty}(1 - q^{i-1/2}\Lambda^{-1}) \cdot q^{-(\tau+1)(s-1/2)^2/2},$$

$$\bar{W}_0 = q^{(\tau+1)(s-1/2)^2/2} \cdot \prod_{i=1}^{\infty}(1 - q^{i-1/2}\Lambda)^{-1} \cdot q^{(\tau^{-1}+1)(s-1/2)^2/2}.$$

Initial values of $L^{1/(\tau+1)}$ and $\bar{L}^{-\tau/(\tau+1)}$

Let L_0 and $\bar{L}_0$ denote the initial values $L|_{\mathbf{t}=\bar{\mathbf{t}}=\mathbf{0}}$ and $\bar{L}|_{\mathbf{t}=\bar{\mathbf{t}}=\mathbf{0}}$ of the Lax operators. We can compute the fractional powers

$$L_0^{1/(\tau+1)} = W_0 \Lambda^{1/(\tau+1)} W_0^{-1},$$
$$\bar{L}_0^{-\tau/(\tau+1)} = \bar{W}_0 \Lambda^{-\tau/(\tau+1)} \bar{W}_0^{-1}$$

with the aid of the foregoing expression of W_0 and $\bar{W}_0$. After some lengthy algebra, we can confirm that

$$L_0^{1/(\tau+1)} = -\bar{L}_0^{-\tau/(\tau+1)} = (1 - q^{(\tau+1)s-\tau-1/2}\Lambda^{-1})\Lambda^{1/(\tau+1)}.$$

End of proof

Both $L^{1/(\tau+1)}$ and $\bar{L}^{-\tau/(\tau+1)}$ satisfy Lax equations of the same form:

$$\frac{\partial L^{1/(\tau+1)}}{\partial t_k} = [B_k, L^{1/(\tau+1)}], \quad \frac{\partial L^{1/(\tau+1)}}{\partial \bar{t}_k} = [\bar{B}_k, L^{1/(\tau+1)}],$$

$$\frac{\partial \bar{L}^{-\tau/(\tau+1)}}{\partial t_k} = [B_k, \bar{L}^{-\tau/(\tau+1)}], \quad \frac{\partial \bar{L}^{-\tau/(\tau+1)}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}^{-\tau/(\tau+1)}].$$

Consequently, since $L^{1/(\tau+1)} = \bar{L}^{-\tau/(\tau+1)}$ at $\mathbf{t} = \bar{\mathbf{t}} = \mathbf{0}$, we can conclude that $L^{1/(\tau+1)} = \bar{L}^{-\tau/(\tau+1)}$ at all times.