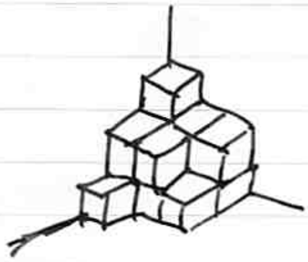
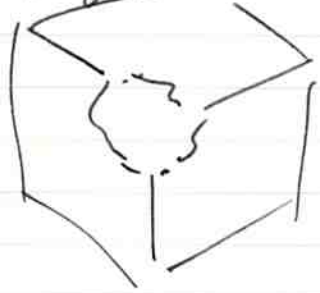
Melting crystal model (random plane partition)

3D Young diagram (plane partition)



$\xrightarrow{\text{complement}}$ crystal corner



$\pi = \emptyset$ $\qquad$ $\pi = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

plane partition

$\pi = (\pi_{ij})_{i,j=0}^{\infty}$ $\qquad$ $\pi_{ij} \geq \pi_{i\,j+1}$, $\quad \pi_{ij} \geq \pi_{i+1\,j}$

$\pi_{ij} \in \mathbb{Z}_{\geq 0}$

$\updownarrow$ 1:1

3D Young diagram $\qquad$ π_{ij} = height of stack at (i,j)

π_{ij}

$0 < q < 1$ $\qquad$ $|\pi| = \sum_{i,j=0}^{\infty} \pi_{ij}$ (volume)

partition function

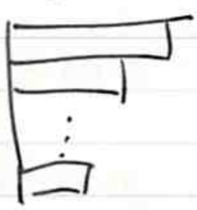
$$Z = \sum_{\pi \in PP} q^{|\pi|}$$

PP: set of all PP's

explicit formula (MacMahon): $Z = \prod_{k=1}^{\infty} (1-q^k)^{-k}$

---

cf. Ordinary Young diagram and partition



$\lambda_1, \lambda_2, \vdots, \lambda_n$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \cdots$

$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$
$= (\lambda_1, \lambda_2, \cdots, \lambda_n, 0, 0, \cdots)$

$$\boxed{Z = \sum_{\lambda \in P} q^{|\lambda|} = \prod_{k=1}^{\infty} (1-q^k)^{-1}}$$

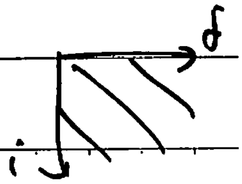
μ_1, μ_2, μ_m

$\mu = (\mu_1, \mu_2, \cdots) = {}^t\lambda$

conjugate partition

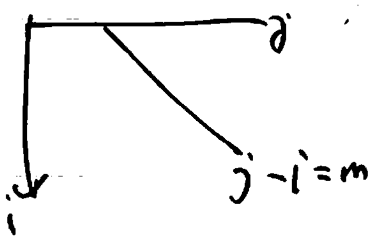
$|\lambda| = \sum_{i \geq 0} \lambda_i \; (= \sum_{j \geq 0} \mu_j)$

diagonal slicing (Okounkov & Reshetikhin)

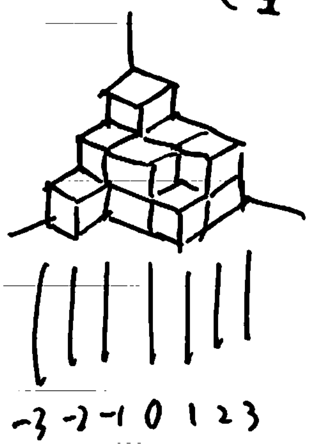


$$\pi = (\pi_{ij})_{i,j\ge 0} \mapsto \pi(m) = \begin{cases} (\pi_{i,i+m})_{i\ge 0} & (m \ge 0) \\ (\pi_{j-m,j})_{j\ge 0} & (m \le 0) \end{cases}$$

ordinary partition
(↔ Young diagram)



