Dispersionless integrable hierarchies revisited

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Contents

1. KP to dKP

KP hierarchy

- Pseudo-differential operators

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots, \ \partial_x = \partial/\partial x,$$
$$B_n = (L^n)_{\geq 0} = \partial_x^n + b_{n2}\partial_x^{n-2} + \cdots + b_{nn}$$

(projection to nonnegative powers of ∂_x)

- Lax and Zakharov-Shabat equations

$$\frac{\partial L}{\partial t_n} = [B_n, L],$$
$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0$$

dKP hierarchy (d = dispersionless)

• Dispersionless limit = Classical limit

$$\partial_x \to p,\ x \to x \text{ (2D phase space)}$$
$$[\partial_x, x] = 1 \to \{p, x\} = 1 \text{ (Poisson bracket)}$$

• Classical counterparts of L and B_n are functions on the 2D phase space (p, x):

$$\mathcal{L} = p + u_2 p^{-1} + u_3 p^{-2} + \cdots ,$$
$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0} = p^n + b_{n2} p^{n-2} + \cdots + b_{nn},$$

(projection to nonnegative powers of p)

• Commutators in Lax and Zakharov-Shabat equations are replaced by Poisson brackets:

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\},$$
$$\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0$$

Comments on Benney equation

• The Benney equation (or hierarchy) is a specialization of the dKP hierarchy:

$$\mathcal{L} = p + \sum_{j=1}^{N} \frac{u_j}{p - v_j}$$

• The Benney hierarchy is the dispersionless limit of the so called "constrained KP hierarchy" or of the "rational reduction (Krichever; Enriquez, Orlov and Rubtsov):

$$\begin{aligned} L &= \partial_x + \sum_{j=1}^{N} \phi_j \cdot \partial_x^{-1} \cdot \psi_j \\ &= \partial_x + \sum_{j=1}^{N} (\partial_x - v_j)^{-1} u_j \end{aligned}$$

• The same reduction of the (one-component) KP hierarchy is also known to be related to the $N+1$-wave system (EOR, loc. cit.), hence to a multi-component KP hierarchy.

• The Benney equation is a quasi-classical limit of the vector (= multi-component) NLS equation (Zakharov):

$$\begin{aligned} \partial_t \psi_j &= -\partial_x^2 \psi_j - 2\psi_j \sum_{k=1}^{N} \phi_k \psi_k, \\ \partial_t \phi_j &= \partial_x^2 \phi_j + 2\phi_j \sum_{k=1}^{N} \phi_k \psi_k \end{aligned}$$

2. Toda to dToda

Toda (2-Toda) hierarchy

- Difference operators

$$
\begin{aligned}
L &= e^{\partial_s} + v_1 + v_2 e^{-\partial_s} + \cdots , \\
\bar{L} &= \bar{v}_0 e^{\partial_s} + \bar{v}_1 e^{2\partial_s} + \cdots , \\
A_n &= (L^n)_{\geq 0} = e^{n\partial_s} + a_{n1} e^{(n-1)\partial_s} + \cdots + a_{nn}, \\
\bar{A}_n &= (\bar{L}^{-n})_{<0} = \bar{a}_{n0} e^{-n\partial_s} + \cdots + \bar{a}_{nn-1} e^{-\partial_s}
\end{aligned}
$$

e^{∂_s} is a shift operator on the lattice ($s \in \mathbf{Z}$). $(\cdot)_{\geq 0}$ and $(\cdot)_{<0}$ are projection to nonnegative and negative powers of e^{∂_s}.

- Lax and Zakharov-Shabat equations

$$\frac{\partial L}{\partial t_n} = [A_n, L],\ \frac{\partial L}{\partial \bar{t}_n} = [\bar{A}_n, L],$$
$$\frac{\partial \bar{L}}{\partial t_n} = [A_n, \bar{L}],\ \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{A}_n, \bar{L}],$$
$$\frac{\partial A_m}{\partial t_n} - \frac{\partial A_n}{\partial t_m} + [A_m, A_n] = 0, \text{ etc.}$$

