Universal Whitham hierarchy and multi-component KP hierarchy

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- K.T. and T. Takebe, Universal Whitham hierarchy, dispersionless Hirota equations and multi-component KP hierarchy, nlin.SI/0608068.
- arxiv 0808.1444 and 1003.5767 for further progress.

Main messages

1. The universal Whitham hierarchy (of genus zero) is a master system of many dispersionless integrable hierarchies (dispersionless KP, dispersioness Toda, etc).

2. It admits several different expressions: Lax equations, Hamilton-Jacobi equations, Hirota equations, etc.

3. It may be thought of as a dispersionless or quasi-classical limit of the multi-component KP hierarchy formulated in a scalar form.

Universal Whitham hierarchy (of genus 0)

Krichever, Comm. Pure. Appl. Math. **47** (1994), 437–475

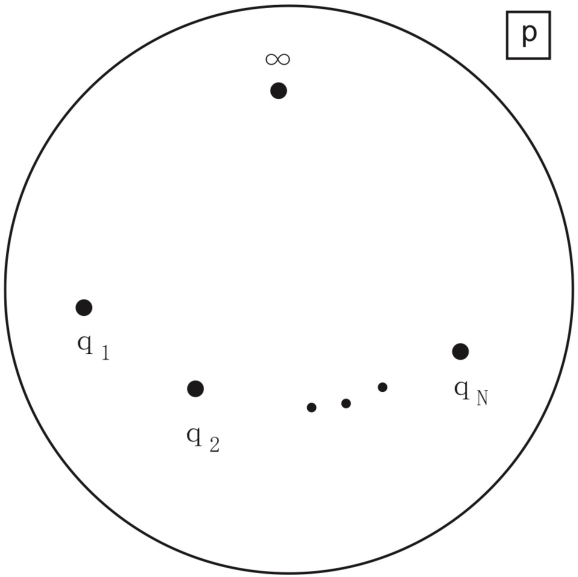
Dynamical variables

marked points $\infty, q_1, \ldots, q_N$

Laurent seires $z_0(p), z_1(p), \ldots, z_N(p)$ $(z_0(p) \sim \mathcal{L} = \mathcal{L}(p))$

$$z_0(p) = p + \sum_{j=2}^{\infty} u_{0j} p^{-j+1},$$

$$z_\alpha(p) = \frac{r_\alpha}{p - q_\alpha} + \sum_{j=1}^{\infty} u_{\alpha j} (p - q_\alpha)^{j-1}$$



$$z_0(p) = p + \sum_{j=2}^{\infty} u_{0j} p^{-j+1},$$

$$z_\alpha(p) = \frac{r_\alpha}{p - q_\alpha} + \sum_{j=1}^{\infty} u_{\alpha j} (p - q_\alpha)^{j-1}$$

u_{0j}, $u_{\alpha j}$ and q_α are dynamical variables.

Spacetime variables

$$\boldsymbol{t} = (\boldsymbol{t}_0, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_N), \quad \boldsymbol{t}_0 = (t_{0n})_{n=1}^{\infty}, \quad \boldsymbol{t}_\alpha = (t_{\alpha n})_{n=0}^{\infty} \ (\alpha = 1, \ldots, N)$$

$t_{01} \sim x$ (spatial variable of dKP hierarchy) dKP = dispersionless KP

$t_{\alpha 0} \sim s$ (spatial variable of dToda hierarchy) dToda = dispersionless Toda

UW hierarchy contains $N+1$ copies of dKP hierarchy and N copies of dToda hierarchy.

p, x

Lax equations

$$\frac{\partial z_\beta(p)}{\partial t_{\alpha n}} = \{\Omega_{\alpha n}(p), z_\beta(p)\} = \frac{\partial \Omega_{\alpha n}(p)}{\partial p}\frac{\partial z_\beta(p)}{\partial t_{01}} - \frac{\partial \Omega_{\alpha n}(p)}{\partial t_{01}}\frac{\partial z_\beta(p)}{\partial p}$$ (Poisson bracket)

$\parallel$

$\partial_{\alpha n} z_\beta(p)$

$$\text{for } \alpha, \beta = 0, 1, \ldots, N,\ n = \begin{cases} 1, 2, \ldots & (\beta = 0) \\ 0, 1, \ldots & (\beta \neq 0) \end{cases}$$