eg. $\pi = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$



$\pi(3) = \emptyset$

$\pi(2) = (2)$

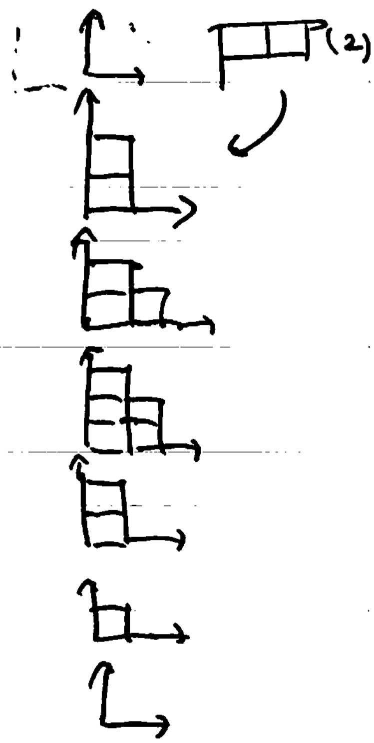
$\pi(1) = (2,1)$

$\pi(0) = (3,2)$

$\pi(-1) = (2)$

$\pi(-2) = (1)$

$\pi(-3) = \emptyset$

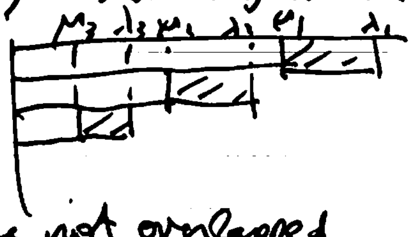


$\boxed{\pi(m)\text{'s are not arbitrary}}$

$$\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots$$

<u>interlacing relation</u> $\lambda \succ \mu \overset{\text{def}}{\iff} \lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots$

$\iff \lambda \supseteq \mu$, $\lambda \setminus \mu$ is a horizontal strip



are not overlapped vertically

coding to triples

$\pi \overset{1:1}{\longleftrightarrow} (\lambda, T, T')$, $\lambda \in P$, $T, T' \in SSTab(\lambda)$

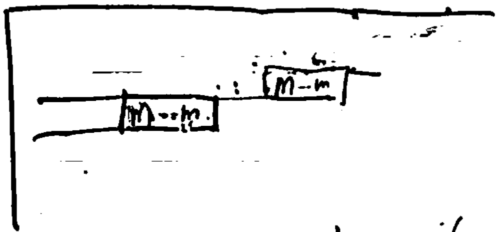
$\{\pi(m)\}_{m \geq 0} \mapsto$ semistandard tableau T ~~of~~ of shape λ

$\{\pi(-m)\}_{m \leq 0} \mapsto$ T'.

T entries $\{1, 2, \cdots\}$

$T = (T(i,j))_{(i,j) \in \lambda}$

$T(i,j) = m \overset{def}{\iff} (i,j) \in \pi(m-1) / \pi(m)$.



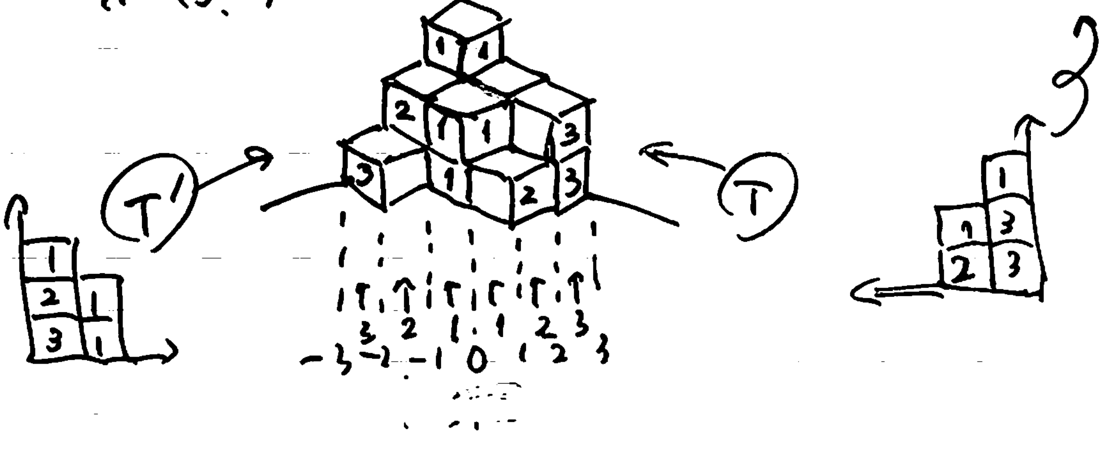
semi-standard $\begin{cases} \cdot \text{ decreasing in } \longrightarrow & (\leftarrow \pi(m) \supset \pi(m+1)) \\ \cdot \text{ strictly decreasing in } \downarrow & (\leftarrow \text{no overlap vertically}) \end{cases}$

[**Remark** This is opposite to usual definition.
("increasing" rather than "decreasing")
but <u>no problem</u>]

$T' = (T'(ij))$ $T'(ij) = m \overset{def}{\iff} (ij) \in \pi(-m+1)/\pi(-m)$

e.g. $\pi = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $T' = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 1 & 1 & \\ \hline \end{array}$ $T = \begin{array}{|c|c|c|} \hline 3 & 3 & 1 \\ \hline 2 & 1 & \\ \hline \end{array}$

