Integrable structure of various melting crystal models

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Contents

References

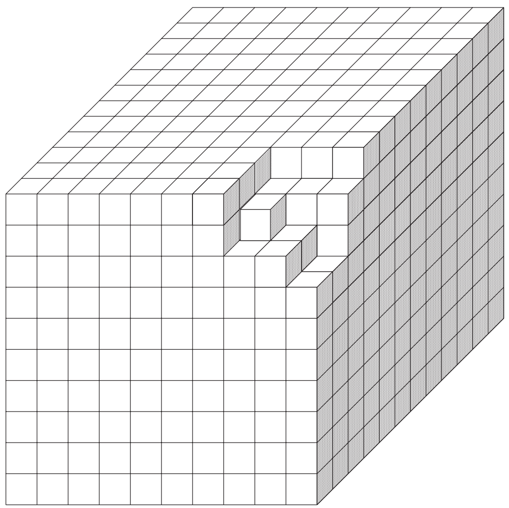
1. K.T. and T. Nakatsu, arXiv:0710.5339 (published)

2. K.T., arXiv:1208.4497, arXiv:1302.6129 (published)

3. K.T., arXiv:1410.5060 (accepted for publication)

1. Ordinary melting crystal model

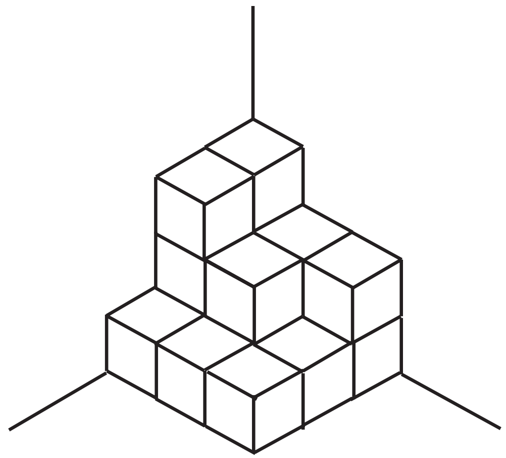
The melting crystal model is a statistical model of a crystal corner in the first octant of the xyz space. The crystal consists of unit cubes, the boundary is a step surface, and the complement in the octant is a 3D Young diagram.



crystal corner

complement

$\longleftrightarrow$



3D Young diagram

Plane partitions and 3D Young diagrams

3D Young diagrams are identified with plane partitions, i.e., non-increasing 2D arrays of non-negative integers:

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{matrix} \pi_{ij} & \geq & \pi_{i,j+1} \\ \rotatebox{90}{$\leq$} & & \\ \pi_{i+1,j} & & \end{matrix}$$

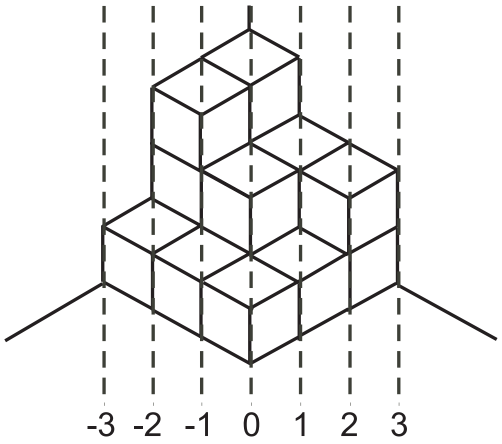
π_{ij} is the height of the stack of cubes on the square $[i-1, i] \times [j-1, j]$ of the xy plane.

Partition function

The Partition function of this model is the sum

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|}, \quad |\pi| = \sum_{i,j=1}^{\infty} \pi_{ij},$$

of the Boltzmann weights $q^{|\pi|}$ $(0 < q < 1)$ over the set $\mathcal{PP}$ of all plane partitions.



This sum can be calculated by the method of diagonal slicing (A. Okounkov and N. Reshetikhin).

$$\pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0, \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0 \end{cases}$$

From plane partitions to semi-standard tableaux

The left and right halves of the diagonal slices give two sequence of Young diagrams growing from $\emptyset$ towards the principal slice $\lambda = \pi(0)$:

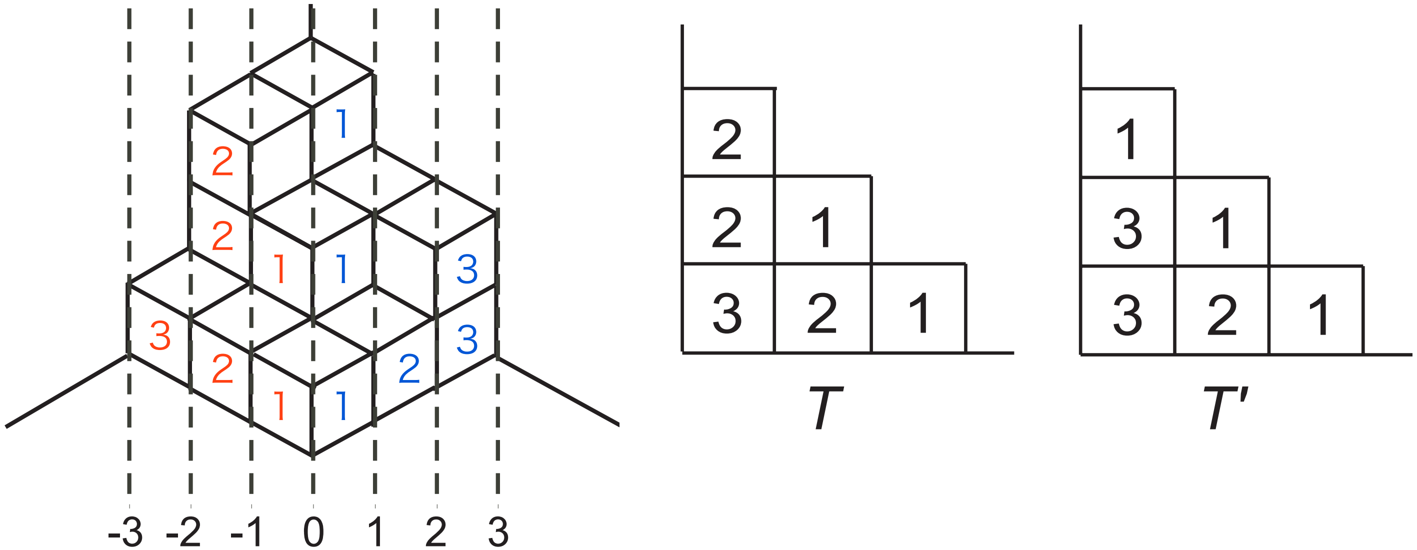
$$\begin{array}{ccccccccccc} \emptyset & \subseteq & \cdots & \subseteq & \pi(-n) & \subseteq & \pi(-(n-1)) & \subseteq & \cdots & \subseteq & \lambda \\ \emptyset & \subseteq & \cdots & \subseteq & \pi(n) & \subseteq & \pi(n-1) & \subseteq & \cdots & \subseteq & \lambda \end{array}$$

Two Young tableaux

$$T = \{T(i,j)\}_{(i,j)\in\lambda}, \quad T' = \{T'(i,j)\}_{(i,j)\in\lambda}$$

of shape λ are determined by inserting the positive integers $n = 1, 2, \ldots$ into the cells (i,j) of the skew Young diagrams $\pi(\pm(n-1))/\pi(\pm n)$.

Example



Left: The entries of the tableaux T, T' can be read out by viewing the 3D Young diagram from the left and right sides, respectively.

Right: The tableaux T, T' are depicted in a position rotated anti-clockwise in 90 degrees.

From plane partitions to semi-standard tableaux (cont'd)

These Young tableaux T, T' are semi-standard tableaux in the sense that the entries are decreasing*) in the horizontal direction and strictly decreasing*) in the vertical direction:

$$\begin{array}{ccc} T(i,j) & \geq & T(i,j+1) \\ \vee & & \\ T(i+1,j) & & \end{array} \qquad \begin{array}{ccc} T'(i,j) & \geq & T'(i,j+1) \\ \vee & & \\ T'(i+1,j) & & \end{array}$$

***) "increasing" and "strictly increasing" in the ordinary definition**

Reduction to sum over triples (λ, T, T')

The foregoing construction is reversible. Namely, any pair of semi-standard tableaux T, T' of shape λ, in turn, determines a plane partition π with $\pi(0) = \lambda$. We thus have a one-to-one correspondence

$$\pi \leftrightarrow (\lambda, T, T'), \quad \pi \in \mathcal{PP}, \quad \lambda \in \mathcal{P}, \quad T, T' \in \mathcal{T}(\lambda).$$

The sum over $\mathcal{PP}$ can be thereby decomposed to a sum over $\mathcal{P}$ and the set $\mathcal{T}(\lambda)$ of all semi-standard tableaux of shape λ:

$$\sum_{\pi \in \mathcal{PP}} (\cdots) = \sum_{\lambda \in \mathcal{P}} \sum_{T, T' \in \mathcal{T}(\lambda)} (\cdots)$$

Reduction to sum over triples (λ, T, T') (cont'd)

The weights $q^{|\pi|}$ can be factorized as $q^{|\pi|} = q^T q^{T'}$, where

$$q^T = \prod_{n=1}^{\infty} q^{(n-1/2)(|\pi(-(n-1))|-|\pi(-n)|)},$$

$$q^{T'} = \prod_{n=1}^{\infty} q^{(n-1/2)(|\pi(n-1)|-|\pi(n)|)}.$$

The partition function thereby takes the partially factorized form

$$Z = \sum_{\lambda \in \mathcal{P}} \left(\sum_{T \in \mathcal{T}(\lambda)} q^T \right) \left(\sum_{T' \in \mathcal{T}(\lambda)} q^{T'} \right).$$

Partial sums over T, T'

The partial sums over T, T' become a special value

$$\sum_{T \in \mathcal{T}(\lambda)} q^{T} = \sum_{T' \in \mathcal{T}(\lambda)} q^{T'} = s_{\lambda}(q^{-\rho})$$

of the Schur function

$$s_{\lambda}(x_1, x_2, \ldots) = \sum_{T \in \mathcal{T}(\lambda)} x^{T}, \quad x^{T} = \prod_{(i,j) \in \lambda} x_{T(i,j)}$$

at

$$q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{i-1/2}, \ldots).$$

Final expression of partition function

The partition function can be reduced to the sum

$$Z = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2$$

over all partitions. By the Cauchy identity

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1},$$

the reduced sum turns into an infinite product (known as the MacMahon function):

$$Z = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1})^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}$$

