

題目:

同変戸田階層とOkounkov–Pandharipande dressing operators
(Equivariant Toda hierarchy and Okounkov–Pandharipande dressing operators)

要旨:

同変戸田階層はGetzlerによってリーマン球面の同変Gromov–Witten不変量における可積分構造として導入されたもので、2次元戸田階層から特殊な簡約条件によって導くことができる。2000年代半ばにOkounkovとPandharipandeはリーマン球面の同変Gromov–Witten不変量をフェルミオン表示することによって、それらの母関数が同変戸田階層の τ 関数になることを示した。その証明の中でフォック空間上のdressing operatorと呼ぶ作用素を用いたが、この作用素の正体はよくわからないままに残った。またLax形式における説明はなされなかった。講演者はこの作用素を2次元戸田階層のLax作用素と同様の意味での差分作用素として再構成し、同変戸田階層が現れる仕組みをLax形式において明らかにすることができた。この講演ではこの結果を紹介し、Hurwitz数の高スピン版に関連すると思われる拡張の試みにも触れる。

(文献: arXiv:2103.10666, 2211.11354)

Plan

- Gromov-Witten partition functions of $\mathbb{CP}^1$
- Toda conjecture
- Equivariant Gromov-Witten invariants of $\mathbb{CP}^1$
- Okounkov-Pandharipande dressing operators
- Equivariant Toda hierarchy
- Reconstruction of OP dressing operators
- Outlook

Gromov-Witten partition functions of $\mathbb{CP}^1$

- GW invariants (connected part)

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle_{g,d}^{\circ} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d)]^{\text{vir}}} \prod_{i=1}^n \psi_i^{k_i} \text{ev}_i^*(\omega)$$

where

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d) \rightarrow \mathbb{CP}^1, (f, C, p_1, \dots, p_n) \mapsto f(p_i),$$

$$\psi_i = c_1(L_i), (L_i)_{(f, C, p_1, \dots, p_n)} = T_{p_i}^* C$$

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle_{g,d}^{\circ} : \text{all curves (connected and disconnected)}$$

$\bar{M}_{g,n}(\mathbb{CP}^1, d)$: moduli space of **stable maps** $f: C \rightarrow \mathbb{CP}^1$
 of degree d from **stable curve** C of genus g
 with n **marked points** $p_1, \dots, p_n$

$\omega \in H^2(\mathbb{CP}^1)$: Kähler class $c_1(\mathcal{O}(1))$

$\tau_k(\omega)$, $k=0, 1, \dots$: **gravitational descendants**

$$\text{cf. } \left\langle \prod_{i=1}^n \tau_{k_i} \right\rangle_g = \int_{[\bar{M}_{g,n}]} \prod_{i=1}^n \psi_{k_i} \quad : \text{GW invariants of single point}$$

$\rightarrow$ Witten-Kontsevich partition function = **KdV tau fn**

- GW/Hurwitz correspondence (Okounkov - Pandharipande
arxiv: 0204305)

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle_{\text{g.d.}}^\bullet = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \prod_{i=1}^n \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}$$

where

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad (\text{partition}),$$

$$\lambda \vdash d \iff |\lambda| = \sum_{i \geq 1} \lambda_i = d,$$

RHS is related to Hurwitz numbers that count ramified coverings of $\mathbb{CP}^1$.

$$p_k(\lambda) = \sum_{i \geq 1} \left((\lambda_i - i + 1/2)^k - (-i + 1/2)^k \right) + (1 - 2^{-k})\zeta(-k),$$

$$\frac{\dim \lambda}{|\lambda|!} = \prod_{(i,j) \in \lambda} \frac{1}{h(i,j)},$$

$$h(i,j) = \lambda_i + {}^t\lambda_j - i - j + 1 \quad (\text{hook length})$$

Correspondence

$$\tau_k(w) \longleftrightarrow \frac{1}{k!} \overline{(k+1)}$$

GW

Hurwitz

- Generating function (partition function)

$$\begin{aligned} \left\langle \exp \left(\sum_{k=0}^{\infty} \tau_k(\omega) t_k \right) \right\rangle^{\bullet} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n=0}^{\infty} \left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle^{\bullet} \prod_{i=1}^n t_{k_i} \\ &= \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \exp \left(\sum_{k=0}^{\infty} \frac{p_{k+1}(\lambda)}{(k+1)!} t_k \right) \end{aligned}$$