- Toda fields $\phi(s)$

$$A_1 = e^{\partial_s} + b(s),\ \bar{A}_1 = c(s)e^{-\partial_s},$$
$$b(s) = \frac{\partial \phi(s)}{\partial t_1},\ c(s) = e^{\phi(s)-\phi(s-1)},$$
$$\partial_{t_1}\partial_{\bar{t}_1}\phi(s) + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0$$

dToda hierarchy

• Classical limit (s becomes continuous)

$$e^{\partial_s} \to P,\ s \to s \text{ (2D phase space)}$$
$$[e^{\partial_s}, s] = e^{\partial_s} \to \{P, s\} = P \text{ (Poisson bracket)}$$
$$L \to \mathcal{L} = P + v_1 + v_2 P^{-1} + \cdots,$$
$$\bar{L} \to \bar{\mathcal{L}} = \bar{v}_0 P + \bar{v}_1 P^2 + \cdots, \text{ etc.}$$

• Commutators in Lax and Zakharov-Shabat equations are replaced by Poisson brackets.

- Continuous Toda fields $\phi(s)$

$$\partial_{t_1}\partial_{\bar{t}_1}\phi(s) + \partial_s(e^{\partial_s\phi(s)}) = 0$$

- Prepotential $F(s)(\sim \log\tau(s))$

$$\phi(s) = \partial_s F(s),\ \partial_{t_1}\partial_{\bar{t}_1}F(s) + e^{\partial_s^2 F(s)} = 0$$

(Boyer-Finley equation)

Relation to genus-zero Whitham hierarchy

• Changing the "momentum" variable to $p := \mathcal{A}_1 = P + b(s)$ yields

$$\begin{aligned}\bar{\mathcal{A}}_1 &= c(s)P^{-1} = \frac{c(s)}{p-b(s)},\\ \mathcal{A}_n &= p^n + b_{n2}p^{n-2} + \cdots + b_{nn},\\ \bar{\mathcal{A}}_n &= \frac{\bar{b}_{n0}}{(p-b(s))^n} + \cdots + \frac{\bar{b}_{nn-1}}{p-b(s)}\end{aligned}$$

Here t_1 plays the role of "position" variable conjugate to p. The t sector, $t = (t_1, t_2, \ldots)$, is a copy of the dKP hierarchy.

• This can be generalized to the charged multi-component KP hierarchy (N.B. Toda hierarchy = charged two-component KP hierarchy)

3. WKB ansatz and S-function

$\hbar$-dependent reformulation of KP hierarchy

1) $\hbar$-dependent reformulation of Lax equations

$$L = \hbar\partial_x + \sum_{j=1}^{\infty} u_{j+1}(x,t,\hbar)(\hbar\partial_x)^{-j},$$

$$\hbar\frac{\partial L}{\partial t_n} = [B_n, L],\ B_n = (L^n)_{\geq 0}$$

2 Assumption of smooth classical limit

$$u_j(x,t,\hbar) = u_j^{(0)}(x,t) + O(\hbar)$$

$\Longrightarrow$ The classical counterpart

$$\mathcal{L} = p + \sum_{j=1}^{\infty} u_{j+1}^{(0)}(x,t)p^{-j},$$

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0} = p^n + \sum_{j=2}^{n} b_{nj}^{(0)}(x,t)p^{n-j}$$

of L and B_n satsify the dKP hierarchy.

WKB ansatz of wave function

- WKB ansatz (Kodama and Gibbons)

$$\Psi = \exp\Big(\hbar^{-1}S + O(1)\Big),\ S = S(x,t,z),$$

to the auxiliary linear problem

$$z\Psi = L(\partial_x)\Psi,\ \hbar\partial_{t_n}\Psi = B_n(\partial_x)\Psi$$

$\Longrightarrow$ Hamilton-Jacobi equations

$$z = \mathcal{L}(\partial_x S) = \partial_x S + u_2^{(0)}(\partial_x S)^{-1} + \cdots ,$$
$$\partial_{t_n} S = \mathcal{B}_n(\partial_x S) = (\partial_x S)^n + \cdots + b_{nn}^{(0)}$$

for the phase function $S = S(x, t, z)$.