Slightest generalization

$$\begin{aligned} Z &= \sum_{\pi \in \mathcal{PP}} q^{|\pi|} Q^{|\pi(0)|} \quad \text{(definition)} \\ &= \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} \\ &= \prod_{n=1}^{\infty} (1 - Qq^n)^{-n}. \end{aligned}$$

This is a kind of deformations of the model by the external potential $|\pi(0)|$ (= area of the principal slice) with the coupling constant $\log Q$. An integrable system emerges in deformations by more complicated external potentials.

External potentials $\Phi_k(\lambda, k)$, $k = 1, 2, \ldots$

Heuristic definition (divergent for $0 < q < 1$):

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} q^{k(\lambda_i + s - i + 1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}$$

True definition (by recombination of terms):

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} (q^{k(\lambda_i + s - i + 1)} - q^{k(s-i+1)}) + \frac{1 - q^{ks}}{1 - q^k} q^k$$

They are q-analogues of the eigenvalues of Casimir operators of $U(\infty)$. The parameter $s \in \mathbb{Z}$ plays the role of lattice coordinate in the underlying Toda hierarchy.

Deformed partition function

$$Z(s,t) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda,s,t)}$$

where $\Phi(\lambda,s,t) = \sum_{k=1}^{\infty} t_k\Phi_k(\lambda,s)$. We find the following integrable structure in this function:

(K.T. and T. Nakatsu, 2007) $Z(s,t)$ is related to a tau function $\tau(s,t)$ of the 1D Toda hierarchy as

$$Z(s,t) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}\right) q^{-s(s+1)(2s+1)/6}\tau(s,\iota(t)),$$

$$\iota(t) = (-t_1, t_2, -t_3, \ldots, (-1)^k t_k, \ldots)$$

Idea of proof

1. Use charged fermions to express $Z(s,t)$ as

$$Z(s,t) = \langle s|\Gamma_{+}(q^{-\rho})Q^{L_0}e^{H(t)}\Gamma_{-}(q^{-\rho})|s\rangle$$

where

- $|s\rangle$ and $\langle s|$ are ground states of the charge-s sector in the fermionic Fock and dual Fock spaces $\mathcal{H}, \mathcal{H}^*$.
- $H(t) = \sum_{k=1}^{\infty} t_k H_k$. H_k's are operators such that $H_k|\lambda, s\rangle = \Phi_k(\lambda, s)|\lambda, s\rangle$ for the excited states $|\lambda, s\rangle$.
- $\Gamma_{\pm}(q^{-\rho})$ are specializations of vertex operators $\Gamma_{\pm}(x)$ for which $\langle s|\Gamma_{+}(x)|\lambda, s\rangle = \langle \lambda, s|\Gamma_{-}(x)|s\rangle = s_{\lambda}(x)$.

Idea of proof (cont'd)

2. Use shift symmetries of a quantum torus algebra to convert $Z(s,t)$ to the 1D Toda tau function

$$\tau(s,t) = \langle s|e^{J_+(t)}g|s\rangle = \langle s|ge^{J_-(t)}|s\rangle,$$

where

- **$J_\pm(t) = \sum_{k=1}^\infty t_k J_{\pm k}$, and J_k's are the well known fermionic realization of the Heisenberg algebra.**
- **g is a somewhat complicated operator:**

  $$g = q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{W_0/2}.$$

  W_0 is the zero-mode of a W_3 algebra.

Implications of shift symmetries

Shift symmetries imply the algebraic relation

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})H_k$$

$$= \left(q^{-W_0/2}(-1)^k J_k q^{W_0/2} + \frac{q^k}{1-q^k}\right)\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$$

between H_k and J_k. This can be exponentiated as

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})e^{H(t)}$$

$$= \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right) q^{-W_0/2} e^{J_+(\iota(t))} q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}).$$

This eventually leads to the relation with $\tau(s,t)$ (See similar calculations in Part 2).

Implications of shift symmetries (cont'd)

Shift symmetries also imply that

$$H_k\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$$

$$= \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\left(q^{-W_0/2}(-1)^k J_{-k} q^{W_0/2} + \frac{q^k}{1-q^k}\right),$$

so that

$$e^{H(t)}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$$

$$= \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right)\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{-W_0/2}e^{J_-(\iota(t))}q^{W_0/2}.$$

This leads to another expression $\tau(s,t) = \langle s|g e^{J_-(t)}|s\rangle$ (hence the 1D reduction of the 2D Toda hierarchy).

2. Modified melting crystal model

Undeformed partition function

$$Z' = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}) Q^{|\lambda|} = \prod_{n=1}^{\infty} (1 + Qq^n)^n$$

where ${}^t\lambda$ denotes the transpose (or conjugate partition) of λ. Formally, this model is obtained from the previous model by replacing

$$s_\lambda(q^{-\rho})^2 \longrightarrow s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}).$$

This model is related to topological string theory on a toric Calabi-Yau threefold called the resolved conifold.

