Hurwitz numbers and integrable hierarchy of Volterra type

Kanehisa Takasaki

Taipei, July 7, 2018

1. Generating functions of Hurwitz numbers
2. Lax equations in single Hurwitz sector
3. Perspective from generalized string equations

1. Generating functions of Hurwitz numbers


Hurwitz numbers

\[ H_d(\mu^{(1)}, \ldots, \mu^{(r)}) = \sum_{(*)} \frac{1}{|\text{Aut}(\pi)|} \]

(*) sum over in-equivalent \(d\)-fold coverings \(\pi : C \to \mathbb{CP}^1\) with ramification profile \((\mu^{(1)}, \ldots, \mu^{(r)})\) over \(r\) points \(P_1, \ldots, P_r \in \mathbb{CP}^1\)

Special Hurwitz numbers

1) Double Hurwitz numbers \(H_d(\mu, \bar{\mu}, 1^{d-2}2, \ldots, 1^{d-2}2)\)
2) Single Hurwitz numbers \(H_d(\mu, 1^{d-2}2, \ldots, 1^{d-2}2)\)
1. Generating functions of Hurwitz numbers

Generating function of double Hurwitz numbers

\[ z(x, \bar{x}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=|\bar{\mu}|=d} H_d(\mu, \bar{\mu}, 1^{d-2}, \ldots, 1^{d-2}) \frac{\beta^r}{r!} Q^d p_\mu \bar{p}_{\bar{\mu}} \]

where \( \beta \) and \( Q \) are parameters, and \( p_\mu \) and \( \bar{p}_{\bar{\mu}} \) are monomials \( p_\mu = p_{\mu_1} p_{\mu_2} \cdots, \bar{p}_{\bar{\mu}} = \bar{p}_{\bar{\mu}_1} \bar{p}_{\bar{\mu}_2} \cdots \) of power sums

\[ p_k = \sum_{i \geq 1} x_i^k, \quad \bar{p}_k = \sum_{i \geq 1} \bar{x}_i^k \]

of \( x = (x_1, x_2, \ldots) \) and \( \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots) \).
Generating function of double Hurwitz numbers

\[ z(x, \bar{x}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu| = |\bar{\mu}| = d} H_d(\mu, \bar{\mu}, 1^{d-2}, \ldots, 1^{d-2}) \frac{\beta^r}{r!} Q^d p_\mu \bar{p}_\mu \]

\[ = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda} s_\lambda(x) s_\lambda(\bar{x}) \]

where

\[ \kappa(\lambda) = \sum_{i \geq 1} \lambda_i (\lambda_i - 2i + 1), \quad |\lambda| = \sum_{i \geq 1} \lambda_i, \]

\( s_\lambda(x) \) and \( s_\lambda(\bar{x}) \) are the Schur functions, and \( \mathcal{P} \) is the set of all partitions of arbitrary lengths.
1. Generating functions of Hurwitz numbers

**Tau functions**

\[ z(x, \bar{x}) = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda} s_\lambda(x) s_\lambda(\bar{x}) \]

\[ \downarrow \quad t_k = p_k/k, \quad \bar{t}_k = -\bar{p}_k/k \]

\[ Z(t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda} S_\lambda(t) S_\lambda(-\bar{t}) \]

(tau function of 2-comp. KP hierarchy)

\( S_\lambda(t) \) are the Schur functions in the \( t \)-variables:

\[ S_\lambda(t) = \det (S_{\lambda_{i-i+j}}(t))_{i,j=1}^{N} \]

\[ \sum_{n=0}^{\infty} S_n(t) z^n = \exp \left( \sum_{k=1}^{\infty} t_k z^k \right) \]
1. Generating functions of Hurwitz numbers

**Tau functions (cont’d)**

\[
Z(t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda} S_\lambda(t) S_\lambda(-\bar{t})
\]

\[s \in \mathbb{Z}\]

\[
Z(s, t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta(\kappa(\lambda)+2s\lambda+(4s^3-s)/12)/2} Q^{\lambda+s(s+1)/2} S_\lambda(t) S_\lambda(-\bar{t})
\]