- $\partial_x S$ can be identified with p:

$$p = \partial_x S = z + O(z^{-1})$$

Solving this relation for z determines $\mathcal{L}$ as a function (Laurent series) of p:

$$z = \mathcal{L} = p + u_2^{(0)} p^{-1} + \cdots$$

Translation to differential forms

• The Hamilton-Jacobi equations can be collected to a single equation of 1-forms:

$$dS = \mathcal{M} d\mathcal{L} + p dx + \sum_{n=1}^{\infty} \mathcal{B}_n dt_n$$

$\mathcal{M}$ is the classical counterpart of the Orlov-Schuman operator M:

$$\mathcal{M} = \frac{\partial S}{\partial z}\Big|_{z=\mathcal{L}} = \sum_{n=2}^{\infty} t_n \mathcal{L}^n + x\mathcal{L} + O(p^{-1})$$

• The 1-form equation implies the 2-form equation

$$\omega = d\mathcal{L} \wedge d\mathcal{M}$$

where $\omega = \sum_{n=2}^{\infty} d\mathcal{B}_n \wedge dt_n + dp \wedge dx$ ($\rightarrow$ twistor theory). The 2-form equation is equivalent to the extended Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\},\ \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\},\ \{\mathcal{L}, \mathcal{M}\} = 1$$

Comments

- $\hbar$-dependence of the tau function:

$$\tau = \exp\Big(\hbar^{-2}F + O(\hbar^{-1})\Big)$$

$F = F(x,t)$ is the prepotential of the dKP hierarchy ($u_2^{(0)} = F_{xx}$, etc.)

- Parallel approach to dToda hierarchy

4. New example from q-difference equations

Ref: K.T, Lett. Math. Phys. (June, 2005), nlin.SI/0412067. This paper deals with a q-analogue of the modified KP hierarchy and Toda hierarchies. We here consider a KP version of these models to illustrate the essential part of ideas.

Tau function and wave function

• Introduce a set of parameters $q = (q_1, q_2, \ldots)$ and independent variables $y = (y_1, y_2, \ldots)$, and deform the KP tau function as

$$\tau_q(t, y) = \tau\Big(t + \sum_{n=1}^{\infty} [y_n]_{q_n}^{(n)}\Big) \quad (t_1 = x)$$

Note that x is now identified with t_1. $[z]_q^{(n)}$ is a q-analogue of the usual $[\cdot]$ symbol:

$$[z]_q^{(n)} = \Big(0, \ldots, 0, z, 0, \ldots, 0, \frac{(1-q)^2 z^2}{2(1-q^2)}, \\ \ldots, 0, \ldots, 0, \frac{(1-q)^k z^k}{k(1-q^k)}, \ldots\Big)$$

Namely, $(1-q)^k z^k / k(1-q-k)$ is placed at the kn-th component for $k = 1, 2, \ldots$. This is a KP version of the q-deformed tau function that Mironov, Morozov and Vinet considered for the Toda hierarchy.

• The associated wave function can be defined in the usual way:

$$\Psi_q = \frac{\tau_q(t - [z^{-1}], y)}{\tau_q(t, y)} \exp\Big(\sum_{n=1}^{\infty} t_n z^n\Big) \prod_{n=1}^{\infty} e_{q_n}^{y_n z^n}$$

The last part consists of the q-exponential functions ($\rightarrow$ quantum di-log)

$$e_q^x = \exp\Big(\sum_{k=1}^{\infty} \frac{(1-q)^k x^k}{k(1-q^k)}\Big) = ((1-q)x; q)_\infty$$

Lax and Zakharov-Shabat equations

- Ψ_q satisfies the q-difference linear equations

$$D_{q_n}(y_n)\Psi_q = C_n\Psi_q$$

C_n is a differential operators of the form

$$C_n = \partial_x^n + c_{n2}\partial_x^{n-2} + \cdots + c_{nn}$$

$D_q(x)$ is the q-difference operator $D_q(x)f(x) = (f(qx) - f(x))/(q - qx)$.

• These linear equations lead to the Lax equations

$$D_{q_n}(y_n)L = C_nL - (T_{q_n}(y_n)L)C_n$$

and the Zakharov-Shabat equations

$$\begin{aligned} &D_{q_n}(y_n)C_m - D_{q_m}(y_n)C_n \\ &+(T_{q_n}(y_n)C_m)C_n - (T_{q_m}(y_m)C_n)C_m = 0 \end{aligned}$$

$T_q(x)$ is the translation operator $T_q(x)f(x) = f(qx)$.

• One can consider the t-flows simultaneously, though we shall suppress them in the following.