Deformed partition function

$$Z'(s,t,\bar{t}) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}) Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda,s,t,\bar{t})},$$

$$\Phi(\lambda,s,t,\bar{t}) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda,s) + \sum_{k=1}^{\infty} \bar{t}_k \Phi_{-k}(\lambda,s).$$

Results obtained in 2012–13 (K.T.)

(i) $Z'(s,t,\bar{t})$ is related to a tau function $\tau'(s,t,\bar{t})$ of the 2D Toda hierarchy.

(ii) This solution of the 2D Toda hierarchy is actually a solution of the Ablowitz-Ladik (or relativistic Toda) hierarchy embedded in the 2D Toda hierarchy.

2.1 Outline of part (i)

Idea of proof of part (i)

Mostly parallel to the case of $Z(s,t)$:

- **Find a fermionic expression of $Z'(s,t,\bar{t})$ in terms of charged free fermions.**

- **Use shift symmetries of a quantum torus algebra to rewrite $Z'(s,t,\bar{t})$.**

Charged fermions

• **Creation-annihilation operators ψ_n, ψ_n^*, $n \in \mathbb{Z}$, with anti-commutation relations**

$$\psi_m \psi_n^* + \psi_n^* \psi_n = \delta_{m+n,0},$$

$$\psi_m \psi_n + \psi_n \psi_m = \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0$$

• **Ground states $\langle s|$, $|s\rangle$ and excited states $\langle \lambda, s|$, $|\lambda, s\rangle$, $\lambda \in \mathcal{P}$, in the charge s sector**

$$\langle s| = \langle -\infty| \cdots \psi_{s-2}^* \psi_{s-1}^* \psi_s^*,$$

$$\langle \lambda, s| = \langle -\infty| \cdots \psi_{\lambda_3+s-2}^* \psi_{\lambda_2+s-1}^* \psi_{\lambda_1+s}^*,$$

$$|s\rangle = \psi_{-s} \psi_{-s+1} \psi_{-s+2} \cdots |-\infty\rangle,$$

$$|\lambda, s\rangle = \psi_{-\lambda_1-s} \psi_{-\lambda_2-s+1} \psi_{-\lambda_3-s+2} \cdots |-\infty\rangle$$

Building blocks of fermionic expression

- **Fermion bilinears**

$$L_0 = \sum_{n\in\mathbb{Z}} n{:}\psi_{-n}\psi_n^*{:}, \quad W_0 = \sum_{n\in\mathbb{Z}} n^2{:}\psi_{-n}\psi_n^*{:},$$

$$H_k = \sum_{n\in\mathbb{Z}} q^{kn}{:}\psi_{-n}\psi_n^*{:}, \quad J_k = \sum_{n\in\mathbb{Z}} {:}\psi_{k-n}\psi_n^*{:}$$

- **Vertex operators**

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right), \; \Gamma'_\pm(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k}\right),$$

$$\Gamma_\pm(x_1, x_2, \ldots) = \prod_{i\geq 1} \Gamma_\pm(x_i), \; \Gamma'_\pm(x_1, x_2, \ldots) = \prod_{i\geq 1} \Gamma'_\pm(x_i)$$

- **Matrix elements**

$$s_\lambda(q^{-\rho}) = \langle s|\Gamma_+(q^{-\rho})|\lambda, s\rangle,$$

$$s_{{}^t\lambda}(q^{-\rho}) = \langle \lambda, s|\Gamma'_-(q^{-\rho})|s\rangle,$$

$$Q^{|\lambda|+s(s+1)/2} = \langle \lambda, s|Q^{L_0}|\lambda, s\rangle,$$

$$\Phi_k(\lambda, s) = \langle \lambda, s|H_k|\lambda, s\rangle$$

Fermionic expression of partition function

$$Z'(s,t,\bar{t}) = \langle s|\Gamma_+(q^{-\rho})Q^{L_0}e^{H(t,\bar{t})}\Gamma'_-(q^{-\rho})|s\rangle,$$

$$H(t,\bar{t}) = H(t) + \bar{H}(\bar{t}), \quad \bar{H}(\bar{t}) = \sum_{k=1}^{\infty} \bar{t}_k H_{-k}$$

Quantum torus algebra

The (centrally extended) quantum torus algebra

$$[V_m^{(k)}, V_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2})(V_{m+n}^{(k+l)} - \delta_{m+n,0}\frac{q^{k+l}}{1-q^{k+l}})$$

is realized by the fermion bilinears

$$V_m^{(k)} = q^{-km/2}\sum_{n\in\mathbb{Z}} q^{kn}{:}\psi_{m-n}\psi_n^*{:}, \quad k, m \in \mathbb{Z}.$$

H_k **and** J_k **are contained therein as**

$$H_k = V_0^{(k)}, \quad J_k = V_k^{(0)}.$$

Shift symmetries

(i) For $k > 0$ and $m \in \mathbb{Z}$,

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\left(V_m^{(k)} - \frac{q^k}{1-q^k}\delta_{m,0}\right)$$
$$= (-1)^k\left(V_{m+k}^{(k)} - \frac{q^k}{1-q^k}\delta_{m+k,0}\right)\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$$

(ii) For $k, m \in \mathbb{Z}$,

$$q^{W_0/2}V_m^{(k)}q^{-W_0/2} = V_m^{(k-m)}$$

Shift symmetries (cont'd)

(iii) For $k > 0$ and $m \in \mathbb{Z}$,

$$\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})\left(V_m^{(-k)} + \frac{1}{1-q^k}\delta_{m,0}\right)$$
$$= \left(V_{m+k}^{(-k)} + \frac{1}{1-q^k}\delta_{m+k,0}\right)\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})$$

• (i) and (ii) are also used in the case of the ordinary melting crystal model. (iii) is a novel one in the modified model.

