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A New Approach to the Self-Dual Yang-Mills Fquations II

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Introduction.

In recent years many mathematicians have come to be interested in the (self-dual) Yang-Mills equations. This is mainly because of profound geometric structure inherent in this nonlinear system, which now becomes an important subject of differential geometry and topology; see, for example, the book of Freed and Uhlenbeck [1] and references cited therein. This nonlinear system however has another significant property—complete integrability. This means that various techniques developed in the study of so called completely integrable systems, such as the celebrated KdV equation, can also be applied to the self-dual Yang-Mills equations. Among these techniques, in particular, the Riemann-Hilbert problem method, whose earliest application to the self-dual Yang-Mills equations goes back to the work of Ward [2] and Zakharov, Shabat [3], has grown up to a very powerful tool; see, for example, Ueno, Nakamura [4], Chau [5], Wu [6] and references cited therein.

In [7] I proposed an alternative method also based on the viewpoint of complete integrability. The main tool used there is an infinite system of differential equations whose unknown functions take their values from an infinite dimensional Grassmann manifold. This method has its origin in the work of Sato [8] who described nonlinear equations related to soliton phenomena as dynamical systems in an infinite dimensional Grassmann manifold.

The present paper is intended to supplement my previous paper by showing further developments as well as the background of the ideas presented there. In Sect. 1 we briefly review the previous paper. Sect. 2 is an introduction to the Grassmann manifold method. In order to illustrate basic ideas, the Riccati equation and its matrix analogues are discussed in detail as examples, and the structure of Grassmann manifolds hidden in these equations is revealed. Based on these arguments, the meaning of the results in [7, 8] is clarified. Sect. 3 deals with the concept of formal loop groups. This provides an algebraic analogue of the Riemann-Hilbert problem, and leads to group-theoretical reconstruction of the results obtained with the Grassmann manifold method. In Sect. 4 we present unified description of the two distinct sets of Riemann-Hilbert transformations introduced by Chau [5] and Wu [6]. The group-theoretical tools developed in Sect. 3 are shown to be also useful in these arguements. In Sect. 5 we seek for directions of further progress of our approach.

§ 1. Brief review of the previous paper.

1.1. Self-dual Yang-Mills equations. In what follows we obey the same notation as used in [7]. In order to see the complete integrability of the self-dual Yang-Mills equations, one needs to consider the equations in complex domains in $\boldsymbol{C}^4$ rather than in real ones. With an appropriate choice of coordinates $(y, z, \bar{y}, \bar{z})$ in $\boldsymbol{C}^4$, the equations can be written as:

$$(1.1) \qquad [\nabla_y, \nabla_z]=0, \quad [\nabla_{\bar{y}}, \nabla_{\bar{z}}]=0, \quad [\nabla_y, \nabla_{\bar{y}}]+[\nabla_z, \nabla_{\bar{z}}]=0,$$

where $\nabla_u=\partial_u+A_u$ $(u=y, z, \bar{y}, \bar{z}, \partial_u=\partial/\partial u)$ denote covariant derivative operators with $\mathfrak{gl}(r, \boldsymbol{C})$-valued connection coefficients (gauge potentials) $A_u=A_u(y, z, \bar{y}, \bar{z})$, and the sign $[\,,\,]$ the commutator $[X, Y]=XY-YX$. Gauge potentials are the unknown functions of this nonlinear system, and we here only consider the case in which $r\geqq 2$. In the above notation gauge transformations can be expressed as:

$$(1.2) \qquad \nabla_u \longrightarrow g^{-1}\circ\nabla_u\circ g=\partial_u+g^{-1}A_u g+g^{-1}\partial_u g, \quad \text{where } g=g(y, z, \bar{y}, \bar{z})$$

is a $\mathrm{GL}(r, \boldsymbol{C})$-valued function.

Applying an appropriate gauge transformation one may eliminate two of the four gauge potentials, say A_y and A_z, so that without loss of generaility one may assume that

$$(1.3) \qquad \nabla_y=\partial_y, \quad \nabla_z=\partial_z, \quad \nabla_{\bar{y}}=\partial_{\bar{y}}+A_{\bar{y}}, \quad \nabla_{\bar{z}}=\partial_{\bar{z}}+A_{\bar{z}}.$$

Thus Eqs. (1.1) reduce to:

$$(1.4) \qquad \partial_{\bar{y}}A_{\bar{z}}-\partial_{\bar{z}}A_{\bar{y}}+[A_{\bar{y}}, A_{\bar{z}}]=0, \qquad \partial_y A_{\bar{y}}+\partial_z A_{\bar{z}}=0.$$

1.2. Infinite system. The object which played a central role in [7] is an infinite system of equations that "*dominates*" Eqs. (1.4) rather than Eqs. (1.4) themselves. This means that there is an surjective map (a "*dominant*" map) from the solution space of this infinite system onto that of Eqs. (1.4). This infinite system has an infinite number of $\mathfrak{gl}(r, \boldsymbol{C})$-valued unknown functions ξ_{ij} indexed by a pair of integers i and j $(-\infty<i<\infty,\ j<0)$, and takes the following form:

$$(1.5a) \qquad -\partial_y\xi_{i+1,j}+\partial_{\bar{z}}\xi_{ij}+\xi_{i,-1}\partial_y\xi_{0,j}=0,$$

$$\partial_z\xi_{i+1,j}+\partial_{\bar{y}}\xi_{ij}-\xi_{i,-1}\partial_z\xi_{0,j}=0 \qquad (-\infty<i<\infty,\ j<0),$$

$$(1.5b) \qquad \xi_{i+1,j}=\xi_{i,j-1}+\xi_{i,-1}\xi_{0,j} \qquad (-\infty<i<\infty,\ j<0),$$

(1.5c) $$\xi_{ij}=\delta_{ij}1_r \qquad (i<0,\ j<0),$$

where δ_{ij} denotes the Kronecker delta and 1_r the $r\times r$ unit matrix. The above infinite system "dominates" Eqs. (1.4) by the following equations:

(1.6) $$A_{\bar{y}}=\partial_z\xi_{0,-1}, \qquad A_{\bar{z}}=-\partial_y\xi_{0,-1}.$$

This fact can be rigorously proved for both formal power series solutions (i.e. solutions with $A_u\in\mathfrak{gl}(r, \boldsymbol{C}[[y, z, \bar{y}, \bar{z}]])$ and local holomorphic solutions; see Sect. 1 of [7].

1.3. Infinite matrix expression. The characteristics of Eqs. (1.4) can be most neatly understood when written in a matrix form. Indeed, the description of solutions and transformation groups presented in [7] deeply depends on this matrix form. Furthermore, this is just the place where the concept of Grassmann manifolds occurs, as we shall see in later sections. This matrix expression of Eqs. (1.4) takes the following form:

(1.7a) $$(-\Lambda\partial_y+\partial_{\bar{z}})\boldsymbol{\xi}=\boldsymbol{\xi}A, \qquad (\Lambda\partial_z+\partial_{\bar{y}})\boldsymbol{\xi}=\boldsymbol{\xi}B,$$

(1.7b) $$\Lambda\boldsymbol{\xi}=\boldsymbol{\xi}C,$$

(1.7c) $$\boldsymbol{\xi}_{(-)}=\mathbf{1},$$

where $\mathbf{1}$ is the $\infty\times\infty$ unit matrix and $\boldsymbol{\xi}$, $\boldsymbol{\xi}_{(-)}$, A, B, C and Λ denote the following infinite matrices:

(1.8) $$\boldsymbol{\xi}=(\xi_{ij})_{i\in\boldsymbol{Z},\, j<0}, \qquad \boldsymbol{\xi}_{(-)}=(\xi_{ij})_{i,\, j<0},$$

(1.9) $$A=(-\partial_y\xi_{i+1,\, j})_{i,\, j<0}=\begin{pmatrix} 0 \\ (-\partial_y\xi_{0,\, j})_{j<0}\end{pmatrix},$$

$$B=(\partial_z\xi_{i+1,\, j})_{i,\, j<0}=\begin{pmatrix} 0 \\ (\partial_z\xi_{0,\, j})_{j<0}\end{pmatrix},$$

$$C=(\xi_{i+1,\, j})_{i,\, j<0}=\begin{pmatrix} \mathbf{1} \\ (\xi_{0,\, j})_{j<0}\end{pmatrix},$$

(1.10) $$\Lambda=(\delta_{i+1,\, j}1_r)_{i,\, j\in\boldsymbol{Z}}, \qquad \boldsymbol{Z}=\text{the totality of integers}.$$

In these formulas, just as in the usual notation for finite matrices, the indices i and j indicate the rows and column where the assigned component is to be placed. For example:

$$\boldsymbol{\xi}=\begin{pmatrix} & -2 & -0 & \\ \cdots & \cdots & \cdots & \\ \cdots & \xi_{-1-2} & \xi_{-1-2} & -1 \\ \cdots & \xi_{0-2} & \xi_{0-1} & 0 \\ \cdots & \xi_{1-2} & \xi_{1-1} & 1 \\ \cdots & \cdots & \cdots & \end{pmatrix}, \qquad \boldsymbol{\xi}_{(-)}=\begin{pmatrix} & -2 & -1 & \\ \cdots & \cdots & \cdots & \\ \cdots & \xi_{-2-2} & \xi_{-2-1} & -2 \\ \cdots & \xi_{-1-2} & \xi_{-1-1} & -1 \end{pmatrix},$$

$$\Lambda=\begin{pmatrix} -1 & 0 & 1 & & \\ \ddots & \ddots & & 0 & \\ 0_r & 1_r & & & -1 \\ & 0_r & 1_r & & 0 \\ & & 0_r & \ddots & 1 \\ 0 & & & \ddots & \end{pmatrix}, \qquad 0_r=\text{the } r\times r \text{ null matrix.}$$

1.4. Description of solutions. Any solution $\boldsymbol{\xi}$ of Eqs. (1.7) is uniquely determined by its initial value $\boldsymbol{\xi}^{(\mathrm{in})}=\boldsymbol{\xi}|_{\bar{y}=\bar{z}=0}$, which therefore can be used as a label for each solution. One of the main results in [7] is an explicit formula that reconstructs $\boldsymbol{\xi}$ from $\boldsymbol{\xi}^{(\mathrm{in})}$ using "*linear algebra of infinite matrices*". Note that the initial value $\boldsymbol{\xi}^{(\mathrm{in})}=(\xi_{ij}^{(\mathrm{in})})_{i\in\mathbf{Z},j<0}$ may be given arbitrarily except that it satisfies the algebraic part of Eqs. (1.7):

(1.11a) $$\Lambda\boldsymbol{\xi}^{(\mathrm{in})}=\boldsymbol{\xi}^{(\mathrm{in})}C^{(\mathrm{in})}, \quad \text{where} \quad C^{(\mathrm{in})}=(\xi_{i+1,j}^{(\mathrm{in})})_{i,j<0},$$

(1.11b) $$\boldsymbol{\xi}_{(-)}^{(\mathrm{in})}=\mathbf{1}, \quad \text{where} \quad \boldsymbol{\xi}_{(-)}^{(\mathrm{in})}=(\xi_{ij}^{(\mathrm{in})})_{i,j<0}.$$

Under these constraints $\boldsymbol{\xi}^{(\mathrm{in})}$ is uniquely determined if $\xi_{0,j}^{(\mathrm{in})}$ $(j<0)$ are assigned, whereas the latter components can be given arbitrarily (see Sect. 1.7). The reconstruction starts from the definition of two matrices $\tilde{\boldsymbol{\xi}}=(\tilde{\xi}_{ij})_{i\in\mathbf{Z},j<0}$ and $\tilde{\boldsymbol{\xi}}_{(-)}=(\tilde{\xi}_{ij})_{i,j<0}$:

(1.12) $$\tilde{\boldsymbol{\xi}}=\exp(\bar{z}\Lambda\partial_y-\bar{y}\Lambda\partial_z)\boldsymbol{\xi}^{(\mathrm{in})}=\sum_{k=0}^{\infty}(\bar{z}\Lambda\partial_y-\bar{y}\Lambda\partial_z)^k\boldsymbol{\xi}^{(\mathrm{in})}/k\,!.$$

If $\bar{y}$ and $\bar{z}$ are sufficiently small, the negative index part $\tilde{\boldsymbol{\xi}}_{(-)}$ is close to the unit matrix and may be expected to be invertible. Using its inverse matrix, the *reconstruction formula* reads:

(1.13) $$\boldsymbol{\xi}=\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}_{(-)}{}^{-1}.$$

The above procedure can be rigorously justified for both formal power series solutions and local holomorpgic solutions; see Sect. 2 of [7].