1. KP tau function

2. $\left\langle \exp \left(s \tau_0(1) + \sum_{k=0}^{\infty} \tau_k(\omega) t_k \right) \right\rangle$ becomes

1D Toda tau function. $1 \in H^0(\mathbb{CP}')$: puncture

Toda Conjecture

types of GW theory	Integrable structure	
absolute stationary	extended 1D Toda	(Getzler (conj.) Dubrovin-Zhang)
relative stationary	2D Toda	(OP)
equivariant	equivariant Toda	(Getzler (conj.) OP, Milanov)
orbifold $\mathbb{C}P^1_{a,b}$	extended / equivariant bigraded Toda	(Milanov-Tseng)

Equivariant Gromov-Witten invariants of $\mathbb{C}P^1$

(Okounkov-Pandharipande)
arXiv:0207233

- torus action

$$\mathbb{C}^* \curvearrowright \mathbb{C}P^1 \quad [z_0 : z_1] \mapsto [z_0 : \lambda z_1] \quad \text{fixed points: } 0, \infty$$

- equivariant cohomology

$$H_{\mathbb{C}^*}^*(\mathbb{C}P^1) : \text{module over } H_{\mathbb{C}^*}^*(pt) = \mathbb{C}[\nu]$$

ν : equivariant parameter

$\nu \rightarrow 0$: non-equivariant limit to ordinary cohomology

- equivariant point classes $\emptyset, \infty \in H_{\mathbb{C}^*}^2(\mathbb{CP}^1)$
- generating functions of equivariant GW invariants

$$F = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} h^{2g-2} q^d \left\langle \exp \left(\sum_{k=0}^{\infty} x_k \tau_k(\emptyset) + \sum_{k=0}^{\infty} x_k^* \tau_k(\infty) \right) \right\rangle_{g,d}^0$$

$$Z = e^F$$

- torus action $\rightarrow$ **localization**

multi-point correlation function

$$G_{g,d}^{\circ}(z_1, \dots, z_n, w_1, \dots, w_m) = \sum_{\text{graph } \Gamma} w_{\Gamma}$$

$w_\Gamma = \text{vertex weights} \cdot \text{edge weights}$



combinatorial numbers

linear Hodge integrals (relation to Hurwitz numbers)

- fermionic formula of generating function

$G(z_1, \dots, z_n, w_1, \dots, w_m, \hbar)$ of θ_g^0 's in terms of

$\mathcal{A}$ -operators $\mathcal{A}(Vz, \hbar w)$ and $\mathcal{A}(-Vz, \hbar w)^*$.

$$\bullet \mathcal{A}(z, w) = \left(\frac{\zeta(w)}{w} \right)^2 \sum_{k \in \mathbb{Z}} \frac{\zeta(w)^k}{(1+z)_k} \xi_k(z)$$

$$\zeta(w) = e^{w/2} - e^{-w/2}, \quad (1+z)_k = \Gamma(1+k+z) / \Gamma(1+z)$$

$$\xi_k(z) = \sum_{n \in \mathbb{Z}} e^{z(n+1/2-k/2)} : \psi_{n-k} \psi_n^* :$$

$$[\psi_m, \psi_n^*]_+ = \delta_{mn}, \quad [\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0$$

$$\text{Remark:} \quad : \psi_m \psi_n^* : \longleftrightarrow E_{mn} \in \mathfrak{gl}(\infty)$$

$$(: \psi_m \psi_n^* :)^* = : \psi_n \psi_m^* : \longleftrightarrow \text{transpose}$$

Okounkov - Pandharipande dressing operators (OP arxiv 0207233)

- fermionic formula of partition function

$$Z = \langle 0 | \exp\left(\sum_{k=0}^{\infty} x_k A_k\right) e^{\alpha_1 \left(\frac{q}{\hbar^2}\right)^H} e^{\alpha_{-1}} \exp\left(\sum_{k=0}^{\infty} x_k^* A_k^*\right) | 0 \rangle$$

$$\alpha_n, n \in \mathbb{Z}, \quad [\alpha_m, \alpha_n] = m \delta_{m+n, 0}$$

$$H = L_0, \quad A_k, A_k^* \notin \text{Span}\{\alpha_n \mid n \in \mathbb{Z}\}$$

$$A(z) = \frac{1}{\hbar} \mathcal{A}(\nu z, \hbar z) = \sum_{k \in \mathbb{Z}} A_k z^{k+1}$$

$$A^*(z) = \frac{1}{\hbar} \mathcal{A}^*(-\nu z, \hbar z) = \sum_{k \in \mathbb{Z}} A_k^* z^{k+1}$$