$\Omega_{0n}(p), \Omega_{\alpha n}(p)$ for $n \neq 0$

They are polynomials in p and $(p-q_\alpha)^{-1}$ of the form

$$\Omega_{0n}(p) = p^n + a_{0n2}p^{n-2} + \cdots + a_{0nn},$$
$$\Omega_{\alpha n}(p) = \frac{a_{\alpha n0}}{(p-q_\alpha)^n} + \frac{a_{\alpha n1}}{(p-q_\alpha)^{n-1}} + \cdots + \frac{a_{\alpha nn-1}}{(p-q_\alpha)},$$

rational functions (genus 0)

and given by the singular part of Laurent expansion of $z_0(p)^n$ and $z_\alpha(p)^n$ (including the constant term for the former):

$$z_0(p)^n = \Omega_{0n}(p) + O(p^{-1}) \quad (p \to \infty),$$
$$z_\alpha(p)^n = \Omega_{\alpha n}(p) + O(1) \quad (p \to q_\alpha).$$

$t_{01} = x$

$\Omega_{10}(p), \ldots, \Omega_{N0}(p)$

$\Omega_{01}(p) = p$

They are exceptional and given by logarithmic functions

$$\Omega_{\alpha 0}(p) = -\log(p - q_\alpha).$$

$t_{\alpha 0} = s_\alpha$

dKP and dToda as particular cases

- **dKP (N=0)** (exeptional case)

$z_0(p)$

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \mathcal{L} = p + \sum_{n=1}^{\infty} u_{n+1} p^{-n}$$

$(p \longleftrightarrow \partial_x)$

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}$$

(Poisson bracket) $\{p, x\} = 1$

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0} = p^n + \sum_{m=0}^{n-2} b_{nm} p^{n-m}$$

$\mathcal{B}_1 = p$

Ω_{n0}

$\mathcal{L} = z$, $\mathcal{B}_n = \Omega_n$ $(N=1)$ (no q_α's) $t_1 = t_{01} = x$

• dToda (N=1) spational (lattice) coord. s

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \tilde{t}_n} = \{\tilde{\mathcal{B}}_n, \mathcal{L}\}, \qquad \mathcal{L} = P + u_1 + u_2 P^{-1} + \cdots,$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{\mathcal{B}_n, \tilde{\mathcal{L}}\}, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{t}_n} = \{\tilde{\mathcal{B}}_n, \tilde{\mathcal{L}}\}, \qquad \tilde{\mathcal{L}}^{-1} = \tilde{u}_0 P^{-1} + \tilde{u}_1 + \tilde{u}_2 P + \cdots,$$

$$\{F, G\} = P\left(\frac{\partial F}{\partial P}\frac{\partial G}{\partial s} - \frac{\partial F}{\partial s}\frac{\partial G}{\partial P}\right)$$

$\{P, s\} = P \quad (P \longleftrightarrow e^{\partial s})$

$\{\log P, s\} = 1$

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0} = P^n + \sum_{m=0}^{n-1} b_{nm} P^{n-m}$$

$$\tilde{\mathcal{B}}_n = (\tilde{\mathcal{L}}^{-n})_{<0} = \sum_{m=0}^{n} \tilde{b}_{nm} P^{m-n}$$

$b = u_1$

Let

$$p = B_1 = P + b \qquad (\text{cf. } p \leftrightarrow \partial_{t_1} = \partial_x,\ P \leftrightarrow e^{\partial_s})$$

Then

$$P = p - b, \qquad \mathcal{L} = p + \boxed{\ } p^{-1} + \boxed{\ } p^{-2} + \cdots$$

$$P^{-1} = \frac{1}{p-b}, \qquad \tilde{\mathcal{L}}^{-1} = \frac{\tilde{u}_0}{p-b} + \bigcirc + \bigcirc (p-b) + \cdots$$

Identify

$$z_0(p) = \mathcal{L}(P), \quad z_1(p) = \tilde{\mathcal{L}}(P)^{-1}, \quad q_1 = b$$

$$\Omega_{0n}(p) = B_n(P), \quad \Omega_{1n}(p) = \hat{B}_n(P),$$

$$\Omega_{10}(p) = \log P = \log(p-b), \quad s = t_{10}$$

Remark $\{\ ,\ \} \to [\ ,\ ]$ $\quad \partial_x = \frac{\partial}{\partial x} \to p$