1.5. Description of transformations. "*Linear algebra of infinite matrices*" also works as a machinery for the construction of transformations that act on the solution space of Eqs. (1.7). As the data for such a transformation, we consider a matrix $P=(p_{ij})_{i,j\in\mathbf{Z}}$ that depends on $(y, z, \bar{y}, \bar{z})$ and satisfies:

(1.14) $$[-\Lambda\partial_y+\partial_{\bar{z}}, P]=0, \quad [\Lambda\partial_z+\partial_{\bar{y}}, P]=0, \quad [\Lambda, P]=0.$$

The last condition in (1.14) shows that P can be written as

(1.15) $$P=(p_{i-j})_{i,\in Z},$$

and if one define a Laurent series $p(\lambda)$ in a new variable λ as

(1.16) $$p(\lambda)=\sum_{j=-\infty}^{\infty} p_j\lambda^j,$$

then the other two conditions in (1.14) are equivalent to:

(1.17) $$(-\lambda\partial_y+\partial_{\bar{z}})p(\lambda)=0, \qquad (\lambda\partial_z+\partial_{\bar{y}})p(\lambda)=0.$$

This data $p(\lambda)$ is just the same as required in the Riemann-Hilbert problem method; see Sect. 4.1. In the framework using the matrix ξ, a transformation of solutions can be obtained as follows. First consider the product matrix $P\xi= ((P\xi)_{ij})_{i\in Z, j<0}$, deviding it into two blocks:

(1.18) $$P\xi=\begin{pmatrix}(P\xi)_{(-)}\\(P\xi)_{(+)}\end{pmatrix}, \quad (P\xi)_{(-)}=((P\xi)_{ij})_{i,j<0}, \quad (P\xi)_{(+)}=((P\xi)_{ij})_{i\geq 0, j<0}.$$

If this product matrix makes sense and P is sufficiently close to the unit matrix, the negative index part $(P\tilde{\xi})_{(-)}$ will be invertible. Again, using its inverse matrix, we define

(1.19) $$P\circ\xi=P\xi(P\xi)_{(-)}^{-1},$$

which becomes *a new solution* of Eqs. (1.7). Some cases where the above construction can be justified are discussed in Sect. 3 of [7].

1.6. Grassmann manifold. The above construction of solutions and transformations is very similar to each other. This reflects the structure of an infinite dimensional Grassmann manifold inherent in Eqs. (1.7). This point will be discussed in detail in Sect. 2.

1.7. Connection with inverse scattering. Usually when one tries to apply the Riemann-Hilbert problem method or any other technique of inverse scattering to a nonlinear system, the most crucial postulate is the presence of a linear system whose integrability conditions coincide with the nonlinear equation in question. For the self-dual Yang-Mills equations such a linear system is:

(1.20) $$(-\lambda\nabla_y+\nabla_{\bar{z}})w=0, \qquad (\lambda\nabla_z+\nabla_{\bar{y}})w=0,$$

where $w=w(y, z, \bar{y}, \bar{z}, \lambda)$ denotes a $GL(r, \boldsymbol{C})$-valued unknown function and λ a parameter moving in the Riemann sphere $\boldsymbol{P}^1$ (see [3-7] and references cited therein). By changing w as $w\to g^{-1}w$ according to gauge transformation (1.2), this linear system becomes gauge-invariant.

In order to connect Eqs. (1.20) with our framework, we fix the gauge as in (1.3), and try to eliminate the gauge potentials from Eqs. (1.20). This leads, as we shall see soon, to an infinite nonlinear system of differential equations to be satisfied by the Laurent coefficients of w arround $\lambda=\infty$. Indeed, because of the gauge-fixing we have adopted, Eqs. (1.20) become:

$$(1.21)\qquad (-\lambda\partial_y+\partial_{\bar z}+A_{\bar z})w=0, \qquad (\lambda\partial_z+\partial_{\bar y}+A_{\bar y})w=0,$$

and one may choose w to have a Laurent expansion arround $\lambda=\infty$ of the following form (see the above references):

$$(1.22)\qquad w=\sum_{n=0}^{\infty} w_n\lambda^{-n}, \qquad w_0=1_r.$$

Putting this expression into Eqs. (1.21) and examining the λ^0 term, one finds:

$$(1.23)\qquad A_{\bar y}=-\partial_z w_1, \qquad A_{\bar z}=\partial_y w_1.$$

Taking them back into Eqs. (1.21), one finally obtains:

$$(1.24a)\qquad (-\lambda\partial_y+\partial_{\bar z}+(\partial_y w_1))w=0, \qquad (\lambda\partial_z+\partial_{\bar y}-(\partial_z w_1))w=0,$$

or equivalently,

$$(1.24b)\qquad -\partial_y w_{n+1}+\partial_{\bar z}w_n+(\partial_y w_1)w_n=0, \qquad \partial_z w_{n+1}-\partial_{\bar y}w_n-(\partial_z w_1)w_n=0.$$

The last equations are what we have sought for; Eqs. (1.24) become equivalent to Eqs. (1.7) when connected by the following relation (see Sect. 1 of [7]):

$$(1.25)\qquad \xi_{0,j}=-w_{-j} \qquad (j<0).$$

This means that *any solution of Eqs.* (1.7) *yields via* (1.25) *a solution of Eqs.* (1.25) *and vice versa.* However the converse will need some more comments, because given a solution of Eqs. (1.24), Eq. (1.25) in itslf gives only a part of the whole components of $\boldsymbol{\xi}$. In fact, the components other than those appearing in (1.25) are determined by the algebraic constraints (1.7b) and (1.7c) in Eqs. (1.7). In other words, Eq. (1.25) defines a one-to-one correspondence between w and $\boldsymbol{\xi}$ with constraints (1.7b) and (1.7c). Note that *this correspondence is purely algebraic.* In particular it holds also for the initial values $w^{(\mathrm{in})}=w|_{\bar y=\bar z=0}$ and $\boldsymbol{\xi}^{(\mathrm{in})}=\boldsymbol{\xi}|_{\bar y=\bar z=0}$; this explains what we remarked in Sect. 1.4 concerning the arbitrariness of $\boldsymbol{\xi}^{(\mathrm{in})}$. The equivalence of Eqs. (1.7) and (1.24) means the equivalence of differential equations (1.7a) and (1.24) under the algebraic one-to-one correspondence of w and $\boldsymbol{\xi}$ as above.

§ 2. Matrix Riccati equations and Grassmann manifolds.

2.1. Matrix Riccati equations. As well known, the ordinary Riccati equation

(2.1) $$\frac{du}{dt}+u^2+a=0, \quad \text{where } a=a(t) \text{ is a given function,}$$

can be transformed into the linear equation

(2.2) $$\frac{d^2v}{d^2t}+av=0$$

by the change of dependent variable as

(2.3) $$u=v^{-1}\frac{dv}{dt}.$$

This machinary combining linear and nonlinear equations can be further generalized to yield the concept of matrix Riccati equations; see, for example, Chau [5], Winternitz [9], Harnad, Saint-Aubin, Shnider [10] and references cited therein. In what follows we briefly review the basic features of matrix Riccati equations.

Let us start from a matrix linear system

(2.4) $$\frac{d\xi}{dt}=L\xi, \quad \text{where } L=L(t) \text{ is an } N\times N \text{ matrix-valued}$$

function with $N\geqq 2$.

We devide N into the sum of two positive integers m and n as $N=m+n$, and in order to make the representation similar to that of Sect. 1, label the rows and columns of L and ξ with integers from $-m$ through $n-1$. Thus:

(2.4) $$\xi=(\xi_i)_{-m\leq i<n}, \qquad L=(l_{ij})_{-m\leq i,j<n}.$$

We furthermore assume that we have a set of m linearly independent column vector solutions $\tilde{\xi}^{(-m)}, \cdots, \tilde{\xi}^{(-1)}$ of Eq. (2.4) for which the $m\times m$ matrix

(2.6) $$\tilde{\boldsymbol{\xi}}_{(-)}=(\tilde{\xi}_i^{(j)})_{-m\leq i,j<0} \quad \text{is invertible.}$$

This matrix is exactly the negative index part of the $N\times m$ matrix

(2.7) $$\tilde{\boldsymbol{\xi}}=(\tilde{\xi}_i^{(j)})_{-m\leq i<n,\,-m\leq j<0}.$$

Now we consider *what equations the* $n\times m$ *matrix*

(2.8) $$\boldsymbol{G}=\tilde{\boldsymbol{\xi}}_{(+)}\tilde{\boldsymbol{\xi}}_{(-)}^{-1}, \qquad \text{where } \tilde{\boldsymbol{\xi}}_{(+)}=(\tilde{\xi}_i^{(j)})_{0\leq i<n,\,-m\leq j<0},$$

will satisfy. The answer is:

(2.9) $$\frac{d\boldsymbol{G}}{dt}=L_2+L_3\boldsymbol{G}-\boldsymbol{G}L_1-\boldsymbol{G}L_4\boldsymbol{G},$$

where $L_1, \cdots, L_4$ denote the four blocks obtained by deviding L with respect to the signature of the row and column indices:

(2.10) $$L=\begin{pmatrix} L_1 & L_4 \\ L_2 & L_3 \end{pmatrix}, \quad L_2=(l_{ij})_{-m\le i,\, j<0}, \quad \text{etc.}$$

When $m=n=1$ and $L=\begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}$, Eq. (2.9) becomes the original Riccati equation (2.1). In general, Eq. (2.9) is called a *matrix Riccati equation*. In the matrix case the role of (2.3) combining linear and nonlinear equations is played by (2.8).

2.2. Derivation of (2.9). Eq. (2.9) may be checked by direct calculation, but a more systematic way is to seek for a differential equations to be satisfied by the $N\times m$ matrix

(2.11) $$\begin{pmatrix} \mathbf{1} \\ \boldsymbol{G} \end{pmatrix}=\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}_{(-)}^{-1}, \quad \text{where } \mathbf{1} \text{ denotes the } m\times m \text{ unit matrix.}$$

In order to derive such an equation, note that from the definition of $\tilde{\boldsymbol{\xi}}$,

(2.12) $$\frac{d\tilde{\boldsymbol{\xi}}}{dt}=L\tilde{\boldsymbol{\xi}}.$$

From this equation,

$$\frac{d(\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}_{(-)}^{-1})}{dt}=L\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}_{(-)}^{-1}+\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}_{(-)}^{-1}A, \qquad A=-\frac{d\tilde{\boldsymbol{\xi}}_{(-)}}{dt}\tilde{\boldsymbol{\xi}}_{(-)}^{-1},$$

therefore,

(2.13) $$\frac{d}{dt}\begin{pmatrix} \mathbf{1} \\ \boldsymbol{G} \end{pmatrix}=L\begin{pmatrix} \mathbf{1} \\ \boldsymbol{G} \end{pmatrix}+\begin{pmatrix} \mathbf{1} \\ \boldsymbol{G} \end{pmatrix}A.$$

The remaining problem is to find an explicit expression of A in terms of $\boldsymbol{G}$, but this can be done quickly becuase Eq. (2.13) splits into the following two equations:

(2.14) $$0=L_1+L_4\boldsymbol{G}+A, \qquad \frac{d\boldsymbol{G}}{dt}=L_2+L_3\boldsymbol{G}+\boldsymbol{G}A.$$

Eliminating A from these equations one obtains Eq. (2.12).

