Extended lattice Gelfand-Dickey hierarchy

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Abstract:
The lattice Gelfand-Dickey hierarchy is a lattice analogue of the Gelfand-Dickey (aka generalized KdV) hierarchy. This integrable hierarchy has an extension by an infinite number of logarithmic flows. These flows are motivated by a possible relation with a kind of Frobenius manifolds and cohomological field theories. The construction of the extended system resembles the extended 1D and bigraded Toda hierarchy, but exhibits several novel features as well. Moreover, this system can be deformed to a generalization of the intermediate long wave hierarchy . This seems to explain an origin of the mysterious logarithmic flows. This talk is based on arXiv:2203.06621 and arXiv:2211.11353.

Gelfand-Dickey (GD) hierarchy

KP hierarchy $\quad \dfrac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \cdots, \quad \partial_x = \dfrac{\partial}{\partial x}$

$L = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + \cdots, \quad B_k = (L^k)_{\geq 0}$ (non-neg. powers of ∂_x)

$\Big\downarrow$ reduction condition $\quad \mathcal{L} = L^N = \partial_x^N + b_2 \partial_x^{N-2} + \cdots + b_N$

GD hierarchy $\quad \dfrac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad \dfrac{\partial \mathcal{L}}{\partial t_{kN}} = 0$

e.g. $N = 2$: KdV

Lattice analogues

- lattice KP hierarchy (aka discrete KP, modified KP, etc)

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \cdots \qquad \Lambda f(s) = f(s+1)$$

$$L = \Lambda + u_1 + u_2 \Lambda^{-1} + \cdots, \qquad \Lambda = e^{\partial_s} \qquad e^{\hbar \partial_s}$$

$$B_k = (L^k)_{\geq 0} = \Lambda^k + b_{k2}\Lambda^{k-2} + \cdots + b_{kk}$$

(non-neg. powers of Λ)

- lattice GD hierarchy

reduction condition

$$\mathcal{L} = L^N = \Lambda^N + b_1 \Lambda^{N-1} + \cdots + b_N \quad (= B_N)$$

$$\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \qquad B_k = (\mathcal{L}^{k/N})_{\geq 0},$$

$$\frac{\partial \mathcal{L}}{\partial t_{kN}} = [B_{kN}, \mathcal{L}] = [\mathcal{L}^k, \mathcal{L}] = 0$$

e.g. $N=2$, $\mathcal{L} = \Lambda^2 + b_1 \Lambda + b_2$ ($\to$ lattice KdV)

Remarks

- b_N is constant : $\frac{\partial b_N}{\partial t_k} = 0$ for all k.

- The ordinary KP and GD are hidden behind :

$$\frac{\partial \Psi}{\partial t_1} = B_1 \Psi = (\Lambda + b)\Psi \longrightarrow \Lambda \Psi = (\partial_{t_1} - b)\Psi$$

elliminate Λ's

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial t_k} &= (\Lambda^k + b_{k1}\Lambda^{k-1} + \cdots + b_{kk})\Psi \\ &= (\partial_{t_1}^k + c_{k2}\partial_{t_1}^{k-2} + \cdots + c_{kk})\Psi \\ z^N \Psi = \mathcal{L}\Psi &= (\partial_{t_1}^N + c_1 \partial_{t_1}^{N-2} + \cdots + c_N)\Psi \end{aligned}\right\} \text{GD}$$

- $\hbar$-dependent formulation $\partial_s \to \hbar\partial_s$, $\partial_{t_k} \to \hbar\partial_{t_k}$

- Frenkel's q-difference GD hierarchy (1996)

$$T = q^{x\partial_x} = e^{(\log q)x\partial_x} \quad (q\text{-shift operator})$$

$$\mathcal{L} = T^N + b_1 T^{N-1} + \cdots + b_N \qquad Tf(x) = f(qx)$$

$$T \leftrightarrow \Lambda = e^{\hbar\partial_s} \text{ with } q = e^{\hbar},\ \log x = s$$

Extension by logarithmic flows

Let $x_1, x_2, \cdots$ be new time variables and consider Lax equations of the form

$$\frac{\partial \mathcal{L}}{\partial x_k} = [(L^{kN} \log L)_{\geq 0}, \mathcal{L}], \quad k = 1, 2, \cdots$$

But what $(L^{kN} \log L)_{\geq 0}$ means? $L^k \log L$ is NOT a genuine difference operator.

Use dressing operator W

$$L = W \Lambda W^{-1}, \quad W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}.$$

$$\log L = W \cdot \partial_s \cdot W^{-1} \qquad (\Lambda = e^{\partial_s})$$

$$= \partial_s - W [\partial_s , W^{-1}]$$

$$= \partial_s - \frac{\partial W}{\partial s} W^{-1}.$$

$$\therefore \; L^{kN} \log L = \mathcal{L}^k \partial_s - \mathcal{L}^k \frac{\partial W}{\partial s} W^{-1}.$$

Let us interprete its $(\ )_{\geq 0}$-part as

$$(L^{kN}\log L)_{\geq 0} = \mathcal{L}^k \partial_s - \left(\mathcal{L}^k \frac{\partial W}{\partial s} W^{-1}\right)_{\geq 0}.$$