- **KP** $\quad \frac{\partial L}{\partial t_k} = [B_k, L], \quad L = \partial_x + \sum_{n=1}^{\infty} u_{n+1} \partial_x^{-n}$

- **Toda** $\quad \frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$

$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}],$$

$$L = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(1-n)\partial_s}, \quad \bar{L}^{-1} = \sum_{n=0}^{\infty} \bar{u}_n e^{(n-1)\partial_s}$$

$e^{\partial_s} f(s) = f(s+1)$

$e^{\partial_s} \to p, \quad \partial_s \to \log p$

Extended Lax formalism

Zakharov–Shabat equations

$$\partial_{\beta n}\Omega_{\alpha m}(p) - \partial_{\alpha m}\Omega_{\beta n}(p) + \{\Omega_{\alpha m}(p), \Omega_{\beta n}(p)\} = 0.$$

$\Updownarrow$

$\omega \wedge \omega = 0$ for

$$\omega = \sum_{n=1}^{\infty} \mathrm{d}\Omega_{0n}(p) \wedge \mathrm{d}t_{0n} + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} \mathrm{d}\Omega_{\alpha n}(p) \wedge \mathrm{d}t_{\alpha n}.$$

By Darboux's theorem, ω can be expressed as

$$\omega = \mathrm{d}z_\beta(p) \wedge \mathrm{d}\zeta_\beta(p), \quad \beta = 0, 1, \cdots, N$$

ζ_β's are dispersionless analogues of the Orlov-Shulman operators. z_β's and ζ_β's satisfy the extended Lax eqs:

$$\partial_{\alpha n} z_\beta(p) = \{\Omega_{\alpha n}(p), z_\beta(p)\}$$

$$\partial_{\alpha n} \zeta_\beta(p) = \{\Omega_{\alpha n}(p), \zeta_\beta(p)\}$$

$$\{z_\beta(p), \zeta_\beta(p)\} = 1.$$

S-function and Hamilton-Jacobi eqs

$$\omega = \mathrm{d}z_\beta(p) \wedge \mathrm{d}\zeta_\beta(p)$$

$n \geq 1$ for $\alpha = 0$, $n \geq 0$ for $\alpha \neq 0$

$$\Rightarrow \quad d\Big(\zeta_\beta(p)\, dz_\beta(p) - \sum_\alpha \sum_n \Omega_{\alpha n}(p)\, dt_{\alpha n}\Big) = 0$$

$$\Rightarrow \quad \zeta_\beta(p)\, dz_\beta(p) - \sum_\alpha \sum_n \Omega_{\alpha n}(p)\, dt_{\alpha n} = d\mathcal{S}_\beta(p)$$

Substitute the inverse function $p = p_\beta(z)$

of $z = z_\beta(p)$, and define $S_\beta(z) = \mathcal{S}_\beta(p(z))$.

Then

$$
\begin{aligned}
\mathrm{d}S_\beta(z) &= S'_\beta(z)\mathrm{d}z + \sum_{n=1}^{\infty} \Omega_{0n}(p_\beta(z))\mathrm{d}t_{0n} \\
&\quad + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} \Omega_{\alpha n}(p_\beta(z))\mathrm{d}t_{\alpha n},
\end{aligned}
$$

hence

$$\partial_{\alpha n} S_\beta(z) = \Omega_{\alpha n}(p_\beta(z)), \quad \alpha = 0, 1, \cdots, N.$$

Moreover, since $\Omega_{01}(p) = p$, $\alpha = 0, n = 1$

$$p_\beta(z) = \partial_{01} S_\beta(z).$$

Consequently, we have the Hamilton-Jacobi eqs

$$\partial_{\alpha n} S_\beta(z) = \Omega_{\alpha n}(\partial_{01} S(z)), \quad \alpha, \beta = 0, 1, \cdots, N$$

