Hurwitz numbers and integrable hierarchy of Volterra type

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1. **Generating functions of Hurwitz numbers**
2. **Lax equations in single Hurwitz sector**
3. **Perspective from generalized string equations**

Ref: K.T., arXiv:1807.00085.

1. Generating functions of Hurwitz numbers

A. Okounkov, Math. Res. Lett. 7 (2000), 447–453

Hurwitz numbers

$$H_d(\mu^{(1)}, \dots, \mu^{(r)}) = \sum_{(*)} \frac{1}{|\mathrm{Aut}(\pi)|}$$

(*) sum over in-equivalent d-fold coverings $\pi : C \to \mathbb{CP}^1$ with ramification profile $(\mu^{(1)}, \dots, \mu^{(r)})$ over r points $P_1, \dots, P_r \in \mathbb{CP}^1$

Special Hurwitz numbers

1) Double Hurwitz numbers $H_d(\mu, \bar{\mu}, 1^{d-2}2, \dots, 1^{d-2}2)$

2) Single Hurwitz numbers $H_d(\mu, 1^{d-2}2, \dots, 1^{d-2}2)$

Generating function of double Hurwitz numbers

$$z(x,\bar{x}) = \sum_{r=0}^{\infty}\sum_{d=0}^{\infty}\sum_{|\mu|=|\bar{\mu}|=d} H_d(\mu,\bar{\mu},\underbrace{1^{d-2}2,\cdots,1^{d-2}2}_{r})\frac{\beta^r}{r!}Q^d p_\mu \bar{p}_{\bar{\mu}}$$

where β and Q are parameters, and p_μ and $\bar{p}_{\bar{\mu}}$ are monomials $p_\mu = p_{\mu_1}p_{\mu_2}\cdots$, $\bar{p}_{\bar{\mu}} = \bar{p}_{\bar{\mu}_1}\bar{p}_{\bar{\mu}_2}\cdots$ of power sums

$$p_k = \sum_{i\ge 1} x_i^k,\ \bar{p}_k = \sum_{i\ge 1}\bar{x}_i^k$$

of $x = (x_1, x_2, \ldots)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$.

Generating function of double Hurwitz numbers

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$$= \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa(\lambda)/2}Q^{|\lambda|}s_\lambda(x)s_\lambda(\bar{x})$$

where

$$\kappa(\lambda) = \sum_{i\geq 1}\lambda_i(\lambda_i - 2i + 1), \quad |\lambda| = \sum_{i\geq 1}\lambda_i,$$

$s_\lambda(x)$ and $s_\lambda(\bar{x})$ are the Schur functions, and $\mathcal{P}$ is the set of all partitions of arbitrary lengths.

Tau functions

$$z(x,\bar{x}) = \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa(\lambda)/2} Q^{|\lambda|} s_\lambda(x) s_\lambda(\bar{x})$$

$$\downarrow \quad t_k = p_k/k, \quad \bar{t}_k = -\bar{p}_k/k$$

$$Z(t,\bar{t}) = \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa(\lambda)/2} Q^{|\lambda|} S_\lambda(t) S_\lambda(-\bar{t})$$

(tau function of 2-comp. KP hierarchy)

$S_\lambda(t)$ are the Schur functions in the t-variables:

$$S_\lambda(t) = \det\left(S_{\lambda_i - i + j}(t)\right)_{i,j=1}^{N},$$

$$\sum_{n=0}^{\infty} S_n(t) z^n = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right)$$

Tau functions (cont'd)

$$Z(t,\bar{t}) = \sum_{\lambda\in\mathcal{P}} e^{\beta\kappa(\lambda)/2} Q^{|\lambda|} S_\lambda(t) S_\lambda(-\bar{t})$$

$$\downarrow s \in \mathbb{Z}$$

$$Z(s,t,\bar{t}) = \sum_{\lambda\in\mathcal{P}} e^{\beta(\kappa(\lambda)+2s|\lambda|+(4s^3-s)/12)/2} Q^{|\lambda|+s(s+1)/2} S_\lambda(t) S_\lambda(-\bar{t})$$

(tau function of 2D Toda hierarchy)

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$$\downarrow \bar{t}_2 = \bar{t}_3 = \cdots = 0$$ (single Hurwitz sector)

$$Z(s,t,\bar{t}_1) = \sum_{\lambda\in\mathcal{P}} e^{\beta(\kappa(\lambda)+2s|\lambda|+(4s^3-s)/12)/2} Q^{|\lambda|+s(s+1)/2} \times \frac{\dim\lambda}{|\lambda|!}(-\bar{t}_1)^{|\lambda|} S_\lambda(t) \quad (\dim\lambda = \#\mathrm{StTab}(\lambda))$$

(tau function of lattice KP hierarchy)

Okounkov and Pandharipande's remark

In a 2001 preprint (arXiv:math/0101147), Okounkov and Pandharipande remarked, with no proof therein, that a generating function of the single Hurwitz numbers yields a solution of the Toda-like field equation

$$\frac{\partial^2 \phi(s)}{\partial t \partial s} + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0.$$

The generating function is essentially the same as $Z(s, t, \bar{t}_1)$.