Classical limit

• Choose $q_n = q^n = e^{-n\beta\hbar}$ (β is a constant), and rescaling $\partial_x \to \hbar\partial_x$, $\partial_{t_n} \to \hbar\partial_{t_n}$ and $D_{q^n}(y_n) \to (1-q^n)D_{q^n}(y_n)$. The auxiliary linear equations thereby take such a form as

$$(1-q^n)D_{q^n}(y_n)\Psi_q = C_n(\hbar\partial_x)\Psi_q$$

• Assume the smooth classical limit of the coefficients of L, C_n, etc. The classical limit of the coefficients of C_n actually coincides with that of B_n:

$$\lim_{\hbar\to 0} C_n(p) = \mathcal{B}_n(p)$$

• Under the WKB ansatz for the wave function $\Psi_q \sim e^{\hbar^{-1}S}$, the action of the q-difference operator turns out to take such a form as

$$(1-q^n)D_{q^n}(y_n)\Psi_q \sim y_n^{-1}(1-e^{-n\beta y_n \partial S/\partial y_n})e^{\hbar^{-1}S}$$

This leas to the differential equation

$$y_n^{-1}(1-e^{-n\beta y_n \partial S/\partial y_n}) = \mathcal{B}_n(\partial_x S)$$

which is the Hamilton-Jacobi equaion for the q-difference linear equation.

• The foregoing Hamilton-Jacobi equations can be rewritten as

$$-n\beta y_n \partial_{y_n} S = \log\Big(1 - y_n \mathcal{B}_n(\partial_x S)\Big)$$

These equations can be converted to dispersionless Lax equations of the form

$$-n\beta y_n \partial_{y_n} \mathcal{L} = \{\log(1 - y_n \mathcal{B}_n(p)),\, \mathcal{L}\}$$

• The unfamiliar logarithmic structure of the Lax equations seems to be related to the integral formula of the di-log function

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = -\int_0^x \log(1-y)\frac{dy}{y}$$

The di-log function is a classical limit of the quantum di-log function $\log(x;q)_\infty$, which appears in the exponent of the q-exponential function e_q^x. Accordingly, the S-function has Laurent expansion of the form

$$S = \sum_{n=2}^{\infty} t_n z^n + xz + \sum_{n=1}^{\infty} \frac{\mathrm{Li}_2(y_n z^n)}{n\beta} + O(z^{-1})$$

• The di-log function is also known to play a role in the "limit shape" of plane partitions (Okounkov and Reshetikhikin).

q-analogue of modified KP and Toda

The Lax equations for these systems are formulated in terms of difference operators in the discrete variable s. Apart from this difference, the whole story is fully parallel.

5. Whitham hierarchy from Multi-component KP hierarchy

The goal of this section is to derive the genus-zero Whitham hierarchy from a multi-component KP hierarchy in classical limit.

Charged tau function

• Tau function of charged $N+1$-component fermion fields (Date, Jimbo, Kashiwara and Miwa; Kac and van de Leur):

$$\tau(s_1, \ldots, s_N, \boldsymbol{t}) = \langle s_1, \ldots, s_N, s_\infty \mid e^{H(\boldsymbol{t})} \mid U \rangle$$

$s_1, \ldots, s_N, s_\infty$ are labels of charges in the Fock space of $N+1$-component charged fermion fields $\psi^{(j)}(z)$, $\psi^{*(j)}(z)$ ($j = 1, \ldots, N, \infty$). The total charge is fixed to be zero

$$s_1 + \cdots + s_N + s_\infty = 0$$

so that s_∞ is determined by the other charges. The time variables $\boldsymbol{t}$ are also grouped as

$$\begin{aligned} \boldsymbol{t} &= (t^{(1)}, \ldots, t^{(N)}, t^{(\infty)}), \\ t^{(j)} &= (t_{j1}, t_{j2}, \ldots), \\ t^{(\infty)} &= (t_{\infty 1}, t_{\infty 2}, \ldots) \end{aligned}$$

• The two component case $\tau(-s, s, \boldsymbol{t})$, $\boldsymbol{t} = (\bar{t}, t)$, is the tau function of the Toda hierarchy (if they do not vanish for all $s \in \mathbf{Z}$). A similar interpretation holds true for the general case (see below).