Remark

$$S_0(z) = \sum_{n=1}^{\infty} t_{0n} z^n + t_{00} \log z - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} v_{0n}, \qquad t_{00} = -\sum_{\alpha=1}^{N} t_{\alpha 0}$$

$$S_\beta(z) = \sum_{n=1}^{\infty} t_{\beta n} z^n + t_{\beta 0} \log z + \phi_\beta - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} v_{\beta n}.$$

$$\zeta_\beta(p_\beta(z)) = \partial_z S_\beta(z)$$

F-function and Hirota eqs

F-function $F = F(\{t_{\beta n}\})$ is defined to be a potential function

$$\partial_{0n} F = v_{0n}, \qquad \partial_{\alpha n} F = v_{\alpha n} \quad (n = 1, 2, \ldots),$$

$$\partial_{\alpha 0} F = -\phi_\alpha + \sum_{\beta=1}^{\alpha} t_{\beta 0} \log(-1) \quad (\alpha = 1, \ldots, N),$$

In a generating functional form,

$$S_0(z) = \sum_{n=1}^{\infty} t_{0n} z^n + t_{00} \log z - D_0(z) F,$$

$$S_\alpha(z) = \sum_{n=1}^{\infty} t_{\alpha n} z^n + t_{\alpha 0} \log z + \phi_\alpha - D_\alpha(z) F,$$

where

$$D_0(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{0n}, \qquad D_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{\alpha n}.$$

F satisfies the dispersionless Hirota equations

$$e^{\hat{D}_0(z)\hat{D}_0(w)\mathcal{F}} = 1 - \frac{\partial_{01}(\hat{D}_0(z) - \hat{D}_0(w))\mathcal{F}}{z-w},$$

$$ze^{\hat{D}_0(z)\hat{D}_\alpha(w)\mathcal{F}} = z - \partial_{01}(\hat{D}_0(z) - \hat{D}_\alpha(w))\mathcal{F},$$

$$e^{\hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F}} = -\frac{zw\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\alpha(w))\mathcal{F}}{z-w},$$

$$\epsilon_{\alpha\beta}e^{\hat{D}_\alpha(z)\hat{D}_\beta(w)\mathcal{F}} = -\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\beta(w))\mathcal{F} \quad (\alpha \neq \beta)$$

where

$$\epsilon_{\alpha\beta} = \begin{cases} +1 & (\alpha \leq \beta), \\ -1 & (\alpha > \beta). \end{cases}$$

$$\hat{D}_0(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{0n}$$

$$\hat{D}_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{\alpha n} + \partial_{\alpha 0}$$

How these stuff are related to multi-component KP hierarchy?

- $\tau = \exp\left(\hbar^{-2} F + O(\hbar^{-1})\right)$

- $\Psi_\beta = \exp\left(\hbar^{-1} S_\beta(z)\right)\left(1 + O(\hbar)\right)$

in an $\hbar$-dependent formulation of the $(1+N)$-component KP hierarchy.

“Charged” $N+1$-component KP hierarchy

• $N+1$ sets of continuous variables $\boldsymbol{t}_0 = (t_{01}, t_{02}, \ldots)$, $\boldsymbol{t}_1 = (t_{11}, t_{12}, \ldots)$, … $\boldsymbol{t}_N = (t_{N1}, t_{N2}, \ldots)$
• N discrete variables $s_1, \ldots, s_N \in \mathbf{Z}$, auxiliary variable $s_0 = -s_1 - \ldots - s_N$
• $\boldsymbol{s} = (s_0, s_1, \ldots, s_N)$ is the charge vector (total charge $= 0$) of a state in the Fock space of $N+1$-component charged fermions $\psi_{\alpha j}, \psi^*_{\alpha j}$, $0 \le \alpha \le N$, $j \in \mathbf{Z}$ (DJKM 81; Kac & van de Leur 93):

• Scalar-valued wave functions (= matrix elements in the 1st row of a matrix-valued wave function)

$$\Psi_0(z) = z^{s_0} e^{\xi(\boldsymbol{t}_0, z)} \frac{e^{-D_0(z)} \tau(\boldsymbol{s}, \boldsymbol{t})}{\tau(\boldsymbol{s}, \boldsymbol{t})},$$

$$\Psi_\beta(z) = \tilde{\epsilon}_\beta(\boldsymbol{s}) z^{s_\beta - 1} e^{\xi(\boldsymbol{t}_\beta, z)} \frac{e^{-D_\beta(z)} \tau(\boldsymbol{s} + \boldsymbol{e}_0 - \boldsymbol{e}_\beta, \boldsymbol{t})}{\tau(\boldsymbol{s}, \boldsymbol{t})}, \quad 1 \le \beta \le N,$$