2.3. Invariance property of an equation that reduces to Eqs. (2.12) and (2.13) in special cases. Generalizing the relation (2.11) that connects the two differential equations (2.12) and (2.13), let us consider

(2.15) $$\boldsymbol{\xi}=\tilde{\boldsymbol{\xi}}K, \quad \text{where } K=K(t) \text{ is an } m\times m \text{ invertible matrix.}$$

Immediately we see that $\boldsymbol{\xi}$ satisfies a differential equation of the form

(2.16) $$\frac{d\boldsymbol{\xi}}{dt}=L\boldsymbol{\xi}+\boldsymbol{\xi}A, \quad \text{where } A=A(t) \text{ is an } m\times m \text{ matrix.}$$

The last term A is in general different from that one in (2.13). Of course if

$K=\mathbf{1}$ then Eq. (2.16) reduces to Eq. (2.12), whereas if $K=\tilde{\boldsymbol{\xi}}_{(-)}^{-1}$ it reduces to Eq. (2.13). Note that as in (2.11), $\boldsymbol{G}$ can be reconstructed from $\boldsymbol{\xi}$ as

(2.17) $$\begin{pmatrix}\mathbf{1}\\ \boldsymbol{G}\end{pmatrix}=\boldsymbol{\xi}\boldsymbol{\xi}_{(-)}^{-1},\quad \text{where } \boldsymbol{\xi}_{(-)} \text{ is the negative index part of } \boldsymbol{\xi}.$$

What is remarkable here is the following invariance property of Eq. (2.16). Note that changing the above K just corresponds to transforming $\boldsymbol{\xi}$ as:

(2.18) $$\boldsymbol{\xi}\longrightarrow\boldsymbol{\xi}H,\quad \text{where } H=H(t) \text{ is } m\times m \text{ and invertible.}$$

If A transforms simultaneously as

(2.19) $$A\longrightarrow H^{-1}AH+H^{-1}\frac{dH}{dt},$$

then Eq. (2.16) preserves its form, i.e. *Eq.* (2.16) *is invariant under such transformations of* $\boldsymbol{\xi}$ *and* A. It is interesting that this transformation takes the same form as gauge transformations.

2.4. Dynamical system in a Grassmann manifold. Geometrically, the above invariance property of Eq. (2.16) is connected with a Grassmann manifold, which in the notation of Sato [8] is denoted by $\mathrm{GM}(m, n)$. $\mathrm{GM}(m, n)$ is by definition the manifold formed by all m-dimensional linear subspaces in $\boldsymbol{C}^{m+n}$. Another equivalent expression is:

(2.20) $$\mathrm{GM}(m, n)=\{\boldsymbol{\xi};\ \text{constant } N\times m \text{ matrices with rank } \boldsymbol{\xi}=m\}/\sim,$$

where the sign $/\sim$ means that for any $H\in\mathrm{GL}(r, \boldsymbol{C})$ two matrices $\boldsymbol{\xi}$ and $H\boldsymbol{\xi}$ are identified to each other and regarded as representing the same point in $\mathrm{GM}(m, n)$. As one sees immediately from the previous arguments, *the three matrices* $\tilde{\boldsymbol{\xi}}=\tilde{\boldsymbol{\xi}}(t)$, $\begin{pmatrix}\mathbf{1}\\ \boldsymbol{G}\end{pmatrix}=\begin{pmatrix}\mathbf{1}\\ \boldsymbol{G}(t)\end{pmatrix}$ *and* $\boldsymbol{\xi}=\boldsymbol{\xi}(t)$ *all correspond to the same point in* $\mathrm{GM}(m, n)$ *that moves as* t *evolves*; thus a dynamical system is defined in $\mathrm{GM}(m, n)$.

From the above geometric viewpoint, matrix Riccati equation (2.9) can be considered an *affine coordinate expression* of the equation of motion of the above dynamical system in an open subset of $\mathrm{GM}(m, n)$. In the notation of Sato [8] this open subset is denoted by $\mathrm{GM}^{\phi}(m, n)$, where ϕ is the empty Young diagram. On $\mathrm{GM}^{\phi}(m, n)$ one has an affine coordinate system with coordinate map:

(2.21) $$\mathrm{GM}^{\phi}(m, n)\longrightarrow \boldsymbol{C}^{mn}$$
$$\boldsymbol{\xi}=\begin{pmatrix}\boldsymbol{\xi}_{(-)}\\ \boldsymbol{\xi}_{(+)}\end{pmatrix}\longrightarrow\boldsymbol{\xi}_{(+)}\boldsymbol{\xi}_{(-)}^{-1}$$

and if this coordinate map is applied to $\boldsymbol{\xi}=\boldsymbol{\xi}(t)$, or equivalently to $\tilde{\boldsymbol{\xi}}=\tilde{\boldsymbol{\xi}}(t)$, the image becomes $\boldsymbol{G}=\boldsymbol{G}(t)$, the unknown function of Eq. (2.9). In other words, the unknown function of matrix Riccati equation (2.9) takes its values essentially from $\mathrm{GM}^{\phi}(m, n)$. If the trajectory of motion intersects with the complement $\mathrm{GM}(m, n)-\mathrm{GM}^{\phi}(m, n)$, the intersection causes singularities of $\boldsymbol{G}=\boldsymbol{G}(t)$, but we shall not go further into the problem of singularities here.

Roughly speaking, the description of nonlinear equations of soliton type by Sato [8] can be derived from the above picture by letting $m, n\to\infty$.

2.5. Interpretation of self-dual Yang-Mills equations. Even apart from the presence of algebraic constraint (1.7b) and the infinite dimensionality of matrices, Eqs. (1.7) have forms considerably different from Eq. (2.13). In particular, they do not fall into the cases discussed by Sato [8]. Nevertheless, a similar machinary also works in the case of Eqs. (1.7). This is due to the fact that *if we forget Eq.* (1.7c) *for the moment and focus our attention on Eqs.* (1.7a) *and* (1.7b), *these equations are invariant under the transformations*

(2.22)
$$\begin{aligned} &\xi \longrightarrow \xi H, \\ &A \longrightarrow H^{-1}AH-H^{-1}C\partial_y H+H^{-1}\partial_{\bar z} H, \\ &B \longrightarrow H^{-1}BH+H^{-1}C\partial_z H+H^{-1}\partial_{\bar y} H, \\ &C \longrightarrow H^{-1}CH, \end{aligned}$$

where $H=H(y, z, \bar y, \bar z)$ *is an* $\infty\times\infty$ *invertible matrix.*

Eq. (1.7c) is fulfilled when we transform the above equations into the $\boldsymbol{G}$-picture as (2.17) shows.

Bearing in mind this invariance property, let us consider the meaning of formulas (1.13) and (1.19). For $\tilde\xi$, one can show the equations

(2.23)
$$(-\Lambda\partial_y+\partial_{\bar z})\tilde\xi=0, \quad (\Lambda\partial_z+\partial_{\bar y})\tilde\xi=0, \quad \Lambda\tilde\xi=\tilde\xi\tilde C,$$

whre $\tilde C$ is an $\infty\times\infty$ invertible matrix whose explicit form is given in Sect. 2.2 of [7]. For $P\xi$, one has

(2.24)
$$(-\Lambda\partial_y+\partial_{\bar z})(P\xi)=P\xi A, \quad (\Lambda\partial_y+\partial_{\bar z})(P\xi)=P\xi B, \quad \Lambda P\xi=P\xi C.$$

Eqs. (2.23) and (2.24) both have the same forms as Eqs. (1.7) except for Eq. (1.7c). Therefore multiplying $\tilde\xi$ and $P\xi$ by their negative index part from the right side, one can change into the $\boldsymbol{G}$-picture; this leads to formulas (1.13) and (1.19).

The above arguments clearly show: *The manifold from which the unknown functions of Eqs.* (1.7) *take their values is essentially an infinite dimensional Grassmann manifold.*

2.6. Special solutions that correspond to the semi-infinite case $m<\infty$, $n=\infty$. We here illustrate the construction of solutions reviewed in Sect. 1.5 in the case where for some integer $m \geqq 1$

$$\xi_{ij}^{(\mathrm{in})}=0 \quad \text{for} \quad i \geqq 0 \quad \text{and} \quad j<-m. \tag{2.25}$$

In this case $\tilde{\boldsymbol{\xi}}$ takes the following form:

$$\tilde{\boldsymbol{\xi}}=\left(\begin{array}{c|c} \ddots \ \ 1 & * \\ \hline 0 & \tilde{\boldsymbol{\xi}}[m] \end{array}\right), \quad \text{where} \quad \tilde{\boldsymbol{\xi}}[m]=(\tilde{\xi}_{ij})_{-m\leq i<\infty,\,-m\leq j<0}. \tag{2.26}$$

Therefore

$$\tilde{\boldsymbol{\xi}}_{(-)}{}^{-1}=\left(\begin{array}{c|c} \ddots \ \ 1 & ** \\ \hline 0 & \tilde{\boldsymbol{\xi}}[m]_{(-)}{}^{-1} \end{array}\right), \quad \text{where} \quad \tilde{\boldsymbol{\xi}}[m]_{(-)}=(\tilde{\xi}_{ij})_{-m\leq i,\,j<0}. \tag{2.27}$$

Thus:

$$\boldsymbol{\xi}=\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}_{(-)}{}^{-1}=\left(\begin{array}{c|c} \ddots \ \ 1 & 0 \\ \hline 0 & \tilde{\boldsymbol{\xi}}[m]\tilde{\boldsymbol{\xi}}[m]_{(-)}{}^{-1} \end{array}\right), \tag{2.28}$$

In particular, one sees that

$$\xi_{ij}=0 \quad \text{for} \quad i \geqq 0 \quad \text{and} \quad j<-m, \tag{2.29}$$

$$\boldsymbol{\xi}[m]\equiv(\xi_{ij})_{-m\leq i<\infty,\,-m\leq j<0}=\tilde{\boldsymbol{\xi}}[m]\tilde{\boldsymbol{\xi}}[m]_{(-)}{}^{-1}. \tag{2.30}$$

Besides, it is not hard to see that

$$\tilde{\boldsymbol{\xi}}[m]=\exp(\bar{z}\Lambda_m\partial_y-\bar{y}\Lambda_m\partial_z)\boldsymbol{\xi}^{(\mathrm{in})}[m], \quad \text{where} \tag{2.31}$$

$$\Lambda_m=(\delta_{i+1,j}1_r)_{-m\leq i,\,j<\infty}, \qquad \boldsymbol{\xi}^{(\mathrm{in})}[m]=(\xi_{ij}^{(\mathrm{in})})_{-m\leq i<\infty,\,-m\leq j<0}.$$

The last two formulas clearly show that we may reformulate the whole calculation by only using *the truncated matrices* $\boldsymbol{\xi}^{(\mathrm{in})}[m]$, $\tilde{\boldsymbol{\xi}}[m]$ *and* $\boldsymbol{\xi}[m]$. This construction enables us to generate in principle all the solutions that satisfy condition (2.29). We can also charcterize these solutions in terms of w (see Sect. 1.7). Indeed, as can be quickly checked by recalling (1.25) and going back to (1.5b) and (1.5c), condition (2.29) is equivalent to w being a polynomial in λ^{-1} of degree $\leqq m$:

$$w_j=0 \quad \text{for} \quad j>m, \tag{2.32}$$

and the same is true for $\boldsymbol{\xi}^{(\mathrm{in})}$ and $w^{(\mathrm{in})}$.