• These relations are proven by straightforward, but somewhat technical calculations based on commutation relations of ψ_n, ψ_n^* and Clifford operators.

Implications of shift symmetries

Shift symmetries imply the algebraic relations

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})H_k$$
$$= \left(q^{-W_0/2}(-1)^k J_k q^{W_0/2} + \frac{q^k}{1-q^k}\right)\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}),$$

$$H_{-k}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})$$
$$= \Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})\left(q^{-W_0/2}J_{-k}q^{W_0/2} - \frac{1}{1-q^k}\right)$$

among the generators of time evolutions.

Implications of shift symmetries (cont'd)

These algebraic relations can be exponentiated as

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})e^{H(t)}$$

$$= \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right) q^{-W_0/2} e^{J_+(\iota(t))} q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$$

and

$$e^{\bar{H}(\bar{t})}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})$$

$$= \exp\left(\sum_{k=1}^{\infty} - \frac{\bar{t}_k}{1-q^k}\right)\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2}e^{J_-(\bar{t})}q^{W_0/2}$$

Rewriting partition function

$$Z'(s,t,\bar{t}) = \langle s|\Gamma_+(q^{-\rho})e^{H(t)}Q^{L_0}e^{\bar{H}(\bar{t})}\Gamma'_-(q^{-\rho})|s\rangle,$$

$$\begin{aligned}
\langle s|\Gamma_+(q^{-\rho})e^{H(t)} &= \langle s|\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})e^{H(t)} \\
&= \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right) \\
&\quad \times \langle s|q^{-W_0/2}e^{J_+(\iota(t))}q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) \\
&= \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right) q^{-s(s+1)(2s+1)/12} \\
&\quad \times \langle s|e^{J_+(\iota(t))}q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})
\end{aligned}$$

Rewriting partition function (cont'd)

$$Z'(s,t,\bar{t}) = \langle s|\Gamma_+(q^{-\rho})e^{H(t)}Q^{L_0}e^{\bar{H}(\bar{t})}\Gamma'_-(q^{-\rho})|s\rangle,$$

$$\begin{aligned}
e^{\bar{H}(\bar{t})}\Gamma'_-(q^{-\rho})|s\rangle &= e^{\bar{H}(\bar{t})}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})|s\rangle \\
&= \exp\left(\sum_{k=1}^{\infty} -\frac{\bar{t}_k}{1-q^k}\right) \\
&\quad \times \Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2}e^{J_-(\bar{t})}q^{W_0/2}|s\rangle \\
&= \exp\left(\sum_{k=1}^{\infty} -\frac{\bar{t}_k}{1-q^k}\right) q^{s(s+1)(2s+1)/12} \\
&\quad \times \Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2}e^{J_-(\bar{t})}|s\rangle
\end{aligned}$$

Partition function as tau function

Thus we arrive at the following result:

(K.T., 2012) **The partition function is related to a tau function $\tau'(s,t,\bar{t})$ of the 2D Toda hierarchy as**

$$Z'(s,t,\bar{t}) = \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k - \bar{t}_k}{1-q^k}\right) \tau'(s,\iota(t),-\bar{t}).$$

The tau function $\tau'(s,t,\bar{t})$ is defined as

$$\tau'(s,t,\bar{t}) = \langle s| e^{J_+(t)} g' e^{-J_-(\bar{t})} |s\rangle,$$

$$g' = q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2}.$$

2.2 Outline of part (ii)

Idea of proof of part (ii)

- **Translate building blocks of the fermionic expression to the language of $\mathbb{Z} \times \mathbb{Z}$ matrices.**

- **Use a matrix factorization problem to determine the initial values of the dressing operators $W, \bar{W}$ ($\mathbb{Z} \times \mathbb{Z}$ matrices) of the 2D Toda hierarchy.**

- **Show that the Lax operators $L, \bar{L}$ ($\mathbb{Z} \times \mathbb{Z}$ matrices) take a special form that characterizes the Ablowitz-Ladik hierarchy in the 2D Toda hierarchy.**

Matrix representation

- Fermion bilinears and $\mathbb{Z} \times \mathbb{Z}$ matrices are related as

$$X = (x_{ij}) = \sum_{i,j \in \mathbb{Z}} x_{ij} E_{ij} \longleftrightarrow \hat{X} = \sum_{i,j \in \mathbb{Z}} x_{ij} {:}\psi_{-i}\psi_j^*{:}$$

This correspondence can be extended to exponentials of fermion bilinears (Clifford operators).