(tau function of 2D Toda hierarchy)
1. Generating functions of Hurwitz numbers

Tau functions (cont’d)

\[ Z(t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta \kappa(\lambda)/2} Q^{\lambda} S_\lambda(t) S_\lambda(-\bar{t}) \]

\[ \downarrow s \in \mathbb{Z} \]

\[ Z(s, t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} e^{\beta (\kappa(\lambda) + 2s|\lambda| + (4s^3 - s)/12)/2} Q^{\lambda} + s(s+1)/2 S_\lambda(t) S_\lambda(-\bar{t}) \]

\[ \downarrow \bar{t}_2 = \bar{t}_3 = \cdots = 0 \quad \text{(single Hurwitz sector)} \]

\[ Z(s, t, \bar{t}_1) = \sum_{\lambda \in \mathcal{P}} e^{\beta (\kappa(\lambda) + 2s|\lambda| + (4s^3 - s)/12)/2} Q^{\lambda} + s(s+1)/2 \]

\[ \times \frac{\dim \lambda}{|\lambda|!} (-\bar{t}_1)^{|\lambda|} S_\lambda(t) \quad (\dim \lambda = \#\text{StTab}(\lambda)) \]

(tau function of lattice KP hierarchy)
Okounkov and Pandharipande’s remark

In a 2001 preprint (arXiv:math/0101147), Okounkov and Pandharipande remarked, with no proof therein, that a generating function of the single Hurwitz numbers yields a solution of the Toda-like field equation

\[ \frac{\partial^2 \phi(s)}{\partial t \partial s} + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0. \]

The generating function is essentially the same as \( Z(s, t, \bar{t}_1) \).
Can we explain this observation?
2. Lax equations in single Hurwitz sector

Single Hurwitz sector

In the single Hurwitz sector \( \bar{t} = (t_1, 0, 0, \ldots) \), the tau function depends on \( t_1 \) through \( e^{\beta s \bar{t}_1} \):

\[
Z(s, t, \bar{t}_1) = e^{\beta(4s^3 - s)/24} Q^{s(s+1)/2} \tilde{Z}(s, t, \bar{t}_1),
\]

\[
\tilde{Z}(s, t, \bar{t}_1) = \sum_{\lambda \in \mathcal{P}} \dim \lambda \frac{e^{\beta \kappa(\lambda)/2} (-Q e^{\beta s \bar{t}_1})^{\lambda}}{|\lambda|!} S_\lambda(t).
\]
2. Lax equations in single Hurwitz sector

Single Hurwitz sector

In the single Hurwitz sector $\bar{t} = (\bar{t}_1, 0, 0, \ldots)$, the tau function depends on $\bar{t}_1$ through $e^{\beta s \bar{t}_1}$:

$$Z(s, t, \bar{t}_1) = e^{\beta (4s^3 - s)/24} Q^{s(s+1)/2} \tilde{Z}(s, t, \bar{t}_1),$$

$$\tilde{Z}(s, t, \bar{t}_1) = \sum_{\lambda \in \mathcal{P}} \dim \lambda \frac{e^{\beta \kappa(\lambda)/2}(-Q e^{\beta s \bar{t}_1})^{\lambda}}{\lambda!} S_{\lambda}(t).$$

Hence

$$\frac{\partial \tilde{Z}(s, t, \bar{t}_1)}{\partial s} = \beta \bar{t}_1 \frac{\partial \tilde{Z}(s, t, \bar{t}_1)}{\partial \bar{t}_1}.$$
2. Lax equations in single Hurwitz sector

Auxiliary linear equations of 2D Toda hierarchy

\[(\partial_{t_k} - B_k) \Psi = 0, \quad k = 1, 2, \ldots, \quad \left(\partial_{\bar{t}_1} - \bar{u}_0 e^{-\partial_s}\right) \Psi = 0\]

where

\[B_k = (L^k)_{\geq 0} = e^{k\partial_s} + b_{k1} e^{(k-1)\partial_s} + \cdots + b_{kk},\]

\[L = e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + \cdots,\]

\[\bar{u}_0 = \frac{Z(s, t, \bar{t}_1)Z(s - 2, t, \bar{t}_1)}{Z(s - 1, t, \bar{t}_1)^2},\]

\[\Psi(s, t, \bar{t}_1, z) = \frac{Z(s - 1, t - [z^{-1}], \bar{t}_1)}{Z(s - 1, t, \bar{t}_1)} z^s e^{\xi(t, z)},\]

\[[x] = (x, x^2/2, \ldots, x^k/k, \ldots), \quad \xi(t, z) = \sum_{k=1}^{\infty} t_k z^k\]
2. Lax equations in single Hurwitz sector