The solutions thus obtained include some interesting classes of special solutions. An example is that discussed in [11]; its original construction used a version of algebro-geometric methods in soliton theory. Another example is formed by solutions for which ξ_{ij}'s, or equivalently w_j's, are all rational in $(y, z, \bar{y}, \bar{z})$. These solutions can be characterized by the rationality of their initial values in (y, z). [*Proof*: From Eq. (1.11) of [7] one can show after some

this part is incorrect

calculation the following identity:

$$\sum_{i=-\infty}^{\infty} \xi_{ij}^{(\mathrm{in})}\lambda^{-i}=\Big(\sum_{i=0}^{\infty} w_i^{(\mathrm{in})}\lambda^{-i}\Big)^{-1}\Big(\sum_{i=0}^{-j-1} w_i^{(\mathrm{in})}\lambda^{-i-j}\Big). \tag{2.33}$$

Because of this, if $w_j^{(\mathrm{in})}$'s are rational and vanish for $j>m$, the right hand side of (2.33) becomes rational in (y, z, λ). On the other hand, from the construction of $\tilde{\xi}_{ij}$

$$\sum_{i=-\infty}^{\infty} \tilde{\xi}_{ij}\lambda^{-i}=\exp(\bar{z}\lambda\partial_y-\bar{y}\lambda\partial_z)\sum_{i=-\infty}^{\infty} \xi_{ij}^{(\mathrm{in})}\lambda^{-i}, \tag{2.34}$$

therefore $\tilde{\xi}_{ij}$'s are also rational in $(y, z, \bar{y}, \bar{z})$. Thus, finally using formula (2.30), one concludes that ξ_{ij}'s are rational in $(y, z, \bar{y}, \bar{z})$.] It seems likely that these rational solutions basically include all the instanton solutions of Atiyah, Drinfeld, Hitchin and Manin [12]; however their characerization in our framework is still an open problem.

2.7. Conclusion of this section. Starting from linear systems we derived matrix Riccati equations with quadratic nonlinearity. The nonlinearity of the self-dual Yang-Mills as well as soliton type equations turned out to have its origin in this quadratic nonlinearity, which geometrically reflects the curved feature of Grassmann manifolds. Tracing back the above process from linear sytems to nonlinear systems, we can find an ultimate form of linearization of these completely integrable systems.

§ 3. Formal loop groups.

3.1. From Riemann-Hilbert problem to its algebraic analogue. The $\boldsymbol{\xi}$-matrix formulation we have viewed is not the only possible approach to the self-dual Yang-Mills equations; as sketched in Sect. 4 of [7], an alternative approach which is in close relationship to the Riemann-Hilbert problem method can be developed. In what follows we shall discuss this in more detail.

In order to motivate our arguments, we start from brief review of the Riemann-Hilbert problem. Take a circle C centered at the origin in the Riemann sphere $\boldsymbol{P}^1$ with affine parameter λ, and let C_+ and C_- denote the connected components of the complement $\boldsymbol{P}^1-C$ for which $C_+\ni 0$ and $C_-\ni\infty$. The Riemann-Hilbert problem is stated as follows: For a given matrix valued, say $\mathrm{GL}(r, \boldsymbol{C})$-valued, analytic function $u(\lambda)$ defined on C, find a pair of $\mathrm{GL}(r, \boldsymbol{C})$-valued holomorphic functions $v(\lambda)$ and $w(\lambda)$ which are defined respectively in a neighborhood of $C\cup C_+$ and in a neighborhood of $C\cup C_-$ and satisfy the equations

$$u(\lambda)=w(\lambda)^{-1}v(\lambda) \qquad (\lambda\in C), \tag{3.1a}$$

$$w(\infty)=1_r. \tag{3.1b}$$

The second equation is a normalization condition which ensures the uniqueness of the solution if it exists. This problem can be reduced to an integral equation, and if $u(\lambda)$ is sufficiently close to the unit matrix, a unique solution exists; see [3-6] and references cited therein.

The above Riemann-Hilbert problem may be viewed as a grouptheoretical decomposition problem as follows. Let us define:

(3.2) $\mathcal{G}_C = \{u$; analytic maps $\lambda \to u(\lambda)$ from C to $\mathrm{GL}(r, \boldsymbol{C})\}$,

$\mathcal{N}_C = \{u \in \mathcal{G}_C$; extendable to a holomorphic map from C_- to $\mathrm{GL}(r, \boldsymbol{C})$ with $u(\infty) = 1_r\}$,

$\mathcal{P}_C = \{u \in \mathcal{G}_C$; extendable to a holomorphic map from C_+ to $\mathrm{GL}(r, \boldsymbol{C})\}$,

$\mathcal{G}_C$ forms a group with pointwise multiplication of maps, which is usually called a loop group; $\mathcal{N}_C$ and $\mathcal{P}_C$ become subgroups of $\mathcal{G}_C$. The Riemann-Hilbert problem means in terms of these groups the problem of decomposing an element u of $\mathcal{G}_C$ as:

(3.3) $$u = w^{-1} v, \quad w \in \mathcal{N}_C, \quad v \in \mathcal{P}_C.$$

If u is sufficiently close to the unit element 1_r of $\mathcal{G}_C$, as we mentioned above, such decomposition is actually possible and unique. By defining some appropriate topology to these groups, such decomposability will be formulated as the openness of the map $\mathcal{N}_C \times \mathcal{P}_C \to \mathcal{G}_C$ sending $(w, v) \in \mathcal{N}_C \times \mathcal{P}_C$ to $w^{-1}v$, but we shall not go further in this direction.

We shall below introduce algebraic analogues of loop groups, which we call *formal loop groups*, and formulate a similar decomposition problem. It will be shown that this decomposition problem always has a unique solution, thus the situations being much simpler for formal loop groups. These formal loop groups will be formed by formal Laurent series in one variable λ with coefficients taken from an associative filtered algebra over the field $\boldsymbol{C}$ of complex numbers or any other field; the algebraic part of the arguments below does not depend on the selection of base fields. Such abstract formulation will not only clarify the algebraic structures concerned, but also be advantageous to various applications not limited to the self-dual Yang-Mills equations; see Sect. 5.

3.2. Associative filtered algebras. Let R be an associative (but in general noncommutative) algebra over $\boldsymbol{C}$ which has a unit element 1 and a decreasing filtration $R = R_0 \supset R_1 \supset R_2 \supset \cdots$ of $\boldsymbol{C}$-vector subspaces with the following properties:

(3.4a) $R_m R_n \subset R_{m+n}$ *for any* m *and* n.

(3.4b) *For any sequence* $a_n \in R_n$ $(n \geqq 0)$ *there exists a unique element* $a \in R$

for which $a-\sum_{n=0}^{N}a_n\in R_{N+1}$ $(N\geqq 0)$; *in what follows this element* a *is denoted by* $\sum_{n=0}^{\infty}a_n$.

It follows immediately that:

(3.4c) *For any sequence* $a_n\in R_n$ $(n\geqq m)$, $\sum_{n=m}^{\infty}a_n$ *belongs to* R_m.

In addition, for simplifying the notation in later arguments we prolong the filtration by defining

$$R_n=R \quad \text{for} \quad n<0, \tag{3.4d}$$

which does not affect properties (3.4a)-(3.4c). An example of R is:

$$\begin{aligned} R&=\mathfrak{gl}(r, \boldsymbol{C}[[y, z, \bar{y}, \bar{z}]]) \\ &=\{\sum a_{ijkm}y^i z^j \bar{y}^k \bar{z}^m;\ a_{ijkm}\in\mathfrak{gl}(r, \boldsymbol{C})\ (i, j, k, m=0, 1, 2, \cdots)\}, \\ &\text{with } R_n=\{\sum a_{ijkm}y^i z^j \bar{y}^k \bar{z}^m;\ a_{ijkm}=0 \text{ if } i+j+k+m<n\}. \end{aligned} \tag{3.5}$$

In the terminology of abstract algebra the property stated in (3.4b) has an equivalent expression as:

(3.6) *The canonical homomorphism* $R\to$ proj. lim. R/R_n *induced by the quotient homomorphisms* $R\to R/R_n$ *is isomorphic.*

As for the invertibility of elements of R, we have the following result:

(3.7) *For any sequence* $a_n\in R_n$ $(n\geqq 0)$, $\sum_{n=0}^{\infty}a_n$ *is invertible in* R *if and only if* a_0 *is invertible in* R.

Proof. Suppose that a_0 is invertible. Since $a=a_0(1+\sum_{n=1}^{\infty}a_0^{-1}a_n)$, in order to prove the invertibility of a one has only to show the invertibility of the second factor of the right hand side. On the other hand, note that for any element r of R_1, $1+r$ is invertible; indeed its inverse element is given by the Neumann series as:

$$(1+r)^{-1}=1-r+r^2-\cdots=\sum_{n=0}^{\infty}(-r)^n \tag{3.8}$$

which actually becomes an elenent of R because of (3.3a) and (3.3b). The invertility of $1+\sum_{n=1}^{\infty}a_0^{-1}a_n$ follows from this fact. Conversely, suppose that $\sum_{n=0}^{\infty}a_n b=1$. Then $a_0=b^{-1}-\sum_{n=1}^{\infty}a_n b^{-1}$, the right hand side being invertible because of what we have just proved above. This proves (3.7).

3.3. Formal loop groups. For an associative filtered albegra R with properties (3.3), we define:

$$\begin{aligned} \mathcal{G}_R&=\{u=\sum u_n\lambda^n;\ u_n\in R_n\ (-\infty<n<\infty),\ u_0 \text{ is invertible in } R\}, \\ \mathcal{N}_R&=\{u=\sum u_n\lambda^n\in\mathcal{G}_R;\ u_n=0\ (n>0),\ u_0=1\}, \end{aligned} \tag{3.9}$$

$$\mathcal{P}_R=\{u=\sum u_n\lambda^n\in\mathcal{G}_R\,;\ u_n=0\ (n<0)\}.$$

We show below that $\mathcal{G}_R$ *forms a group with respect to the usual multiplication of Laurent series,* and that $\mathcal{N}_R$ *and* $\mathcal{P}_R$ *become subgroups of* $\mathcal{G}_R$.

Product. For two elements $u^{(1)}=\sum u_n^{(1)}\lambda^n$ and $u^{(2)}=\sum u_n^{(2)}\lambda^n$ of $\mathcal{G}_R$, we define their product as:

$$u^{(1)}u^{(2)}=\sum u_n\lambda^n,\quad \text{where } u_n=\sum_{m=-\infty}^{\infty}u_m^{(1)}u_{n-m}^{(2)}. \tag{3.10}$$

Then $u^{(1)}u^{(2)}$ *becomes an element of* $\mathcal{G}_R$.

Proof. From (3.4a) and (3.4d), one sees that

$$u_m^{(1)}u_{n-m}^{(2)}\in R_m\cap R_n \qquad \text{for } m\geqq 0,$$
$$u_m^{(1)}u_{n-m}^{(2)}\in R_{n-m}\cap R_n \qquad \text{for } m\leqq n,$$
$$u_m^{(1)}u_{n-m}^{(2)}\in R_n \qquad \text{otherwise.}$$

Therefore by virtue of (3.4b) and (3.4c) $\sum u_m^{(1)}u_{n-m}^{(2)}$ defines an element of R_n. Furthermore, because of the fact that

$$u_0=u_0^{(1)}u_0^{(2)}+\sum_{m=1}^{\infty}(u_m^{(1)}u_{-m}^{(2)}+u_{-m}^{(1)}u_m^{(2)}),$$
$$u_0^{(0)}u_0^{(2)} \qquad \text{is invertible,}$$
$$u_m^{(1)}u_{-m}^{(2)}+u_{-m}^{(1)}u_m^{(2)}\in R_m \qquad (m=1,\ 2,\ \cdots),$$

and (3.7), it turns out that u_0 is invertible. This proves (3.10).