where $\boldsymbol{e}_\alpha = (\ldots, 0, 1, 0, \ldots)$ (1 in the α-th component),

$$\xi(\boldsymbol{t}, z) = \sum_{n=1}^{\infty} t_n z^n, \quad [z^{-1}] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots, \frac{z^n}{n}, \ldots \right)$$

$$\tilde{\epsilon}_\beta(\boldsymbol{s}) = (-1)^{s_1 + \cdots + s_\beta}$$

- Auxiliary linear equations for scalar wave functions

(1) $\partial_{0n}\Psi_\beta(z) = B_{0n}(\partial_{01})\Psi_\beta(z)$,

(2) $\partial_{\alpha n}\Psi_\beta(z) = B_{\alpha n}(\partial_{\alpha 1})\Psi_\beta(z)$,

(3) $\partial_{01}\Psi_\beta(z) = (e^{-\partial_{\alpha 0}} + q_\alpha)\Psi_\beta(z)$,

(4) $\partial_{\alpha 1}\Psi_\beta(z) = r_\alpha e^{\partial_{\alpha 0}}\Psi_\beta(z)$,

(5) $((\partial_{01} - q_\alpha)\partial_{\alpha 1} - r_\alpha)\Psi_\beta(z) = 0, \quad 1 \le \alpha \le N,\ 0 \le \beta \le N,$

where $e^{\partial_{\alpha 0}}$ stand for the shift operators $e^{\partial/\partial s_\alpha}$ in $\boldsymbol{s}$,

$$e^{\pm\partial_{\alpha 0}} f(\boldsymbol{s}) = f(\boldsymbol{s} \mp \boldsymbol{e}_0 \pm \boldsymbol{e}_\alpha) = f(s_0 \mp 1, \ldots, s_\alpha \pm 1, \ldots).$$

Quasi-classical approximation

- $\partial_{\alpha n} \to \hbar \partial_{\alpha n}$, $e^{\pm \partial_{\alpha 0}} \to e^{\pm \hbar \partial_{\alpha 0}}$

- $\hbar \partial_{\alpha n} \Psi_\beta(z) = B_{\alpha n}(\hbar \partial_{01}) \Psi_\beta(z)$

  $\Psi_\beta(z) \sim e^{\hbar^{-1} S_\beta(z)}$

  $$\longrightarrow \quad \partial_{\alpha n} S_\beta(z) = \lim_{\hbar \to 0} B_{\alpha n}(\partial_{01} S_\beta(z))$$

  $$= \Omega_{\alpha n}(\partial_{01} S_\beta(z))$$

Hamilton-Jacobi equation!

- $\partial_{01} S_\beta(z) = p_\beta(z)$

- $\hbar \partial_{01} \Psi_\beta(z) = \left(e^{\hbar \partial_{\alpha 0}} + q_\alpha\right) \Psi_\beta(z)$

$$\to \quad \partial_{01} S_\beta(z) = e^{\partial_{\alpha 0} S_\beta(z)} + q_\alpha$$

$$\to \quad \partial_{\alpha 0} S_\beta(z) = \log\left(\partial_{01} S_\beta(z) - q_\alpha\right) = \Omega_{\alpha 0}\left(p_\beta(z)\right)$$

- $\left(\left(\partial_{\alpha 0} - q_\alpha\right) \partial_{\alpha 1} - r_\alpha\right) \Psi_\beta(z) = 0$ (2D Schrödinger eq)

$$\to \quad \partial_{\alpha 1} S_\beta(z) = \frac{r_\alpha}{p_\beta(z) - q_\alpha} = \Omega_{\alpha 1}\left(p_\beta(z)\right)$$

Remark

$$((\partial_{01} - q_\alpha)\partial_{\alpha 1} - r_\alpha)\Psi(z) = 0$$

$$\to \quad \partial_{\alpha 1}\Psi(z) = (\partial_{01} - q_\alpha)^{-1} r_\alpha \Psi(z)$$

$$\to \quad \partial_{\alpha n}\Psi(z) = (\text{diff. op. in } t_{\alpha 1})\Psi(z)$$

$$= (\text{pseudo-diff. op. in } t_{01})\Psi(z)$$