- dressing operators V, V^* (W and W^* in OP's notation)

$$\langle 0 | V = \langle 0 |, \quad V^* | 0 \rangle = | 0 \rangle$$

$$V^{-1} A_k V = \tilde{A}_k \in \text{span} \{ \alpha_n \mid n \geq 1 \}$$

$$V^* A_k^* V^{*-1} = \tilde{A}_k^* \in \text{span} \{ \alpha_{-n} \mid n \geq 1 \}$$

$$\begin{aligned} \bullet \quad Z &= \langle 0 | \exp \left(\underbrace{\sum_{k=0}^{\infty} x_k \tilde{A}_k}_{= \sum_{k=1}^{\infty} t_k \alpha_k} \right) V^{-1} e^{\alpha_1 \left(\frac{q}{t^2} \right)^H} e^{\alpha_{-1}} V^{*-1} \exp \left(\underbrace{\sum_{k=0}^{\infty} x_k^* \tilde{A}_k^*}_{= - \sum_{k=1}^{\infty} \bar{t}_k \alpha_{-k}''} \right) | 0 \rangle \end{aligned}$$

2-KP tau function !

Equivariant Toda hierarchy

$$(Q = q/\hbar^2)$$

$$\tau(s, \hbar, \bar{\hbar}) = Q^{s^2/2} Z(t_1 + \frac{s}{\hbar}, t_2, \dots, \bar{t}_1 + \frac{s}{\bar{\hbar}}, \bar{t}_2, \dots)$$

is a tau function of the **2D Toda hierarchy**
that satisfies the reduction condition

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial \bar{t}_1} - \nu \partial_s \right) (\tau Q^{-s^2/2}) = 0$$

to the equivariant Toda hierarchy.

Remark: Okounkov and Pandharipande's reasoning
is not very clear.

- Lowest equation (equivariant Toda equation)

$$\frac{\partial^2 \log Z(t_1, \bar{t}_1)}{\partial t_1 \partial \bar{t}_1} + Q \frac{Z(t_1 + \frac{1}{Q}, \bar{t}_1 + \frac{1}{Q}) Z(t_1 - \frac{1}{Q}, \bar{t}_1 - \frac{1}{Q})}{Z(t_1, \bar{t}_1)^2} = 0$$

$\uparrow$ from $Q^{S^2/2}$

- Question: What about Lax formalism?

cf. Milanov and Tseng, 0707.3172, Appendix
good review of the equiv. Toda hierarchy.

- 2D Toda hierarchy in Lax formalism ($\hbar=1$)

$$L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n}, \quad \bar{L}^{-1} = \sum_{n=0}^{\infty} \bar{u}_n \Lambda^{n-1} \quad \Lambda = e^{\partial_s}$$

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$

$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}],$$

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{< 0}.$$

- Reduction condition to **orbifold** generalization of equivariant Toda hierarchy :

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \nu \log Q$$

($a=b=1$ for ordinary equivariant Toda hierarchy)

Remark : Definition of $\log L$ and $\log \bar{L}$:

$$L = W \Lambda W^{-1}, \quad \bar{L} = \bar{W} \Lambda \bar{W}^{-1} \quad (W, \bar{W} : \text{dressing operators})$$

$$\log L = W \log \Lambda W^{-1} = \partial_s - \frac{\partial W}{\partial s} W^{-1}, \quad \log \Lambda = \partial_s$$

$$\log \bar{L} = \bar{W} \log \Lambda \bar{W}^{-1} = \partial_s - \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}. \quad \Lambda = e^{\partial_s}$$

- Reduced form of tau function

$$\tau = Q^{S^2/2} Z(t_1, \dots, t_a + \frac{S}{V}, \dots, \bar{t}_1, \dots, \bar{t}_b + \frac{S}{V}, \dots)$$

Reconstruction of OP dressing operators

(T. 2103.10666)