Invertibility. *Any element of* $\mathcal{G}_R$ *has its inverse element in* $\mathcal{G}_R$.

Proof. For any element $u=\sum u_n\lambda^n$ we decompose it as:

$$u=u_-+u_+,\quad u_-=\sum_{n<0}u_n\lambda^n,\quad u_+=\sum_{n\geqq 0}u_n\lambda^n. \tag{3.11}$$

The second term u_+ itself belongs to $\mathcal{G}_R$, and it is not hard to check that u_+ is invertible in $\mathcal{G}_R$. Indeed, u_+ is a formal power series in λ with invertible leading term u_0, therefore its invertibility is due to well known facts; other conditions the coefficients of u_+^{-1} should satisfy can also be checked easily. Thus

$$u_+^{-1}\ \text{lies in } \mathcal{G}_R, \tag{3.12}$$

and one may write

$$u=(1+u_-u_+^{-1})u_+. \tag{3.13}$$

The remaining problem therefore is to prove the invertibility of the first factor on the right hand side of (3.13). This can be proved by using the Neumann series

(3.14) $$\sum_{m=0}^{\infty}(-u_-u_+^{-1})^m=1-u_-u_+^{-1}+(u_-u_+^{-1})^2-\cdots.$$

Indeed, writing each term of this Neumann series as

(3.15) $$(u_-u_+^{-1})^m=\sum_{n=-\infty}^{\infty}a_n^{(m)}\lambda^n,$$

one can easily check from (3.11) and (3.12) that

(3.16) $$a_n^{(m)}\in R_{n+m},$$

therefore because of (3.4b) and (3.4c) $\sum_{m=0}^{\infty}a_n^{(m)}$ becomes an element of R_n and, in particular, because of (3.7) it is invertible in R when $n=0$; thus one sees that Neumann series (3.14) belongs to $\mathcal{G}_R$ and gives an explicit expression of the inverse element of $1+u_-u_+^{-1}$. This proves the invetibility of u in $\mathcal{G}_R$.

3.4. Decomposition in $\mathcal{G}$. The main result of this section is:

(3.17) *Any element u of $\mathcal{G}_R$ can be uniquely decomposed as*

$$u=w^{-1}v,\quad w\in\mathcal{N}_R,\quad v\in\mathcal{P}_R.$$

We show below a proof of this result, which at the same time provides an explicit expression of the factors w and v. For the proof we again use "*linear algebra of infinite matrices*" that played crucial roles in the $\mathfrak{z}$-matrix formulation. As we shall see later (see Sect. 3. 6), the above decomposition is in close connection with the $\mathfrak{z}$-matrix formulation.

Proof of (3.17). **Step 1.** The uniqueness of v and w can be checked as follows. Suppose that another pair $w'\in\mathcal{N}_R$ and $v'\in\mathcal{P}_R$ satisfies $u=w'^{-1}v'$. Then $w'w^{-1}=v'v^{-1}\in\mathcal{N}_R\cap\mathcal{P}_R$. On the other hand $\mathcal{N}_R\cap\mathcal{P}_R=\{1\}$, therefore $w'w^{-1}=v'v^{-1}=1$. This proves the uniqueness.

Step 2. The construction of w and v starts from rewriting the equation $wu=v$ into the matrix form

(3.18) $$(w_{-j})_{j\in\boldsymbol{Z}}(u_{j-i})_{i,j\in\boldsymbol{Z}}=(v_j)_{j\in\boldsymbol{Z}},$$

where u_n, v_n, w_n $(n\in\boldsymbol{Z})$ denote the Laurent coefficients of u, v, and w:

(3.19) $$u=\sum_{n=-\infty}^{\infty}u_n\lambda^n,\quad v=\sum_{n=0}^{\infty}v_n\lambda^n,\quad w=\sum_{n=0}^{\infty}w_n\lambda^{-n},$$

$$\text{with}\quad w_0=1,\quad v_n=w_n=0\quad\text{for } n<0.$$

We devide the matrix $(u_{j-i})_{i,j\in\boldsymbol{Z}}$ into four blocks as in (2.10):

(3.20) $$(u_{j-i})_{i,j\in Z}=\begin{pmatrix} U_1 & U_4 \\ U_2 & U_3 \end{pmatrix}, \quad U_1=(u_{j-i})_{i,j<0}, \quad \text{etc.}$$

Step 3. From Eq. (3.18) one can at least formally derive explicit formulas for v and w. Indeed, taking into account the last conditions in (3.19), one can split Eq. (3.18) into the following two equations:

(3.21a) $$(w_{-j})_{j<0}U_1+(u_j)_{j<0}=0,$$

(3.21b) $$(w_{-j})_{j<0}U_4+(u_j)_{j\geq 0}=(v_j)_{j\geq 0}.$$

Therefore if U_1 is shown to be invertible, one obtains in particular the following formula:

(3.22) $$(w_{-j})_{j<0}=(-u_j)_{j<0}U_1^{-1}.$$

For such $w=1+\sum_{n=1}^{\infty}w_n\lambda^{-1}$, the product wu clearly belongs to $\mathcal{P}_R$. Thus the problem comming next is to justify formula (3.22).

Step 4. In order to consider the above problem from a more general view-point, we introduce the set $\mathcal{H}$ formed by all matrices $H=(h_{ij})_{i,j<0}$ that satisfy the conditions

(3.23a) $h_{ij}\in R_{j-j}$ for $i, j<0$,

(3.23b) the principal diagonal components h_{ii} are invertible in R.

What we want to prove is:

(3.24) *$\mathcal{H}$ forms a group with respect to matrix multiplication.*

Indeed, from this it follows immediately that formula (3.22) makes sense.

Step 5. Assuming for the moment that (3.24) is true, let us check the validity of formula (3.22). If one applies (3.24) to the matrix U_1 which clearly belongs to $\mathcal{H}$, it follows that U_1 is invertible and the components of its inverse matrix also satisfy conditions (3.23). On the other hand another problem in justifying formula (3.22) is that the definition of each component of the product matrix $(-u_j)_{j<0}U_1^{-1}$ involves an infinite series because of the infinite dimensionality of the matrices concerned; but due to what we have viewed above about the inverse matrix of U_1, this infinite series turns out to become a well defined element of R as assumptions (3.4) assure. Thus we see that formula (3.22) is valid. Therefore the remaining problem is to prove (3.24).

Step 6. Finally we verify (3.24) and complete the proof of (3.17). The proof given here is almost a repetition of the arguments we used in the latter half of Sect. 3.3. First, for any element $H=(h_{ij})_{i,j<0}$ of $\mathcal{H}$ we decompose it as:

(3.25) $$H=H_-+H_+,\quad H_+=(h_{ij}\theta(j-i)),\quad H_-=(h_{ij}(1-\theta(j-i))),$$

where θ denotes the Heaviside function, $\theta(x)=1$ for $x\geqq 0$ and $\theta(x)=0$ for $x<0$. Since H_+ is an upper triangular matrix with invertible principal diagonal components, it is invertible, and besides one can show easily that

(3.26) H_+^{-1} is an upper triangular matrix and belongs to $\mathcal{H}$.

Therefore

(3.27) $$H=(\mathbf{1}+H_-H_+^{-1})H_+,$$

so one has only to check the invertibility of $\mathbf{1}+H_-H_+^{-1}$ in $\mathcal{H}$. This can be checked by using the Neumann series

(3.28) $$\sum_{n=0}^{\infty}(-H_-H_+^{-1})^n=\mathbf{1}-H_-H_+^{-1}+(H_-H_+^{-1})^2-\cdots.$$

Indeed, writing the components of each term as

(3.29) $$(-H_-H_+^{-1})=(a_{ij}^{(n)})_{i,j<0},$$

one sees from (3.25) and (3.26) that

(3.30) $$a_{ij}^{(n)}\in R_{j-i+n},$$

therefore the above Neumann series turns out to become an element of $\mathcal{H}$ and gives an explicit expression of the inverse element of $\mathbf{1}+H_-H_+^{-1}$. This completes the proof of (3.24).

3.5. Applications to self-dual Yang-Mills equations. We here show applications of the above result to the description of solutions and transformation groups of the self-dual Yang-Mills equations. In these applications the basic algebra R is set equal to that one shown in (3.5), or its appropriate extensions including additional parameters other than $(y, z, \bar{y}, \bar{z})$. Except that everything is formulated in terms of formal loop groups, the arguments we adopt below are almost the same as those of the Riemann-Hilbert problem method.

Description of solutions. Let $w^{(\mathrm{in})}$ be an arbitrary formal power series of the form

(3.31) $$w^{(\mathrm{in})}=1+\sum_{n=1}^{\infty}w_n^{(\mathrm{in})}\lambda^{-n},\qquad w_n^{(\mathrm{in})}\in\mathfrak{gl}(r, \boldsymbol{C}[[y, z, \bar{y}, \bar{z}]]),$$

and define

(3.32) $$u=\exp(\bar{z}\lambda\partial_y-\bar{y}\lambda\partial_z)w^{(\mathrm{in})}=\sum_{k=0}^{\infty}(\bar{z}\lambda\partial_y-\bar{y}\lambda\partial_z)^k w^{(\mathrm{in})}/k!,$$

whicn is an element of $\mathcal{G}_R$ with $R=\mathfrak{gl}(r, \boldsymbol{C}[[y, z, \bar{y}, \bar{z}]])$ as in (3.5). We now decompose u as (3.17) shows:

(3.33) $$\exp(\bar{z}\lambda\partial_y-\bar{y}\lambda\partial_z)w^{(\mathrm{in})}=w^{-1}v, \qquad w\in\mathcal{N}_R, \quad v\in\mathcal{P}_R.$$

Then the conclusion is:

(3.34) *w solves Eqs.* (1.24) *under the initial condition*

$$w|_{\bar{y}=\bar{z}=0}=w^{(\mathrm{in})}.$$

Proof of (3.34). The initial condition can be checked immediately; so we here derive Eqs. (1.24). From the construction,

(3.35) $$(-\lambda\partial_z+\partial_{\bar{y}})u=0, \qquad (\lambda\partial_y+\partial_{\bar{z}})u=0.$$

By the substitution $u=w^{-1}v$, one obatins:

(3.36) $$(-\lambda\partial_z+\partial_{\bar{y}})w\cdot w^{-1}=(-\lambda\partial_z+\partial_{\bar{y}})v\cdot v^{-1},$$
$$(\lambda\partial_y+\partial_{\bar{z}})w\cdot w^{-1}=(\lambda\partial_y+\partial_{\bar{z}})v\cdot v^{-1}.$$

Now examine the both hand sides of the last equations. The left hand side is a formal power series in λ^{-1}, whereas the right hand side a formal power series in λ. Therefore they should be independent of λ, i.e. become elements of R. This means that Eqs. (1.21) are satisfied. Eliminating gauge potientials as we did in Sect. 1.7, we obtain Eqs. (1.24).