Wave functions

- An $N+1$ tuple of wave functions are defined as follows:

$$\Psi_j = \frac{\tau(\ldots, s_j - 1, \ldots, t^{(j)} - [z^{-1}], \ldots)}{\tau(\ldots, s_j, \ldots, t^{(j)}, \ldots)} z^{s_j} e^{\xi_j},$$

$$\Psi_\infty = \frac{\tau(\ldots, t^{(\infty)} - [z^{-1}])}{\tau(\ldots, t^{(\infty)})} z^{s_\infty} e^{\xi_\infty},$$

$$\xi_j = \sum_{n=1}^{\infty} t_{jn} z^n, \ \xi_\infty = \sum_{n=1}^{\infty} t_{\infty n} z^n$$

• These scalar wave functions appear in the last low of an $(N+1) \times (N+1)$ matrix wave function (DJKP, KvL, loc. cit.):

$$\mathbf{\Psi} = \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ z^{-1}\Psi_1 & \cdots & z^{-1}\Psi_N & \Psi_\infty \end{pmatrix}$$

They have Laurent expansion of the form

$$\Psi_j = (e^{\phi_j} + O(z^{-1}) z^{s_j} e^{\xi_j},$$
$$\Psi_\infty = (1 + O(z^{-1})) z^{s_\infty} e^{\xi_\infty}$$

As it will turn out below, ϕ_j's are Toda fields.

• Following Segal and Wilson, one can also characterize these wave functions by the condition

$$(\Psi_1, \ldots, \Psi_N, \Psi_\infty) \in U$$

that the row vector of Laurent series lies in a fixed linear space U (a point of an infinite dimensional Grassmann manifold).

Toda-like auxiliary linear equations

• The wave functions $\Psi_1, \ldots, \Psi_N, \Psi_\infty$ turn out to satisfy linear equations of Toda type:

$$\partial_{t_{\infty n}}\Psi = A_{jn}(e^{\partial/\partial s_j})\Psi,\ \partial_{t_{jn}}\Psi = \bar{A}_{jn}(e^{\partial/\partial s_j})\Psi$$

Note that all of Ψ_j's satisfy the same equations. A_{jn} and $\bar{A}_{jk}$ are difference operators of the form

$$A_{jn}(e^{\partial/\partial s_j}) = e^{-n\partial_{s_j}} + \cdots + a_{jnn},$$
$$\bar{A}_{jn}(e^{\partial/\partial s_j}) = \bar{a}_{jn0}e^{n\partial_{s_j}} + \cdots + \bar{a}_{jnn-1}e^{\partial/\partial s_j}$$

In particular,

$$A_{j1} = e^{-\partial/\partial s_j} + b_j,\ \bar{A}_{j1} = c_j e^{\partial/\partial s_j}$$

• One can see, from the linear equation for $\Psi = \Psi_j$, that b_j can be expressed in terms of ϕ_j as

$$b_j = \partial_{t_{\infty 1}} \phi_j, \ c_j = e^{\phi_j(\ldots, s_j, \ldots) - \phi_j(\ldots, s_j+1, \ldots)}$$

The zero-curvature equation

$$\frac{\partial A_{j1}}{\partial t_{j1}} - \frac{\partial \bar{A}_{j1}}{\partial t_{\infty 1}} + [A_{j1}, \bar{A}_{\infty 1}] = 0$$

thereby reduces to the Toda field equation

$$\partial_{t_{j1}} \partial_{t_{\infty 1}} \phi_j + e^{\phi_j(\ldots, s_j - 1, \ldots) - \phi_j(\ldots, s_j, \ldots)} - e^{\phi_j(\ldots, s_j, \ldots) - \phi_j(\ldots, s_j+1, \ldots)} = 0$$

in the $(t_{j1}, t_{\infty 1}, s_j)$ sector.

This is a generalization of the well known fact that Toda equations emerge in the Schlesinger transformations of isomonodromic deformations. Shifting $s_j \to s_j \pm 1$, $s_\infty \to s_\infty \mp 1$ amounts to an elementary Schlesinger transformation.