- Matrix representation of building blocks of $Z'(s, t, \bar{t})$:

$$L_0 = \Delta, \quad W_0 = \Delta^2, \quad H_k = q^{k\Delta}, \quad J_k = \Lambda^k,$$
$$\Gamma_{\pm}(z) = (1 - z\Lambda^{\pm 1})^{-1}, \quad \Gamma'_{\pm}(z) = 1 + z\Lambda^{\pm 1}$$

where $\Delta = \sum_{i \in \mathbb{Z}} i E_{ii}$, $\Lambda = \sum_{i \in \mathbb{Z}} E_{i,i+1}$.

Digression: Encounter with quantum dilogarithm

The matrix representation of $\Gamma_\pm(q^{-\rho})$ and $\Gamma'_\pm(q^{-\rho})$ are matrix-valued quantum dilogarithm:

$$\Gamma_\pm(q^{-\rho}) = \prod_{i=1}^{\infty}(1 - q^{i-1/2}\Lambda^{\pm 1})^{-1},$$

$$\Gamma'_\pm(q^{-\rho}) = \prod_{i=1}^{\infty}(1 + q^{i-1/2}\Lambda^{\pm 1}).$$

Remark: Quantum dilogarithmic function

$$\prod_{i=1}^{\infty}(1-q^{i-1/2}z)^{-1} = 1+\sum_{k=1}^{\infty}\frac{q^{k/2}z^k}{(1-q)(1-q^2)\cdots(1-q^k)}$$

Digression: Encounter with theta function

The vertex operators show up in g and g' in a pair as $\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$ and $\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})$. Jacobi's triple product formula

$$\vartheta(z) = \prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1+q^{n-1/2}z)\prod_{n=1}^{\infty}(1+q^{n-1/2}z^{-1})$$

suggests a link with the theta function.

Remark:

- **Takuya Okuda, arXiv:hep-th/0409270, unitary matrix model with a theta function in the integrand**
- **John Harnad, private communication on another approach to the melting crystal model**

Matrix factorization problem

In principle, all solutions of the 2D Toda hierarchy can be captured by the factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k\Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k\Lambda^{-k}\right) = W^{-1}\bar{W}.$$

• U is a $\mathbb{Z}\times\mathbb{Z}$ matrix that corresponds to the generating operator g of a tau function.

• The problem is to find $\mathbb{Z}\times\mathbb{Z}$ matrices W and $\bar{W}$ that are lower triangular and upper triangular, respectively, and satisfy the factorization relation.

• W and $\bar{W}$ are dressing operators that define the Lax operators $L = W\Lambda W^{-1}$ and $\bar{L} = \bar{W}\Lambda\bar{W}^{-1}$.

Initial values of $W, \bar{W}$

The generating operator g' of $\tau'(s,t,\bar{t})$ has the matrix representation

$$U' = q^{\Delta^2/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{\Delta}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-\Delta^2/2}.$$

Since

$$\Gamma_+(q^{-\rho})Q^{\Delta}\Gamma'_-(q^{-\rho}) = \Gamma'_-(Qq^{-\rho})Q^{\Delta}\Gamma_+(Qq^{-\rho}),$$

this matrix can be factorized to a product of lower and upper triangular matrices as

$$U' = q^{\Delta^2/2}\Gamma_-(q^{-\rho})\Gamma'_-(Qq^{-\rho}) \quad \leftarrow \text{lower triangular}$$

$$\times\, Q^{\Delta}\Gamma_+(Qq^{-\rho})\Gamma'_+(q^{-\rho})q^{-\Delta^2/2}. \quad \leftarrow \text{upper triangular}$$

Initial values of $W, \bar{W}$ (cont'd)

Inserting $q^{-\Delta^2/2}q^{\Delta^2/2} = 1$ in the middle, one can interpret this factorization as solving the matrix factorization problem for U' at the initial time $t = \bar{t} = 0$:

$$W(0,0) = q^{\Delta^2/2}\Gamma'_-(Qq^{-\rho})^{-1}\Gamma_-(q^{-\rho})^{-1}q^{-\Delta^2/2},$$

$$\bar{W}(0,0) = q^{\Delta^2/2}Q^{\Delta}\Gamma_+(Qq^{-\rho})\Gamma'_+(q^{-\rho})q^{-\Delta^2/2}.$$

These explicit forms of the initial values of W and $\bar{W}$ enable us to calculate the initial values of L and $\bar{L}^{-1}$ as well:

$$L(0,0) = W(0,0)\Lambda W(0,0)^{-1},$$

$$\bar{L}(0,0)^{-1} = \bar{W}(0,0)\Lambda^{-1}\bar{W}(0,0)^{-1}$$

Calculating initial values of Lax operators

$$
\begin{aligned}
L(0,0) &= q^{\Delta^2/2}\Gamma'_-(Qq^{-\rho})^{-1}\Gamma_-(q^{-\rho})^{-1}q^{-\Delta^2/2}\Lambda q^{\Delta^2/2} \\
&\qquad \times \Gamma_-(q^{-\rho})\Gamma'_-(Qq^{-\rho})q^{-\Delta^2/2} \\
&= q^{\Delta^2/2}\Gamma'_-(Qq^{-\rho})^{-1}\Gamma_-(q^{-\rho})^{-1}\Lambda q^{\Delta-1/2} \\
&\qquad \times \Gamma_-(q^{-\rho})\Gamma'_-(Qq^{-\rho})q^{-\Delta^2/2} \\
&= q^{\Delta^2/2}\Lambda\Gamma'_-(Qq^{-\rho})^{-1}\Gamma_-(q^{-\rho})^{-1}q^{\Delta-1/2}\cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma_-(q^{-\rho})^{-1}q^{\Delta}\Gamma_-(q^{-\rho}) \\
&\quad = \prod_{i=1}^{\infty}(1-q^{i-1/2}\Lambda^{-1})\cdot\prod_{i=1}^{\infty}(1-q^{i+1/2}\Lambda^{-1})^{-1}\cdot q^{\Delta} \\
&\quad = (1-q^{1/2}\Lambda^{-1})q^{\Delta},
\end{aligned}
$$

Calculating initial values of Lax operators (cont'd)

$$\Gamma'_-(Qq^{-\rho})^{-1} q^{\Delta} \Gamma'_-(Qq^{-\rho})$$

$$= \prod_{i=1}^{\infty}(1 + Qq^{i-1/2}\Lambda^{-1})^{-1} \cdot \prod_{i=1}^{\infty}(1 + Qq^{i+1/2}\Lambda^{-1}) \cdot q^{\Delta}$$

$$= (1 + Qq^{1/2}\Lambda^{-1})^{-1} q^{\Delta},$$

hence

$$L(0,0) = q^{\Delta^2/2}(\Lambda - q^{1/2})(1 + Qq^{1/2}\Lambda^{-1})^{-1} q^{\Delta - 1/2} q^{-\Delta^2/2}.$$

In much the same way,

$$\bar{L}(0,0)^{-1} = q^{\Delta^2/2} Q^{\Delta}(1 + q^{-1/2}\Lambda^{-1})(1 - Qq^{1/2}\Lambda)^{-1} \times q^{-\Delta} Q^{-\Delta} q^{-\Delta^2/2}.$$

Initial values of Lax operators

Thus, after some more algebra, the initial values of L and $\bar{L}^{-1}$ turn out to take a factorized form:

$$L(0,0) = (\Lambda - q^{\Delta})(1 + Qq^{\Delta-1}\Lambda^{-1})^{-1},$$
$$\bar{L}(0,0)^{-1} = -(1 + Qq^{\Delta-1}\Lambda^{-1})(\Lambda - q^{\Delta})^{-1}.$$

Remark: Associativity breaks down partly in the set of $\mathbb{Z} \times \mathbb{Z}$ matrices. In particular,

$$\bar{L}(0,0) = (\bar{L}(0,0)^{-1})^{-1} \neq -L(0,0).$$

Structure of Lax operators

As observed by Brini et al., arXiv:1002.0582, the factorized form of the Lax operators is preserved by time evolutions of the 2D Toda hierarchy. We thus arrive at the following conclusion.

(K.T., 2013) The Lax operators have the factorized form

$$L = BC^{-1}, \quad \bar{L}^{-1} = -CB^{-1},$$

$$B = \Lambda - b, \quad C = 1 + c\Lambda^{-1},$$

b and c are diagonal matrices.

According to Brini et al., this factorized form characterizes the Ablowitz-Ladik hierarchy.

3. Orbifold melting crystal model

Partition functions of undeformed model

$$Z_{a,b} = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1 q^{-\rho}, \ldots, p_a q^{-\rho}) s_\lambda(r_1 q^{-\rho}, \ldots, r_b q^{-\rho}) Q^{|\lambda|}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{n=1}^{\infty} (1 - p_i r_j Q)^{-n},$$

$$Z'_{a,b} = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1 q^{-\rho}, \ldots, p_a q^{-\rho}) s_{{}^{\mathrm{t}}\lambda}(r_1 q^{-\rho}, \ldots, r_b q^{-\rho}) Q^{|\lambda|}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{n=1}^{\infty} (1 + p_i r_j Q)^{n}.$$

$p_1, \ldots, p_a$ and $r_1, \ldots, r_b$ are parameters of the model.

Reparametrization of parameters

Introduce new set of parameters $P_1, P_2, \ldots, P_{a-1}$ and $R_1, R_2, \ldots, R_{b-1}$ as

$$p_i = P_i \cdots P_{a-1} \quad \text{for } i = 1, 2, \ldots, a-1, \quad p_a = 1,$$
$$r_j = R_j \cdots R_{b-1} \quad \text{for } j = 1, 2, \ldots, b-1, \quad r_b = 1.$$

Under this reparametrization, the partition functions can be expressed in the following fermionic form:

$$Z_{a,b} = \langle 0|\Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho})$$
$$\times Q^{L_0}\Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho}) \cdots P_1^{L_0}\Gamma_-(q^{-\rho})|0\rangle,$$
$$Z'_{a,b} = \langle 0|\Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho})$$
$$\times Q^{L_0}\Gamma'_-(q^{-\rho})R_{b-1}^{L_0}\Gamma'_-(q^{-\rho}) \cdots P_1^{L_0}\Gamma'_-(q^{-\rho})|0\rangle.$$