Description of transformation groups. Let $w\in\mathcal{N}_R$ be a solution of Eqs. (1.24), and p an element of $\mathcal{G}_R$ that satisfies Eqs. (1.17). We set

(3.37) $$u=pw^{-1}$$

and again perform the decomposition of (3.17); in this case we denote the (w, v) pair as $(p\circ w, v)$:

(3.38) $$pw^{-1}=(p\circ w)^{-1}v, \qquad p\circ w\in\mathcal{N}_R. \quad v\in\mathcal{P}_R.$$

The conclusion here is:

(3.39) *$p\circ w$ becomes a new solution of Eqs.* (1.24).

In particular, the transformations $w\to p\circ w$ obtained as above form a transformation group on the space of $\mathcal{N}_R$-solutions to Eqs. (1.24).

Proof of (3.39). From the assumptions,

(3.40) $$(-\lambda\partial_y+\partial_{\bar{z}}+A_{\bar{z}})(wp^{-1})=0, \qquad (\lambda\partial_z+\partial_{\bar{y}}+A_{\bar{y}})(wp^{-1})=0,$$

where $A_{\bar{y}}$ and $A_{\bar{z}}$ are the gauge potentials corresponding to w. By the substitution $pw^{-1}=(p\circ w)^{-1}v$,

(3.41) $$(-\lambda\partial_y+\partial_{\bar{z}}+A_{\bar{z}}))p\circ w)\cdot(p\circ w)^{-1}=(-\lambda\partial_y+\partial_{\bar{z}}+A_{\bar{z}})v\cdot v^{-1},$$
$$(\lambda\partial_z+\partial_{\bar{y}}+A_{\bar{y}})(p\circ w)(p\circ w)^{-1}=(\lambda\partial_z+\partial_{\bar{y}}+A_{\bar{y}})v\cdot v^{-1},$$

and examining the both hand sides one sees that they are independend of λ. This proves (3.39).

Parametric solutions and transformations. In the above arguments the algebra R is set equal to that one shown in (3.5), but this is not the only one choice. The same arguments are also valid when one replaces the above R by its extensions including some additional parameters other than $(y, z, \bar{y}, \bar{z})$ and modifies the filtration appropriately. This leads to the description of various parametric solutions and transformations; the examples of papametric solutions presented in Sect. 3.3 of [7] can be reconstructed from the above view point.

3.6. Connection with Grassmann manifold method. The decomposition equation $u=w^{-1}v$ in (3.17) has the following equivalent expression:

(3.42) $$(u_{j-i})_{i\in Z, j<0}=(w^*_{i-j})_{i\in Z, j<0}(v_{j-i})_{i, j<0},$$

where w^*_n's denote the Laurent coefficients of w^{-1},

(3.43) $$w^{-1}=\sum_{n=0}^{\infty} w^*_n\lambda^{-n}, \quad w^*_n=0 \quad \text{for } n<0.$$

Eq. (3.42) means that $(u_{j-i})_{i\in Z, j<0}$ and $(w^*_{i-j})_{i\in Z, j<0}$ correspond to the same point in an infinite dimensional Grassmann manifold, whereas $(v_{j-i})_{i, j<0}$ plays the role of H-matrices that define the equivelence relation $\sim$; see Sect. 2.4. This provides a very clear interpretation of decomposition problem (3.17) from the viewpoint of Grassmann manifolds.

In order to show more direct connection with the ξ-matrix, one has to rewrite Eq. (3.42) further. Note that from (3.42),

(3.44) $$U_1=(u_{j-i})_{i, j<0}=(w^*_{i-j})_{i, j<0}(v_{j-i})_{i, j<0}.$$

Also note that because of the triangularity of $(w^*_{i-j})_{i, j<0}$,

(3.45) $$(w^*_{i-j})_{i, j<0}{}^{-1}=(w_{i-j})_{i, j<0}.$$

Combining these facts with Eq. (3.42), one obtains:

(3.46) $$(u_{j-i})_{i\in Z, j<0}U_1{}^{-1}=\begin{pmatrix} 1 \\ U_2U_1{}^{-1} \end{pmatrix}=(w^*_{i-j})_{i\in Z, j<0}(w_{j-i})_{i, j<0}.$$

Relation to the ξ-matrix formulation manifests itself in the last equation. Indeed, according to Proposition 3 in [7], the right hand side of Eq. (3.46) is exactly the ξ-matrix that corresponds to w via the one-to-one correspondence mentioned in Sect. 1.7. In particular, extracting the 0^{th} row from Eq. (3.46) one can recover formula (3.22).

3.7. Analytical version. The basic ideas developed in Sect. 3.4 for solving the decomposition problem are also applicable to analytical situations such as Riemann-Hilbert problem (3.1), though we omit the details here. It turns out in particular that as far as u of the left hand side of (3.1) is sufficiently close to the unit matrix, formula (3.22) makes sense even analytically and provides an explicit expression for the solution of the original Riemann-Hilbert problem.

Similar arguments can be applied to analytical elements (i. e. elements with some convergence domains) of a formal loop group, say $\mathcal{G}_{\mathfrak{gl}(r,C[[y,z,\bar{y},\bar{z}]])}$. Indeed, it follows that if u is such an analytical element, then the factors w and v also become analytical; in other words, the decomposition problem can ce solved within the subset of such analytical elements. Decomposition theorems of this type can be formulated for various formal loop groups.

3.8 Connection with groups of formal microdifferential operations. In order to give a group-theoretical interpretation to the Grassmann manifold method of Sato [8], Mulase [13] introduced a group of formal microdifferential operators and established a decomposition theorem. Formal microdifferential operators considered there take forms such as $u=\sum_{n=-\infty}^{\infty}u_n(\partial/\partial x)^n$, where the coefficients u_n are taken from a filtered algebra on which $\partial/\partial x$ acts as a derivation operator. The decomposition theorem of Mulase then shows that such u can be uniquely decomposed as:

$$u=w^{-1}v,\quad w=\sum_{n=0}^{\infty}w_n(\partial/\partial x)^{-n},\quad w_0=1,\quad v=\sum_{n=0}^{\infty}v_n(\partial/\partial x)^n. \tag{3.47}$$

The situations concerning formal loop groups can be recovered if one only consider the case in which $\partial u_n/\partial x=0$ for every n; indeed, via the replacement $\partial/\partial x\leftrightarrow\lambda$ such u's may be identified with elemtents of formal loop groups.

3.9. Conclusion of this section. We considered a decomposition problem in formal loop groups. Although this decomposition plays the same role as the Riemann-Hilbert problem in applications to completely integrable systems, our formal loop group approach seems to be advantageous in some respects. First, in constrast to the Riemann-Hilbert problem, the decomposition problem in a formal loop group always has a unique solution, thus the situations being much simpler. Second, one has an explicit formula for the solution. By seeking for its origin we found its relation to the Grassmann manifold method. Third, considerably large part of remarkable features of completely integrable systems is usually concerned with their formal aspects rather than analytical ones, and the formal loop group approach provides a suitable language for describing them.

§4. Riemann-Hilbert transformations.

4.1. Motivation. If $\mathcal{G}_R$ is replaced by $\mathcal{G}_C$, the construction of transformations presented in Sect. 3.5. reduces exactly to the Riemann-Hilbert transformations of Ueno and Nakamura [4]. To be more pricise, we start from a solution $w=w(y, z, \bar{y}, \bar{z}, \lambda)$ of Eqs. (1.24) with $w(y, z, \bar{y}, \bar{z}, \cdot)\in\mathcal{N}_C$ and a $GL(r, \boldsymbol{C})$-valued holomorphic function $p=p(y, z, \bar{y}, \bar{z}, \lambda)$ satisfying Eqs. (1.17) with $p(y, z, \bar{y}, \bar{z}, \cdot)\in\mathcal{G}_C$, where $w(y, z, \bar{y}, \bar{z}, \cdot)\in\mathcal{N}_C$ etc. means that w belongs to $\mathcal{N}_C$ for any fixed value of $(y, z, \bar{y}, \bar{z})$ etc. Then by solving the Riemann-Hilbert problem (see Sect. 3.1)

$$(4.1)\quad pw^{-1}=(p\circ w)^{-1}v\ (\lambda\in C),\quad p\circ w(y, z, \bar{y}, \bar{z}, \cdot)\in\mathcal{N}_C,\quad v(y, z, \bar{y}, \bar{z}, \cdot)\in\mathcal{P}_C$$

one obtains a transformation $w\rightarrow p\circ w$ sending w into a new solution $p\circ w$ of Eqs. (1.24).

Here however arises a problem. As clearly pointed out by Wu [6], the above set of Riemann-Hilbert transformations involves fairly trivial ones that do not change gauge potentials. Indeed, if $p(y, z, \bar{y}, \bar{z}, \cdot)$ lies in $\mathcal{N}_C$, then

$$(4.2)\qquad p\circ w=wp^{-1},$$

the corresponding gauge potentials being the same as those of w.

In order to fill up this gap, Chau [5] and Wu [6] (see also Wu and Ge [14]) introduced another set of Riemann-Hilbert transformations, and mixing these two families of transformations (to be more precise, their infinitesimal generators) they obtained symmetry algebras that act non-trivially on gauge potentials. They also remarked that the symmetry algebras thus obtained are just the same as those constructed in Refs. [15] without using the Riemann-Hilbert problem.

In what follows we try to reformulate the above enlarged set of Riemann-Hilbert transformations as transformations acting on a pair of w-functions rather than a single one. Here a central role will be played by the direct product of two loop groups. Such a viewpoint considerably clarifies the group-theoretical meaning of the above transformations.

4.2. Riemann-Hilbert transformations acting on a pair of w-functions. For the definition of the above two distinct sets of transformations, Chau and Wu used two w-functions with different analytical properties. Inspired by this, we here describe these transformations as transformations acting on a pair, not a single one, of w-functions.

As such a pair, we here take the one to be obtained from the Riemann-Hilbert problem (3.1) in which u is assumed to depend also on $(y, z, \bar{y}, \bar{z})$ with $u(y, z, \bar{y}, \bar{z}, \cdot)\in\mathcal{G}_C$ and to satisfy the equations

$$(4.3)\qquad (-\lambda\partial_y+\partial_{\bar{z}})u=0,\qquad (\lambda\partial_z+\partial_{\bar{y}})u=0.$$

To make the notation more suitable for later arguments, we write the (w, v) pair of this Riemann-Hilbert problem as $w^{(0)}=w^{(0)}(y, z, \bar{y}, \bar{z}, \lambda)$ and $w^{(\infty)}=w^{(\infty)}(y, z, \bar{y}, \bar{z}, \lambda)$:

$$(4.4)\qquad u=w^{(\infty)-1}w^{(0)},\quad w^{(\infty)}(y,z,\bar{y},\bar{z},\cdot)\in\mathfrak{N}_C,\quad w^{(0)}(y,z,\bar{y},\bar{z},\cdot)\in\mathfrak{P}_C.$$

By means of the same arguments as used in the proof of (3.34), we see that *$w^{(\infty)}$ and $w^{(0)}$ satisfy Eqs.* (1.21) *for a common set of gauge potentials.* It should be noted that this is exactly the way by which Ward [2] and Zakharov and Shabat [3] constructed solutions.

Now we consider the effect of changing u as:

(4.5) $u\rightarrow puq^{-1}$, where $p=p(y,z,\bar{y},\bar{z},\lambda)$ and $q=q(y,z,\bar{y},\bar{z},\lambda)$ are solutions of Eqs. (1.17) for which $(p(y,z,\bar{y},\bar{z},\cdot),\ q(y,z,\bar{y},\bar{z},\cdot))\in\mathfrak{G}_C\times\mathfrak{G}_C$.