Can we explain this observation?

2. Lax equations in single Hurwitz sector

Single Hurwitz sector

In the single Hurwitz sector $\bar{t} = (\bar{t}_1, 0, 0, \ldots)$, the tau function depends on $\bar{t}_1$ through $e^{\beta s}\bar{t}_1$:

$$Z(s, t, \bar{t}_1) = e^{\beta(4s^3 - s)/24} Q^{s(s+1)/2} \tilde{Z}(s, t, \bar{t}_1),$$

$$\tilde{Z}(s, t, \bar{t}_1) = \sum_{\lambda \in \mathcal{P}} \frac{\dim \lambda}{|\lambda|!} e^{\beta\kappa(\lambda)/2} (-Q e^{\beta s} \bar{t}_1)^{|\lambda|} S_\lambda(t).$$

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Hence

$$\frac{\partial \tilde{Z}(s,t,\bar{t}_1)}{\partial s} = \beta\bar{t}_1 \frac{\partial \tilde{Z}(s,t,\bar{t}_1)}{\partial \bar{t}_1}$$

Auxiliary linear equations of 2D Toda hierarchy

$$(\partial_{t_k} - B_k)\,\Psi = 0, \quad k = 1, 2, \ldots, \quad \left(\partial_{\bar{t}_1} - \bar{u}_0 e^{-\partial_s}\right)\Psi = 0$$

where

$$B_k = (L^k)_{\geq 0} = e^{k\partial_s} + b_{k1} e^{(k-1)\partial_s} + \cdots + b_{kk},$$

$$L = e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + \cdots ,$$

$$\bar{u}_0 = \frac{Z(s, t, \bar{t}_1) Z(s-2, t, \bar{t}_1)}{Z(s-1, t, \bar{t}_1)^2},$$

$$\Psi(s, t, \bar{t}_1, z) = \frac{Z(s-1, t-[z^{-1}], \bar{t}_1)}{Z(s-1, t, \bar{t}_1)} z^s e^{\xi(t,z)},$$

$$[x] = (x, x^2/2, \ldots, x^k/k, \ldots), \quad \xi(t, z) = \sum_{k=1}^{\infty} t_k z^k$$

Emergence of logarithmic Lax operator

$$\left(\partial_{\bar{t}_1} - \bar{u}_0 e^{-\partial_s}\right)\Psi = 0$$

$$\downarrow \qquad \frac{\partial \tilde{Z}(s,t,\bar{t}_1)}{\partial s} = \beta \bar{t}_1 \frac{\partial \tilde{Z}(s,t,\bar{t}_1)}{\partial \bar{t}_1}$$

$$\left(\partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial_s}\right)\Psi = (\log z)\Psi \quad \text{(eigenvalue problem!)}$$

Emergence of new Lax operator

$$\mathfrak{L} = \partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial_s}$$

Emergence of logarithmic Lax operator

$$\left(\partial_{\bar{t}_1} - \bar{u}_0 e^{-\partial_s}\right)\Psi = 0$$

$$\downarrow \qquad \frac{\partial \tilde{Z}(s,t,\bar{t}_1)}{\partial s} = \beta\bar{t}_1 \frac{\partial \tilde{Z}(s,t,\bar{t}_1)}{\partial \bar{t}_1}$$

$$\left(\partial_s - \beta\bar{t}_1\bar{u}_0 e^{-\partial_s}\right)\Psi = (\log z)\Psi$$

Since $L\Psi = z\Psi$, the new Lax operator is the logarithm of L:

$$\mathfrak{L} = \partial_s - \beta\bar{t}_1\bar{u}_0 e^{-\partial_s} = \log L$$

Reduced Lax equations

$$\frac{\partial \mathfrak{L}}{\partial t_k} = [B_k, \mathfrak{L}], \quad k = 1, 2, \ldots$$

The lowest equation

$$\left[\partial_{t_1} - e^{\partial_s} - u_1,\, \partial_s - ve^{-\partial_s}\right] = 0, \quad v = \beta \bar{t}_1 \bar{u}_0$$

turns into the Toda-like field equation

$$\boxed{\frac{\partial^2 \phi(s)}{\partial t_1 \partial s} + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0}$$

by letting $u_1(s) = \partial_{t_1}\phi(s)$, $v(s) = e^{\phi(s)-\phi(s-1)}$.