Deformed partition functions

We now deform these partition functions by replacing $\langle 0| \to \langle s|$, $|0\rangle \to |s\rangle$ and inserting $e^{H(t)}$ and $e^{H(t,\bar{t})}$, respectively:

$$Z_{a,b}(s,t) = \langle s|\Gamma_+(q^{-\rho})P_1^{L_0}\cdots\Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho})$$
$$\times Q^{L_0}e^{H(t)}\Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\cdots P_1^{L_0}\Gamma_-(q^{-\rho})|s\rangle,$$
$$Z'_{a,b}(s,t,\bar{t}) = \langle s|\Gamma_+(q^{-\rho})P_1^{L_0}\cdots\Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho})$$
$$\times Q^{L_0}e^{H(t,\bar{t})}\Gamma'_-(q^{-\rho})R_{b-1}^{L_0}\Gamma'_-(q^{-\rho})\cdots P_1^{L_0}\Gamma'_-(q^{-\rho})|s\rangle.$$

This amounts to multiplying the Boltzmann weights with $Q^{s(s+1)/2}e^{\Phi(\lambda,s,t)}$ and $Q^{s(s+1)/2}e^{\Phi(\lambda,s,t,\bar{t})}$.

Relation to tau functions

(K.T., 2014)

$$Z_{a,b}(s,t) = f_{a,b}(s,t)\tau_{a,b}(s,T,0) = f_{a,b}(s,t)\tau_{a,b}(s,0,-\bar{T}),$$

$$Z'_{a,b}(s,t,\bar{t}) = f'_{a,b}(s,t,\bar{t})\tau'_{a,b}(T,-\bar{T}),$$

$$T = (\underbrace{0,\ldots,0}_{a-1},T_1,\underbrace{0,\ldots,0}_{a-1},T_2,\ldots\underbrace{0,\ldots,0}_{a-1},T_k,\ldots),$$

$$\bar{T} = (\underbrace{0,\cdots,0}_{b-1},\bar{T}_1,\underbrace{0,\cdots,0}_{b-1},\bar{T}_2,\ldots,\underbrace{0,\cdots,0}_{b-1},\bar{T}_k,\ldots)$$

where $\tau_{a,b}(s,t,\bar{t})$ and $\tau'_{a,b}(s,t,\bar{t})$ are 2D Toda tau functions, $f_{a,b}(s,t,\bar{t})$ and $f'_{a,b}(s,t,\bar{t})$ are simple functions, and $T_k, \bar{T}_k \propto t_k$ for Z_{ab} and $T_k \propto t_k$, $\bar{T}_k \propto \bar{t}_k$ for Z'_{ab}.

Relation to tau functions (cont'd)

The tau functions $\tau_{a,b}(s,t,\bar{t})$ and $\tau'_{a,b}(s,t,\bar{t})$ are defined by the following generating operators:

$$\begin{aligned} g &= q^{W_0/2a}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})P_1^{L_0}\cdots\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})P_{a-1}^{L_0} \\ &\quad\times\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) \\ &\quad\times R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\cdots R_1^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{W_0/2b}, \\ g' &= q^{W_0/2a}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})P_1^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\cdots P_{a-1}^{L_0} \\ &\quad\times\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho}) \\ &\quad\times R_{b-1}^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})\cdots R_1^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2b}. \end{aligned}$$

Lax operators

(K.T., 2014) The Lax operators of $Z_{a,b}(s,t)$ satisfy the algebraic relation

$$L^a = D^{-1}\bar{L}^{-b}$$

where D is a constant. Both sides of this relation become a multi-diagonal matrix of the form

$$\mathfrak{L} = \Lambda^a + \alpha_1\Lambda^{a-1} + \cdots + \alpha_{a+b}\Lambda^{-b},$$

α_i's are diagonal matrices

This implies that the underlying integrable structure is the bigraded Toda hierarchy of type (a,b).

(K.T., 2014) **The Lax operators of $Z'_{a,b}(s,t,\bar{t})$ have the factorized form**

$$L^a = BC^{-1}, \quad \bar{L}^{-b} = DCB^{-1}$$

where D is a constant, and

$$B = \Lambda^a + \beta_1 \Lambda^{a-1} + \beta_a,$$

$$C = 1 + \gamma_1 \Lambda^{-1} + \cdots + \gamma_b \Lambda^{-b},$$

β_i's and γ_j's are diagonal matrices.

This implies that the underlying integrable structure is the rational reduction of bi-degree (a,b) studied by A. Brini et al., arXiv:1401.5725.

Conclusion

All melting crystal models considered here fall into particular reductions of the 2D Toda hierarchy:

Melting crystal model	Integrable structure
ordinary model $Z(s,t)$	1D Toda hierarchy
modified model $Z'(s,t)$	Ablowitz-Ladik hierarchy
orbifold model $Z_{a,b}(s,t)$	bi-graded Toda hierarchy
orbifold model $Z'_{a,b}(s,t,\bar{t})$	(a,b) rational reduction

The relation between the ordinary and modified models resembles that of the Hermitian and unitary matrix models.