This of course causes change of $(w^{(\infty)}, w^{(0)})$, which we write as:

$$(4.6)\qquad (w^{(\infty)}, w^{(0)})\longrightarrow((p,q)\circ w^{(\infty)},\ (p,q)\circ w^{(0)}).$$

Transformation of the $(w^{(\infty)}, w^{(0)})$ pair thus obtained are what we have sought for. Note that *if one forgets the functional dependence on* $(y,z,\bar{y},\bar{z})$, *both* (4.5) *and* (4.6) *may be viewed as group action of the direct product group $\mathfrak{G}_C\times\mathfrak{G}_C$ on, respectively, $\mathfrak{G}_C$ and $\mathfrak{N}_C\times\mathfrak{P}_C$.*

Let us examine the above transformations further. The Riemann-Hilbert problem that characterizes transformation (4.6) is:

$$(4.7)\qquad puq^{-1}=((p,q)\circ w^{(\infty)})^{-1}((p,q)\circ w^{(0)})\qquad(\lambda\in C).$$

By the substitution $u=w^{(\infty)-1}w^{(0)}$, Eq. (4.7) becomes:

$$(4.8)\qquad ((p,q)\circ w^{(\infty)})pw^{(\infty)-1}=((p,q)\circ w^{(0)})qw^{(0)-1}\qquad(\lambda\in C).$$

If $q=1$, the right hand side of Eq. (4.8) lies in $\mathfrak{P}_C$ for any value of $(y,z,\bar{y},\bar{z})$; then Eq. (4.8) reduces to Eq. (4.1) and one has:

$$(4.9)\qquad (p,1)\circ w^{(\infty)}=p\circ w^{(\infty)}.$$

On the other hand if $p=1$, one obtains another type of transformations, the roles of $w^{(\infty)}$ and $w^{(0)}$ being exchanged in that case. Obviously, the action of $(p,1)$ and that of $(1,q)$ commute with each other, and their composition yields a general transformation induced by (p,q).

4.3. In case the domains where $w^{(\infty)}$ and $w^{(0)}$ are defined do not overlap. Even in this case we can define similar transformations acting on the pair $(w^{(\infty)}, w^{(0)})$. Accordingly, $w^{(\infty)}$ and $w^{(0)}$ are simply assumed to satisfy Eqs. (1.21) for a common set of gauge potentials together with the constraint $w^{(\infty)}(y,z,y,z,\infty)=1_r$:

$$(4.10)\qquad (-\lambda\partial_y+\partial_{\bar{z}}+A_{\bar{z}})w=0,\quad(\lambda\partial_z+\partial_{\bar{y}}+A_{\bar{y}})w=0\quad(w=w^{(\infty)},\ w^{(0)}),$$

and we here do not assume that $w^{(\infty)}$ and $w^{(0)}$ are those obtained from Riemann-Hilbert problem (4.4).

The construction of transformations consists of the following steps.

Step 1. Take small circles $C^{(\infty)}$ and $C^{(0)}$ respectively centered at ∞ and 0 in the Riemann sphere $\boldsymbol{P}^1$ so that $(w^{(\infty)}(y, z, \bar{y}, \bar{z}, \cdot), w^{(0)}(y, z, \bar{y}, \bar{z}, \cdot)) \in \mathcal{N}_{C^{(\infty)}} \times \mathcal{P}_{C^{(0)}}$. Note that the complementary set $\boldsymbol{P}^1 - (C^{(\infty)} \cup C^{(0)})$ splits into three connected components, the disc $C_+^{(0)}$, the annulus $C_-^{(0)} \cap C_+^{(\infty)}$, and the disc $C_-^{(\infty)}$.

Step 2. Take two $\mathrm{GL}(r, \boldsymbol{C})$-valued functions $p = p(y, z, \bar{y}, \bar{z}, \lambda)$ and $q = q(y, z, \bar{y}, \bar{z}, \lambda)$ with the following properties:

(4.11a) $$(p(y, z, \bar{y}, \bar{z}, \cdot), q(y, z, \bar{y}, \bar{z}, \cdot)) \in \mathcal{G}_{C^{(\infty)}} \times \mathcal{G}_{C^{(0)}},$$

(4.11b) *p and q satisfy* Eqs. (1.17).

Step. 3. Solve the following Riemann-Hilbert problem which is defined with respect to the disconnected curve $C^{(\infty)} \cup C^{(0)}$:

(4.12)
$$p w^{(\infty)-1} = ((p, q) \circ w^{(\infty)})^{-1} v \qquad (\lambda \in C^{(\infty)}),$$
$$q w^{(0)-1} = ((p, q) \circ w^{(0)})^{-1} v \qquad (\lambda \in C^{(0)}),$$
$$((p, q) \circ w^{(\infty)})(y, z, \bar{y}, \bar{z}, \cdot) \in \mathcal{N}_{C^{(\infty)}},$$
$$((p, q) \circ w^{(0)})(y, z, \bar{y}, \bar{z}, \cdot) \in \mathcal{P}_{C^{(0)}},$$
$$v(y, z, \bar{y}, \bar{z}, \cdot) \in \mathcal{G}_{C^{(0)}, C^{(\infty)}},$$

where

(4.13) $$\mathcal{G}_{C^{(0)}, C^{(\infty)}} = \{v\,;\ \text{holomorphic maps } \lambda \to v(\lambda) \text{ of } C_-^{(0)} \cap C_+^{(\infty)} \text{ into } \mathrm{GL}(r, \boldsymbol{C})\}.$$

If the above Riemann-Hilbert problem has a solution $((p, q) \circ w^{(\infty)}, (p, q) \circ w^{(0)})$, it is unique and becomes a new pair of w-functions, i.e. $(p, q) \circ w^{(\infty)}$ and $(p, q) \circ w^{(0)}$ satisfy linear equations of the form

(4.14) $$(-\lambda \partial_y + \partial_{\bar{z}} + A_{\bar{z}}^{p,q}) w = 0, \qquad (\lambda \partial_z + \partial_{\bar{y}} + A_{\bar{y}}^{p,q}) w = 0,$$

with *transformed gauge potentials* $A_{\bar{y}}^{p,q}$ and $A_{\bar{z}}^{p,q}$. Indeed, from (4.12),

(4.15)
$$(-\lambda \partial_y + \partial_{\bar{z}} + A_{\bar{z}})((p, q) \circ w) \cdot ((p, q) \circ w)^{-1} = (-\lambda \partial_y + \partial_{\bar{z}} + A_{\bar{z}}) v \cdot v^{-1},$$
$$(\lambda \partial_z + \partial_{\bar{y}} + A_{\bar{y}})((p, q) \circ w) \cdot ((p, q) \circ w)^{-1} = (\lambda \partial_z + \partial_{\bar{y}} + A_{\bar{y}}) v \cdot v^{-1},$$
$$\text{for} \quad w = w^{(\infty)}, w^{(0)},$$

and examining the both hand sides just in the same way as we proved (3.39), we find that they are independent of λ; this means that Eqs. (4.14) are really satisfied. Thus by means of Riemann-Hilbert problem (4.12) we obtain a transformation $(w^{(\infty)}, w^{(0)}) \to ((p, q) \circ w^{(\infty)}, (p, q) \circ w^{(0)})$ of the pair of w-functions.

It should be noted that also in the present set-up, *if one forgets the functional* *dependence on* $(y, z, \bar{y}, \bar{z})$, *the above transformations can be regarded as group action of the direct product group* $\mathcal{G}_{C^{(\infty)}} \times \mathcal{G}_{C^{(0)}}$ *on its subgroup* $\mathcal{N}_{C^{(\infty)}} \times \mathcal{P}_{C^{(0)}}$. This generalizes the description of transformations in Sect. 4.2, the latter being recovered when $C^{(\infty)}$ and $C^{(0)}$ are set equal to C. However, one should be careful about the fact that it may happen that the Riemann-Hilbert problem defining a transformation does not have a solution; of course if p and q are sufficiently close to the unit matrix, it certainly has a solution, but otherwise the existence of a solution depends on (p, q). This means that in a mathematically rigorous sense the above action on $\mathcal{N}_{C^{(\infty)}} \times \mathcal{P}_{C^{(0)}}$ is meaningful *only for a subset* (*group germ*) of the whole group $\mathcal{G}_{C^{(\infty)}} \times \mathcal{G}_{C^{(0)}}$.

4.4. Formal loop group approach. The above construction of transformations can be generalized to the level of formal loop groups. We here define:

(4.16) $$\mathcal{G}_R^{(\infty)} = \mathcal{G}_R, \quad \mathcal{N}_R^{(\infty)} = \mathcal{N}_R, \quad \mathcal{P}_R^{(\infty)} = \mathcal{P}_R \quad \text{(see Sect. 3.3)},$$

(4.17) $$\mathcal{G}_R^{(0)} = \{u = \sum u_n \lambda^n ;\ u_n \in R_{-n} (-\infty < n < \infty),\ u_0 \text{ is invertible}\},$$
$$\mathcal{N}_R^{(0)} = \{u = \sum u_n \lambda^n \in \mathcal{G}_R^{(0)} ;\ u_n = 0\ (n > 0),\ u_0 = 1\},$$
$$\mathcal{P}_R^{(0)} = \{u = \sum u_n \lambda^n \in \mathcal{G}_R^{(0)} ;\ u_n = 0\ (n < 0)\},$$

(4.18) $$\mathcal{G}_R^{(0\infty)} = \{u = \sum u_n \lambda^n ;\ u_n \in R_{|n|} (-\infty < n < \infty),\ u_0 \text{ is invertible}\}.$$

It can be checked in the same way as the arguments in Sect. 3.3 that *these sets of formal Laurent series all form groups with respect to the usual multiplication*; besides, *they are in the following subgroup relation*:

(4.19) $$\begin{array}{ccccccc} & & \mathcal{N}_R^{(\infty)} & \supset & \mathcal{N}_R^{(0)} & & \\ & \cap & & \cap & & \cap & \\ \mathcal{G}_R^{(\infty)} & & \supset & \mathcal{G}_R^{(0\infty)} & \subset & & \mathcal{G}_R^{(0)} \\ & \cup & & \cup & & \cup & \\ & & \mathcal{P}_R^{(\infty)} & \subset & \mathcal{P}_R^{(0)} & & \end{array}$$

Replacing the groups $\mathcal{G}_{C^{(\infty)}}$, $\mathcal{G}_{C^{(0)}}$, $\mathcal{N}_{C^{(\infty)}}$, $\mathcal{P}_{C^{(0)}}$ and $\mathcal{G}_{C^{(\infty)}, C^{(0)}}$ respectively by their formal counterparts $\mathcal{G}_R^{(\infty)}$, $\mathcal{G}_R^{(i)}$, $\mathcal{N}_R^{(\infty)}$, $\mathcal{P}_R^{(i)}$ and $\mathcal{G}_R^{(0\infty)}$, we obtain formal loop group analogues of the transformation discussed in Sect. 4.2 and 4.3. Here the algebra R is, as in Sect. 3.5, set equal to $\mathfrak{gl}(r, \boldsymbol{C}[[y, z, \bar{y}, \bar{z}]])$ or its appropriate extensions including some additional parameters. The pair $(w^{(\infty)}, w^{(0)})$ to be transformed is an element of $\mathcal{N}_R^{(\infty)} \times \mathcal{P}_R^{(0)}$ satisfying Eqs. (4.10); the data (p, q) of a transformation is taken from $\mathcal{G}_R^{(\infty)} \times \mathcal{G}_R^{(0)}$ and assumed to satisfy Eqs. (1.17). Then, considering a decomposition problem of the same form as (4.12) for the above formal loop groups and solving it (this is always possible; see below), we obtain a transformation $(w^{(\infty)}, w^{(0)}) \rightarrow ((p, q) \circ w^{(\infty)}, (p, q) \circ w^{(0)})$.