After all, this turns out to be a correct interpretation. Thus we re-define the Lax eqs of the logarithmic flows as:

$$\frac{\partial \mathcal{L}}{\partial x_k} = [\mathcal{L}^k \partial_s + P_k, \mathcal{L}],$$

$$P_k = -\left(\mathcal{L}^k \frac{\partial W}{\partial s} W^{-1}\right)_{\geq 0}.$$

Remarks

- The eqs of motion for the last term b_N of $\mathcal{L}$ read
$$\frac{\partial b_N}{\partial t_k} = 0, \qquad \frac{\partial b_N}{\partial x_k} = b_N^k \frac{\partial b_N}{\partial s}.$$

- Buryak and Rossi (arxiv: 1806.09825) proposed the $N=2$ (lattice KdV) case as an integrable structure of an exotic cohomological field theory and considered an extension of the lattice KdV hierarchy (without a Lax ~~from~~ form of the extended flows). Our logarithmic flows seem to match their proposal. At least the eqs of motion of b_N agree with theirs.

- Recent work of Liu-Qu-Wang-Zhang (arxiv: 2402.00373) seems to study a similar issue.

- Introducing yet another independent variable y and replacing $\mathcal{L}^k \partial_s$ and $\frac{\partial W}{\partial s}$ by $\mathcal{L}^k \partial_y$ and $\frac{\partial W}{\partial y}$, we obtain a (2+1)D or toroidal-algebraic extension:

$$\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}],$$

$$\frac{\partial \mathcal{L}}{\partial x_k} = [\mathcal{L}^k \partial_y + P_k, \mathcal{L}],$$

$$P_k = -\left(\mathcal{L}^k \frac{\partial W}{\partial y} W^{-1}\right)_{\geq 0}$$

Bilinear eqs for τ-function

- ext. lattice GD hierarchy: $\tau = \tau(s, \mathbf{t}, \mathbf{x})$

$$\oint z^{mN+(s'-s)} e^{\xi(\mathbf{t}'-\mathbf{t}, z)} \tau(s'-\xi(\alpha, z^N), \mathbf{t}'-[z^{-1}], \mathbf{x}+\alpha)$$

$$\times \tau(s-\xi(\beta, z^N), \mathbf{t}+[z^{-1}], \mathbf{x}+\beta) \frac{dz}{2\pi i} = 0$$

for $m, s'-s \in \mathbb{Z}_{\geq 0}$, where

$$\xi(\mathbf{t}, z) = \sum_{k=1}^{\infty} t_k z^k, \quad [z^{-1}] = \left(z^{-1}, \frac{z^{-2}}{2}, \dots, \frac{z^{-k}}{k}, \dots\right)$$

- (2+1)D lattice GD hierarchy: $\tau = \tau(s, \underset{\sim}{y}, t, x)$

$$\oint z^{mN+(s'-s)} e^{\xi(t'-t,z)} \tau(s', y-\xi(\alpha, z^N), t'-[z^{-1}], x+\alpha)$$
$$\times \tau(s, y-\xi(\beta, z^N), t+[z^{-1}], x+\beta) \frac{dz}{2\pi i} = 0$$

- (2+1)D GD hierarchy $\tau = \tau(y, t, x)$

$$\oint z^{mN} e^{\xi(t'-t,z)} \tau(y-\xi(\alpha, z^N), t'-[z^{-1}], x+\alpha)$$
$$\times \tau(y-\xi(\beta, z^N), t+[z^{-1}], x+\beta) \frac{dz}{2\pi i} = 0$$

Generalized ILW hierarchy

(ILW = Intermediate Long Wave)

is the lattice KP hierarchy under the reduction condition

$$(L^N - \nu \log L)_{<0} = 0$$

where ν is a non-zero parameter.

Reduced Lax operator

$$\mathcal{L} = \Lambda^N + b_1 \Lambda^{N-1} + \cdots + b_N - \nu \partial_s$$

$N = 1$: ILW eq & hierarchy

(Ablowitz-Kodama-Satsuma, Chen-Lee, Degasperis-Lebedev-Olshanetsky-Pakuliak, Tutiya-Satsuma, Shiraishi-Tutiya)

$\gamma \to 0$ limit

- As $\gamma \to 0$, the constraint reduces to

$$(L^N)_{<0} = 0$$

hence the gen. ILW hierarchy reduces to the lattice GD hierarchy. Actually, this limit is not so naive.

- The kN-th flows become trivial
($\frac{\partial L}{\partial t_{kN}} = [L^{kN}, L] = 0$) in the naive limit.
However, they turn into the logarimic flows
in the scaling limit by setting

$$t_k = \begin{cases} T_k & \text{if } k \not\equiv 0 \mod N, \\ X_{k/N}/\nu & \text{if } k \equiv 0 \mod N. \end{cases}$$

and letting $\nu \to 0$.

Remarks

- The structure of the gen. ILW hierarchy resembles the equivariant Toda hierarchy.

  The parameter ν plays the role of equivariant parameter of the equivariant cohomology of $\mathbb{C}P^1$ (Getzler, Okounkov - Pandharipande).

- equivariant (bigraded) Toda hierarchy

$\nu \to 0$ $\Big\downarrow$ scaling limit

extended (bigraded) Toda hierarchy

for non-equivariant Gromov-Witten theory of (2-point orbifold of) $\mathbb{CP}^1$.