Emergence of logarithmic Lax operator

\[ \left( \partial_{\bar{t}_1} - \bar{u}_0 e^{-\partial_s} \right) \Psi = 0 \]

\[ \downarrow \quad \frac{\partial \tilde{Z}(s, t, \bar{t}_1)}{\partial s} = \beta \bar{t}_1 \frac{\partial \tilde{Z}(s, t, \bar{t}_1)}{\partial \bar{t}_1} \]

\[ \left( \partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial_s} \right) \Psi = (\log z) \Psi \quad \text{(eigenvalue problem!)} \]

Emergence of new Lax operator

\[ \mathfrak{L} = \partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial_s} \]
 Emergence of logarithmic Lax operator

\[ \left( \partial_{\tilde{t}_1} - \tilde{u}_0 e^{-\partial_s} \right) \Psi = 0 \]

\[
\downarrow \quad \frac{\partial \tilde{Z}(s, t, \tilde{t}_1)}{\partial s} = \beta \tilde{t}_1 \frac{\partial \tilde{Z}(s, t, \tilde{t}_1)}{\partial \tilde{t}_1}
\]

\[ \left( \partial_s - \beta \tilde{t}_1 \tilde{u}_0 e^{-\partial_s} \right) \Psi = (\log \, z) \Psi \]

Since \( L \Psi = z \Psi \), the new Lax operator is the logarithm of \( L \):

\[ \mathcal{L} = \partial_s - \beta \tilde{t}_1 \tilde{u}_0 e^{-\partial_s} = \log L \]
2. Lax equations in single Hurwitz sector

Reduced Lax equations

\[ \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad k = 1, 2, \ldots \]

The lowest equation

\[ \left[ \partial_{t_1} e^{\partial_s} - u_1, \partial_s - ve^{-\partial_s} \right] = 0, \quad v = \beta \bar{t}_1 \bar{u}_0 \]

turns into the Toda-like field equation

\[ \frac{\partial^2 \phi(s)}{\partial t_1 \partial s} + e^{\phi(s+1) - \phi(s)} - e^{\phi(s) - \phi(s-1)} = 0 \]

by letting \( u_1(s) = \partial_{t_1} \phi(s), \ v(s) = e^{\phi(s) - \phi(s-1)} \).
Bogoyavlensky-Itoh equations

The $p$-step Bogoyavlensky-Itoh (aka hungry Lotka-Volterra) equation (Bogoyavlensky 1987, Itoh 1987)

$$\frac{dv_k}{dt} = v_k \left( \sum_{i=1}^{p} v_{k-i} - \sum_{i=1}^{p} v_{k+i} \right), \quad k \in \mathbb{Z}$$
Bogoyavlensky-Itoh equations

The $p$-step Bogoyavlensky-Itoh (aka hungry Lotka-Volterra) equation (Bogoyavlensky 1987, Itoh 1987)

$$\frac{dv_k}{dt} = v_k \left( \sum_{i=1}^{p} v_{k-i} - \sum_{i=1}^{p} v_{k+i} \right), \quad k \in \mathbb{Z}$$

In the large-$p$ (continuum) limit as $p \to \infty$, $k/(p+1) \sim s \in \mathbb{R}$ and $t$ being rescaled by $p+1$ (Bogoyavlensky 1988, Itoh 1988),

$$\frac{1}{p+1} \sum_{i=1}^{p} v_{k+i} \sim \int_{s}^{s+1} v(s')ds'$$

$$\frac{\partial v(s)}{\partial t} = v(s) \left( \int_{s-1}^{s} v(s')ds' - \int_{s}^{s+1} v(s')ds \right)$$