Let us examine in more detail the decomposition problem needed in the above procedure. The problem is to find, for a given element $(U^{(\infty)}, U^{(0)}) \in \mathcal{G}_R^{(\infty)} \times \mathcal{G}_R^{(0)}$, a triple $(W^{(\infty)}, W^{(0)}, v)$ that satisfies the following conditions:

$$U^{(\infty)} = W^{(\infty)-1}v, \qquad U^{(0)} = W^{(0)-1}v, \tag{4.20}$$
$$W^{(\infty)} \in \mathcal{N}_R^{(\infty)}, \quad W^{(0)} \in \mathcal{P}_R^{(0)}, \quad v \in \mathcal{G}_R^{(0\infty)}.$$

In the application to the above mentioned transformations we set $U^{(\infty)} = pw^{(\infty)-1}$, $U^{(0)} = qw^{(0)-1}$. Therefore a question coming next is whether decomposition problem (4.20) has a solution. The answer is *yes*:

(4.21) *For any* $(U^{(\infty)}, U^{(0)}) \in \mathcal{G}_R^{(\infty)} \times \mathcal{G}_R^{(0)}$, *there is a unique triple* $(W^{(\infty)}, W^{(0)}, v)$ *that satisfies* (4.20).

4.5. Proof of (4.21). Checking the uniqueness is simple. Suppose that another triple $(W'^{(\infty)}, W'^{(0)}, v')$ satisfies (4.20). Then $W'^{(0)}W^{(0)-1} = v'v^{-1} = W'^{(\infty)}W^{(\infty)-1} \in \mathcal{P}_R^{(0)} \cap \mathcal{G}_R^{(0\infty)} \cap \mathcal{N}_R^{(\infty)}$; on the other hand, $\mathcal{P}_R^{(0)} \cap \mathcal{G}_R^{(0\infty)} \cap \mathcal{N}_R^{(\infty)} = \{1\}$. Therefore $W'^{(\infty)} = W^{(\infty)}$, $W'^{(0)} = W^{(0)}$ and $v' = v$. This proves the uniqueness.

The existence can be proved by successively performing the decomposition discussed in Sect. 3.4 as follows. Note that this method can also be applied the analytical case, i.e. Riemann-Hilbert problem (4.12).

Step 1. Decompose $U^{(\infty)} \in \mathcal{G}_\infty^{(\infty)}$ as:

$$U^{(\infty)} = U_1^{-1}V_1, \qquad U_1 \in \mathcal{N}_R^{(\infty)}, \quad V_1 \in \mathcal{P}_R^{(\infty)}.$$

This is due to the decomposition presented in Sect. 3.4.

Step 2. Decompose $U^{(0)}V_1^{-1} \in \mathcal{G}_R^{(0)}$ as:

$$U^{(0)}V_1^{-1} = U_2^{-1}V_2, \qquad U_2 \in \mathcal{P}_R^{(0)}, \quad V_2 \in \mathcal{N}_R^{(0)}.$$

This decomposition is also possible. Indeed, replacing λ by λ^{-1}, one may reduce the problem into a decomposition problem in $\mathcal{G}_R^{(\infty)}$.

Step 3. Finally, define:

$$W^{(\infty)} = V_2U_1, \quad W^{(0)} = U_2, \quad v = V_2V_1.$$

Taking into account the subgroup relation as shown in (4.19), one can easily check that the above triple $(W^{(\infty)}, W^{(0)}, v)$ becomes a solution of (4.20). This completes the proof of (4.21).

4.6. Grassmann manifold approach. We viewed in Sect. 3.6 that decomposition problem (3.17) in formal loop groups is in close connection with the Grassmann manifold method. In this respect one may naturally expect that another proof of (4.21) will be obtained from the viewpoint of Garssmann manifolds. Of course the proof presented in Sect. 4.5 is certainly connected with some Grassmann manifolds via the decomposition problems in $\mathcal{G}_R^{(\infty)}$ and in $\mathcal{G}_R^{(0)}$, but this seems to be still indirect. If possible, one may as well seek for a more direct proof using a Grassmann manifold. Indeed, slight modification of the arguments in Sect. 3.6 enables us to develop such an approach to decomposition problem (4.20).

As in the arguments of Sect. 3.6, we start from rewriting the two equations in (4.20) into the following matrix equation:

$$(4.22)\quad \left(\begin{array}{c|c} U_{j-i}^{(\infty)} & U_{-i-j-1}^{(\infty)} \\ \hline U_{i+j+1}^{(0)} & U_{i-j}^{(0)} \end{array}\right)_{i\in Z,\, j<0} = \left(\begin{array}{c|c} W_{i-j}^{*(\infty)} & W_{i+j+1}^{*(\infty)} \\ \hline W_{i+j+1}^{*(0)} & W_{i-j}^{\infty(0)} \end{array}\right)_{i\in Z,\, j<0} \left(\begin{array}{c|c} v_{j-1} & v_{-i-j-1} \\ \hline v_{i+j+1} & v_{i-j} \end{array}\right)_{i,\, j<0},$$

where the components of the above matrices denote the coefficients of the Laurent expansion

$$(4.23)\quad \begin{aligned} &U^{(\infty)}=\sum U_n^{(\infty)}\lambda^n, \quad U^{(0)}=\sum U_n^{(0)}\lambda^n, \\ &W^{(\infty)-1}=\sum W_n^{*(\infty)}\lambda^{-n}, \quad W^{(0)-1}=\sum W_n^{*(0)}\lambda^n, \quad v=\sum v_n\lambda^n. \end{aligned}$$

Note here that the size of rows and columns of the matrices in (4.22) is the double of those that appear in Sect. 3.6; except for this difference, Eq. (4.22) has almost the same form as Eq. (3.42).

In view of the arguments in Sect. 3.6, what we have to do next is to multiply the left hand side of Eq. (4.22) by its negative index part. Performing this calculation and writing the result in terms of the coefficients of the Laurent expansion

$$(4.24)\quad \begin{aligned} &W^{(\infty)}=\sum W_n^{(\infty)}\lambda^{-n}, \quad W^{(0)}=\sum W_n^{(0)}\lambda^n, \\ &\hat{W}^{(\infty)}\equiv W_0^{(0)-1}W^{(\infty)}=\sum \hat{W}_n^{(\infty)}\lambda^{-n}, \quad \hat{W}^{(\infty)-1}=\sum \hat{W}_n^{*(\infty)}\lambda^{-n}, \\ &\hat{W}^{(0)}\equiv W_0^{(0)-1}W^{(0)}=\sum \hat{W}^{(0)}\lambda^n, \quad \hat{W}^{(0)-1}=\sum \hat{W}_n^{*(0)}\lambda^n, \end{aligned}$$

we obtain:

$$(4.25)\quad \begin{aligned} &\left(\begin{array}{c|c} U_{j-i}^{(\infty)} & U_{-i-j-1}^{(\infty)} \\ \hline U_{i+j+1}^{(0)} & U_{i-j}^{(0)} \end{array}\right)_{i\in Z,\, j<0} \left(\begin{array}{c|c} U_{j-i}^{(\infty)} & U_{-i-j-1}^{(\infty)} \\ \hline U_{i+j+1}^{(0)} & U_{i-j}^{(0)} \end{array}\right)_{i,\, j<0}^{-1} \\ &= \left(\begin{array}{c|c} W_{i-j}^{*(\infty)} & W_{i+j+1}^{*(\infty)} \\ \hline W_{i+j+1}^{*(0)} & W_{i-j}^{*(0)} \end{array}\right)_{i\in Z,\, j<0} \left(\begin{array}{c|c} W_{i-j}^{(\infty)} & 0 \\ \hline 0 & W_{i-j}^{(0)} \end{array}\right)_{i,\, j<0} \\ &= \left(\begin{array}{c|c} W_{i-j}^{*(\infty)} & \hat{W}_{i+j+1}^{*(\infty)} \\ \hline \hat{W}_{i+j+1}^{*(0)} & \hat{W}_{i-j}^{*(0)} \end{array}\right)_{i\in Z,\, j<0} \left(\begin{array}{c|c} W_{i-j}^{(\infty)} & 0 \\ \hline 0 & \hat{W}_{i-j}^{(0)} \end{array}\right)_{i,\, j<0}. \end{aligned}$$

In particular, we obtain the following explicit formula for the solution of decomposition problem (4.20):

$$(4.26)\qquad \left(\begin{array}{c|c} -W^{(\infty)}_{-j} & \hat{W}^{(\infty)}_{-j-1} \\ \hline W^{(0)}_{-j-1} & -\hat{W}^{(0)}_{-j} \end{array}\right)_{j<0} = \left(\begin{array}{c|c} U^{(\infty)}_{j} & U^{(\infty)}_{-j-1} \\ \hline U^{(0)}_{j+1} & U^{(0)}_{-j} \end{array}\right)_{j<0} \left(\begin{array}{c|c} U^{(\infty)}_{j-i} & U^{(\infty)}_{-i-j-1} \\ \hline U^{(0)}_{i+j+1} & U^{(0)}_{i-j} \end{array}\right)^{-1}_{i,j<0}.$$

4.7. Conclusion of this section. We have viewed how an enlarged family of Riemann-Hilbert transformations can be formulated as transformations acting on a pair of w-functions. A key idea was to use the direct product of two loop groups, or formal loop groups, rather than a single one, and the transformations were then realized as group action of elements of the direct product group on its subgroup. This considerably clarifies the group-theoretical structure of the above transformations. In particular, the set of their infinitesimal generators can accordingly be embedded into the direct sum of two loop algebras; this seems to explain the meaning of *"richer group structure"* of Riemann-Hilbert transformations pointed out by Wu and Ge [6, 13]. Finally it should be noted that also in the present section, as well as in the previous section, the use of formal loop groups much simplifies the description.

§ 5. Further developments of our approach.

A number of applications and generalizations may be expected; among them we first point out that our approach can also be applied to the supersymmetric Yang-Mills equations. Recently there appeared several papers that deal with supersymmetric gauge fields from the viewpoint of complete integrability; see Volovich [16], Devchand [17], Chau, Ge, Popowicz [18], Chau [19] and references cited therein. Our approach can readily be applied to these supersymmetric cases with slightest modification; what one has to do is just to change the basic algebra R (see Sect. 3.2) appropriately so that it includes anti-commuting variables as well as ordinary ones. In particular the coefficients of Laurent series and the components of infinite matrices that appear in the arguments depend on both type variables.

Another application would be expected to higher dimensional generalizations of gauge field equations. In a recent paper Ward [20] presented some examples of such integrable gauge field equations and gave a solution technique that generalizes his previous work [2]. Another example in eight dimensions can be found in a paper of Witten [21], and it was recently solved by Suzuki [22] using the Grassmann manifold method. The paper of Witten is also concerned with a geometric interpretation of supersymmetric gauge fields from the viewpoint of twistor theory; its relation to complete integrability is discussed in [19]. In a sense supersymmetric extension is a sort of higher dimensional extension which involves anti-commuting coordinates besides ordinary ones, so some unified viewpoint might be found in this direction.

We finally note that the present approach will be of much use in studying the Einstein equations as well, especially in their self-dual sector. In this respect recent work of Boyer and Plebanski [23] seems to provide very interesting material. One of their remarkable conclusions is that an infinite dimensional group of loop type appears also here and plays the role of transformation group for the self-dual Einstein Eqtations. This loop group however is considerably different from the previous ones which we discussed for Yang-Mills fields. Indeed, this loop group is formed by maps on a circle with values in the group of canonical transformations; note that the latter group in itself is infinite dimensional. Thus the situations become much more complicated than in the case of Yang-Mills fields in which related loop groups are obtained from finite dimensional matrix groups such as GL(r, $\boldsymbol{C}$). Accordingly our tools such as formal loop groups should be appropriately modified; research in this direction is now in progress. It seems likely that the self-dual Einstein equations may be viewed as a completely integrable system of new type.

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