Bogoyavlensky-Itoh equations

The p-step Bogoyavlensky-Itoh (aka hungry Lotka-Volterra) equation (Bogoyavlensky 1987, Itoh 1987)

$$\frac{dv_k}{dt} = v_k \left(\sum_{i=1}^{p} v_{k-i} - \sum_{i=1}^{p} v_{k+i} \right), \quad k \in \mathbb{Z}$$

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In the large-p (continuum) limit as $p \to \infty$, $k/(p+1) \sim s \in \mathbb{R}$ and t being rescaled by $p+1$ (Bogoyavlensky 1988, Itoh 1988),

$$\frac{1}{p+1} \sum_{i=1}^{p} v_{k+i} \sim \int_{s}^{s+1} v(s')ds',$$

$$\frac{\partial v(s)}{\partial t} = v(s) \left(\int_{s-1}^{s} v(s')ds' - \int_{s}^{s+1} v(s')ds \right)$$

Bogoyavlensky-Itoh equations (cont'd)

$$\frac{\partial \log v(s)}{\partial t} = \int_{s-1}^{s} v(s')ds' - \int_{s}^{s+1} v(s')ds$$

$$\downarrow \quad v(s) = e^{\phi(s)-\phi(s-1)}$$

$$\frac{\partial^2 \phi(s)}{\partial t \partial s} = e^{\phi(s)-\phi(s-1)} - e^{\phi(s+1)-\phi(s)}$$

(Toda-like field equation)

Continuum limit of Lax operator

$$\mathfrak{L} = e^{\partial_k} - v_k e^{-p\partial_k}$$

$\downarrow \quad p \to \infty,\ k/(p+1) \sim s,$ subtracting 1 from $\mathfrak{L}$

$$\mathfrak{L} = \partial_s - v(s)e^{-\partial_s}$$

3. Perspective from generalized string equations

Lax and Orlov-Schulman operators of 2D Toda hierarchy

$$L = e^{\partial_s} + u_1 + u_2 e^{-\partial_s} + \cdots ,$$

$$\bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \cdots ,$$

$$M = \sum_{k=1}^{\infty} k t_k L^k + s + \sum_{n=1}^{\infty} v_n L^{-n},$$

$$\bar{M} = -\sum_{k=1}^{\infty} k \bar{t}_k \bar{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n$$

satisfy the commutation relations $[L, M] = L$, $[\bar{L}, \bar{M}] = \bar{L}$ and Lax equations of the same form:

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad \ldots (L \to \bar{L}, M, \bar{M}) \ldots$$

Generalized string equation for double Hurwitz numbers

The Lax and Orlov-Schulman operators for the double Hurwitz numbers satisfy the generalized string equaitons (K.T., J. Geom. Phys. 62 (2012), 1135–1156)

$$\boldsymbol{L = Qe^{\beta\bar{M}}\bar{L}, \quad \bar{L}^{-1} = QL^{-1}e^{\beta M}}$$

Remark **They determine a solution of the 2D Toda hierarchy uniquely (at least in the dispersionless limit).**

Generalized string equation for double Hurwitz numbers

These equations can be converted to the logarithmic form

$$\log L = \beta\bar{M} + \log\bar{L} - \beta/2 + \log Q,$$
$$\log\bar{L} = \log L - \beta M - \beta/2 - \log Q$$

Remark $\log L$ and $\log\bar{L}$ can be defined with the aid of dressing operators:

$$L = We^{\partial_s}W^{-1}, \quad \bar{L} = \bar{W}e^{\partial_s}\bar{W}^{-1},$$
$$\log L = W\partial_s W^{-1}, \quad \log\bar{L} = \bar{W}\partial_s\bar{W}^{-1}$$

Recovering the reduced Lax operator $\mathfrak{L}$

The first equation

$$\log L = \beta \bar{M} + \log \bar{L} - \beta/2 + \log Q$$

implies

$$(\log L)_{<0} = (\beta \bar{M})_{<0} = -\beta \sum_{k=1}^{\infty} k \bar{t}_k \left(\bar{L}^{-k}\right)_{<0}.$$

In the single Hurwitz sector $\bar{t} = (\bar{t}_1, 0, 0, \ldots)$,

$$(\log L)_{<0} = -\beta \bar{t}_1 \bar{u}_0 e^{-\partial_s},$$

hence

$$\log L = \partial_s - \beta \bar{t}_1 \bar{u}_0 e^{-\partial_s}.$$

Conclusion

- **Okounkov and Pandharipande remarked that a generating function of the single Hurwitz numbers yields a solution of the Toda-like field equation**

$$\frac{\partial^2 \phi(s)}{\partial t \partial s} + e^{\phi(s+1)-\phi(s)} - e^{\phi(s)-\phi(s-1)} = 0.$$

- **This is the lowest equation of the continuum version of the Bogoyavlensky-Itoh hierarchy. We have explained how this integrable structure and its unusual Lax operator**

$$\mathfrak{L} = \partial_s - v e^{-\partial_s}$$

emerge in the machinery of the 2D Toda hierarchy.