- Redefine V and $\bar{V}$ as difference operators

$$V = 1 + \sum_{n=1}^{\infty} v_n \Lambda^{-n}, \quad \bar{V} = 1 + \sum_{n=1}^{\infty} \bar{v}_n \Lambda^n,$$

$$v_n = v_n(s), \quad \bar{v}_n = \bar{v}_n(s),$$

that satisfy the intertwining relations

$$(\Lambda^a + H - \nu \log \Lambda)V = V(\Lambda^a - \nu \log \Lambda),$$

$$\bar{V}(\Lambda^{-b} + H - \nu \log \Lambda) = (\Lambda^{-b} - \nu \log \Lambda)\bar{V}.$$

$$H = S + \frac{1}{2} \sim L_0$$

Remark: Construction of $\bar{V}$ is parallel ($\bar{V}^* \sim V$)

- The equations for V can be reduced to the equations

$$[\Lambda^a, V_0] + HV_0 = 0,$$

$$[\Lambda^a, V_0^{-1}V_k] = V_0^{-1}[\log \Lambda, V_{k-1}]. \quad k \geq 1.$$

for the coefficients of the V -expansion $V = \sum_{k=0}^{\infty} v^k V_k$.

- We can solve these equations (not uniquely) as

$$V_0 = 1 + \sum_{n=1}^{\infty} v_{0n} \Lambda^{-n}, \quad V_k = \sum_{n=ka}^{\infty} v_{kn} \Lambda^{-n}, \quad k \geq 1.$$

where v_{kn} 's are polynomials in S .

1. Equation for the coeff. of $V_0 = 1 + \sum_{n=1}^{\infty} v_{0n} \lambda^{-n}$:

$$v_{0,n+a}(s+a) - v_{0,n+a}(s) = -H v_{0n}(s)$$

$n = 0$:

$$v_{0a}(s+a) - v_{0a}(s) = -H v_{00}(s) = -s \oplus 1/2$$

Solution :

$$v_{0a}(s) = -\frac{1}{2a}(s+a)s \oplus \frac{1}{2a}s.$$

$n = 1$:

$$v_{0,2a}(s+a) - v_{0,2a}(s) = -H v_{0a}(s).$$

Solution :

$\therefore v_{0,2a}(s) = \text{polynomial of degree 4 in } s.$

$$\begin{cases} u_{0,n} \text{ are polynomials in } s, \\ u_{0,n} = 0 \text{ for } n \not\equiv 0 \pmod{a}. \end{cases}$$

We can use the Bernoulli polynomials to solve difference equations.

2. Equations for the coeff. of $V_0^{-1} V_k = \sum_{n=ka}^{\infty} v'_{kn} \Lambda^{-n}$:

$$v'_{k,n+a}(s+a) - v'_{k,n+a}(s) = f_{kn}(s)$$

where $V_0^{-1}[\log \Lambda, V_{k-1}] = \sum_{n=(k-1)a}^{\infty} f_{kn} \Lambda^{-n}.$

f_{kn} is a polynomial in s .

- Fermionic form of OP tau function amounts to solving the factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1} \bar{W},$$

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n,$$

where

$$U = V^{-1} e^{\Lambda^a/a} Q^H e^{\Lambda^{-b}/b} \bar{V}^{-1} \quad . \quad \left(H = s + \frac{1}{2} \right)$$

cf.

$$\tau = \langle s | \exp\left(\sum_{k=1}^{\infty} t_k \alpha_k\right) \overbrace{V^{-1} e^{\Lambda^a/a} Q^H e^{\Lambda^{-b}/b} \bar{V}^{-1}}^{\text{factorization}} \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \alpha_{-k}\right) | s \rangle$$

- U satisfies the intertwining relation

$$(\Lambda^a - \nu \log \Lambda)U = U(\Lambda^{-b} - \nu \log \Lambda - \nu \log Q).$$

The intertwining relation implies the reduction condition

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \nu \log Q.$$

This explains how the equivariant Toda hierarchy emerges in the Lax formalism.

Outlook

- "Higher spin" analogue

r -spin

$$((\Lambda^a + H)^r - r \log \Lambda) V = V (\Lambda^{ar} - r \log \Lambda), \dots$$

- $r \rightarrow 0$ limit: $L^a = \bar{L}^{-b}$

Bigraded Toda hierarchy of type (a, b) emerges.

Additional (logarithmic) flows can be derived.

(T. 2299.11353)

- Explicit construction of $V_0 = \lim_{r \rightarrow 0} V$ by

Bernoulli polynomials. (cf. Alexandrov's work)