Classical limit

• The prescription for classical limit is parallel to the previous cases: Rescale

$$e^{\partial/\partial s_j} \to e^{\hbar\partial/\partial s_j},\ \partial_{t_{jn}} \to \hbar\partial_{t_{jn}},\ \phi_j \to \hbar^{-1}\phi_j$$

and set the WKB ansatz

$$\Psi_j \sim e^{\hbar^{-1}S_j}$$

• The phase functions $S = S_1, \ldots, S_N, S_\infty$ satisfy the Hamilton-Jacobi equations

$$\partial_{t_{\infty n}} S = \mathcal{A}_{jn}(e^{\partial S/\partial s_j}),\ \partial_{t_{jn}} S = \bar{\mathcal{A}}_{jn}(e^{\partial S/\partial s_j})$$

They can be collected to the single 1-form equation

$$\begin{aligned} dS &= \sum_{j=1}^{N}\sum_{n=1}^{\infty} \bar{\mathcal{A}}_{jn}(e^{\partial S/\partial s_j})dt_{jn} + \sum_{j=1}^{N} \frac{\partial S}{\partial s_j} ds_j \\ &\quad + \sum_{n=1}^{\infty} \mathcal{A}_{jn}(e^{\partial S/\partial s_j})dt_{\infty n} + \frac{\partial S}{\partial z} dz \end{aligned}$$

Deriving Whitham hierarchy

- The Hamilton-Jacobi equations for $t_{\infty 1}$ read

$$\partial S/\partial t_{\infty 1} = \mathcal{A}_{j1}(e^{\partial S/\partial s_j}) = e^{-\partial S/\partial s_j} + b_j$$

This can be solved for $\partial S/\partial s_j$ as

$$\partial S/\partial s_j = -\log(\partial S/\partial t_{\infty 1} - b_j)^{-1}$$

One can thereby eliminate $\partial S/\partial s_j$ from the other Hamilton-Jacobi equations.

• This is also a place where the variable p is to be introduced as

$$p = \partial S / \partial t_{\infty 1}$$

Solving this relation for z gives a function

$$z = \mathcal{L}(s_1, \ldots, s_N, \boldsymbol{t}, p)$$

that corresponds to the $\mathcal{L}$ function of the dKP and Toda hierarchies. Actually, there are $N+1$ such functions $\mathcal{L}_1, \ldots, \mathcal{L}_N, \mathcal{L}_\infty$ to be obtained from $S = S_1, \ldots, S_N, S_\infty$. They turn out to be Laurent series of p or of $p - b_j$, respectively:

$$\mathcal{L}_\infty = p + O(p^{-1}) \ (p \to \infty),$$
$$\mathcal{L}_j = \frac{c_j}{p - b_j} + O(1) \ (p \to b_j)$$

• In the new independent variables $s_1, \ldots, s_N, \boldsymbol{t}$ and p, the foregoing equation for S now takes the form

$$dS = \sum_{j=1}^{N} \sum_{n=1}^{\infty} \Omega_{jn} dt_{jn} + \sum_{j=1}^{N} \Omega_{j0} ds_j + \sum_{n=1}^{\infty} \Omega_{\infty n} dt_{\infty n} + \mathcal{M} d\mathcal{L}$$

where

$$\Omega_{jn} = \bar{\mathcal{A}}_{jn}((p - b_j)^{-1}),\ \Omega_{j0} = -\log(p - b_j),$$
$$\Omega_{\infty n} = \mathcal{A}_{jn}((p - b_j)^{-1}),\ \mathcal{M} = \partial S / \partial z|_{z=\mathcal{L}}$$

• Recalling the structure of A_{jn} and $\bar{A}_{jn}$, one can see that

$$\Omega_{jn} = \frac{\bar{a}_{jn0}}{(p-b_j)^n} + \cdots + \frac{\bar{a}_{jnn-1}}{p-b_j},$$
$$\Omega_{\infty n} = p^n + \cdots \text{ (polynomial in } p\text{)}$$

This is exactly the setup of the genus-zero Whitham hierarchy!

• It is also instructive to comapre this result with the $N+1$-field system realized as a special case of the constrained one-component KP hierarchy (EOR, loc. cit.). The lowest members, Ω_{j1} and $\Omega_{\infty 1}$, of Ω_{jn}'s are given by

$$\Omega_{j1} = \frac{c_j}{p-b_j},\ \Omega_{j\infty} = p$$

They can be interpreted as the classical limit of $(\partial_x - b_j)^{-1}$ and ∂_x.

• In particular, there is the "quantum-classical correspondence"

$$L = \partial_x + \sum_{j=1}^{N} (\partial_x - b_j)^{-1} c_j \ \leftrightarrow \ \mathcal{L} = p + \sum_{j=1}^{N} \frac{c_j}{p - b_j}$$

between the Lax operator of the constrained KP hierarchy and the Lax function of the Benney hierarchy. We have derived this correspondence via the multi-component KP hierarchy.