16
2. Lax equations in single Hurwitz sector

Bogoyavlensky-Itoh equations (cont’d)

\[
\frac{\partial \log v(s)}{\partial t} = \int_{s-1}^s v(s') ds' - \int_s^{s+1} v(s') ds
\]
\[
\Downarrow \quad v(s) = e^{\phi(s) - \phi(s-1)}
\]
\[
\frac{\partial^2 \phi(s)}{\partial t \partial s} = e^{\phi(s) - \phi(s-1)} - e^{\phi(s+1) - \phi(s)}
\]

(Toda-like field equation)

Continuum limit of Lax operator

\[
\mathcal{L} = e^{\partial_k} - v_k e^{-p \partial_k}
\]
\[
\Downarrow \quad p \to \infty, \quad k/(p+1) \sim s, \quad \text{subtracting 1 from } \mathcal{L}
\]
\[
\mathcal{L} = \partial_s - v(s) e^{-\partial_s}
\]
3. Perspective from generalized string equations

Lax and Orlov-Schulman operators of 2D Toda hierarchy

\[ L = e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + \cdots , \]
\[ \bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \cdots , \]
\[ M = \sum_{k=1}^{\infty} k t_k L^k + s + \sum_{n=1}^{\infty} v_n L^{-n}, \]
\[ \bar{M} = -\sum_{k=1}^{\infty} k \bar{t}_k \bar{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n \]

satisfy the commutation relations \([L, M] = L, [\bar{L}, \bar{M}] = \bar{L}\) and Lax equations of the same form:

\[ \frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial t_k} = [\bar{B}_k, L], \quad \ldots (L \rightarrow \bar{L}, M, \bar{M}) \ldots \]
Generalized string equation for double Hurwitz numbers

The Lax and Orlov-Schulman operators for the double Hurwitz numbers satisfy the generalized string equations (K.T., J. Geom. Phys. 62 (2012), 1135–1156)

\[ L = Q e^{\beta \tilde{M}} \tilde{L}, \quad \tilde{L}^{-1} = Q L^{-1} e^{\beta M} \]

**Remark** They determine a solution of the 2D Toda hierarchy uniquely (at least in the dispersionless limit).
3. Perspective from generalized string equations

Generalized string equation for double Hurwitz numbers

These equations can be converted to the logarithmic form

\[
\begin{align*}
\log L &= \beta \bar{M} + \log \bar{L} - \beta/2 + \log Q, \\
\log \bar{L} &= \log L - \beta M - \beta/2 - \log Q
\end{align*}
\]

Remark  \( \log L \) and \( \log \bar{L} \) can be defined with the aid of dressing operators:

\[
\begin{align*}
L &= W e^{\partial_s} W^{-1}, \\
\bar{L} &= \bar{W} e^{\partial_s} \bar{W}^{-1}, \\
\log L &= W \partial_s W^{-1}, \\
\log \bar{L} &= \bar{W} \partial_s \bar{W}^{-1}
\end{align*}
\]
3. Perspective from generalized string equations

Recovering the reduced Lax operator $\mathcal{L}$

The first equation

$$\log L = \beta \bar{M} + \log \bar{L} - \beta/2 + \log Q$$

implies

$$(\log L)_{<0} = (\beta \bar{M})_{<0} = -\beta \sum_{k=1}^{\infty} k \bar{t}_k \left( \bar{L}^{-k} \right)_{<0}.$$ 

In the single Hurwitz sector $\bar{t} = (\bar{t}_1, 0, 0, \ldots)$,

$$(\log L)_{<0} = -\beta \bar{t}_1 \bar{u}_0 e^{-\partial_s},$$

hence

$$\log L = \partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial_s}.$$
Conclusion

- Okounkov and Pandharipande remarked that a generating function of the single Hurwitz numbers yields a solution of the Toda-like field equation

\[
\frac{\partial^2 \phi(s)}{\partial t \partial s} + e^{\phi(s+1) - \phi(s)} - e^{\phi(s) - \phi(s-1)} = 0.
\]

- This is the lowest equation of the continuum version of the Bogoyavlensky-Itoh hierarchy. We have explained how this integrable structure and its unusual Lax operator

\[
\mathcal{L} = \partial_s - ve^{-\partial_s}
\]

emerge in the machinery of the 2D Toda hierarchy.