$\lambda = (3, 2)$

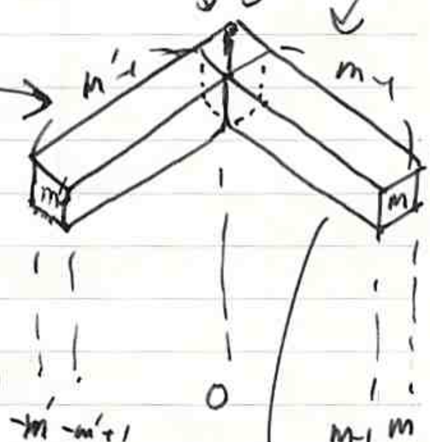


4. Partial sum of partition function

$$Z = \sum_{\pi \in PP} q^{|\pi|} \qquad \pi \longleftrightarrow (\lambda, T, T')$$

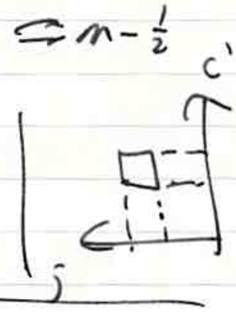
$$|\pi| = \sum_{m=1}^{\infty} |\pi(m-1)/\pi(m)| + |\pi(0)| + \sum_{m=1}^{\infty} |\pi(-m+1)/\pi(-m)|$$

$$|\pi(0)| = \tfrac{1}{2}|\pi(0)| + \tfrac{1}{2}|\pi'(0)|$$



$(i,j) \in \lambda$ contributes to the volume by $m - \frac{1}{2} = T(i,j) - \frac{1}{2}$

volume $= m-1+\frac{1}{2}$ $= m - \frac{1}{2}$



$$|\pi| = \sum_{(i,j)\in\lambda} \left(T(i,j) - \tfrac{1}{2}\right) + \sum_{(i,j)\in\lambda} \left(T'(i,j) - \tfrac{1}{2}\right)$$

$$q^{|\pi|} = \prod_{(i,j)\in T} q^{T(i,j)-1/2} \times \prod_{(i,j)\in T'} q^{T'(i,j)-1/2}$$

$\prod_{(i,j)\in T} q^{T(i,j)-1/2}$ ← weight of T $W(T)$

$\prod_{(i,j)\in T'} q^{T'(i,j)-1/2}$ ← weight of T' $W(T')$

factorization to partial sum

$$Z = \sum_{\lambda \in P} \left(\sum_{T \in SSTab(\lambda)} W(T)\right) \left(\sum_{T' \in SSTab(\lambda)} W(T')\right)$$

8/9

Partial sums for T, T' are given by a special value of Schur functions

Schur functions $$s_\lambda(x_1, x_2, \cdots) = \sum_{T \in SSTab(\lambda)} \underbrace{\prod_{(i,j)\in\lambda} x_{T(i,j)}}_{W(T)}$$

non trivial fact: symmetric function

e.g. $\begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 1 & 1 & \\ \hline \end{array} \to x_1^3 x_2 x_3$

Changing "increasing" to "decreasing" is not problematic

$$\sum_{T \in SSTab(\lambda)} W(T) = s_\lambda(q^{1/2}, q^{3/2}, \cdots, q^{m-1/2}, \cdots)$$

Consequently,

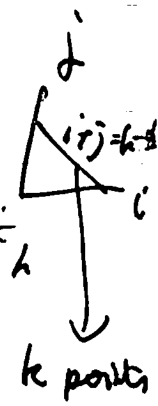
$$Z = \sum_{\lambda\in P} s_\lambda(\underbrace{q^{1/2}, q^{3/2}, \dots}_{q^{-\rho}})^2 , \qquad \rho = \left(-\tfrac{1}{2}, -\tfrac{3}{2}, \dots, \tfrac{1}{2}-m, \dots\right)$$

$\boxed{\text{Cauchy identity}}$ (ref. Macdonald's book)

- $$\sum_{\lambda\in P} s_\lambda(x_1,x_2,\dots)\, s_\lambda(y_1,y_2,\dots) = \prod_{i,j\geq 1}(1-x_iy_j)^{-1}$$
- $$\sum_{\lambda\in P} s_\lambda(x_1,x_2,\dots)\, s_{{}^t\lambda}(y_1,y_2,\dots) = \prod_{i,j\geq 1}(1+x_iy_j)$$
- etc... (relative version)

By this identity

$$\sum_{\lambda\in P} s_\lambda(q^{-\rho})^2 = \prod^{\infty} (1 - \underbrace{q^{i-1/2}\cdot q^{j-1/2}}_{q^{i+j-1}})^{-1}$$

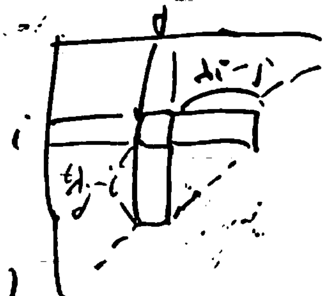


$$= \sum_{k=1}^{\infty}(1-q^k)^{-k}$$

(MacMahon function)

New hook formula — q-analogue of S_∞ plancherel measure

$$s_\lambda(q^{-\rho}) = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda}\left(q^{-h(i,j)/2} - q^{h(i,j)/2}\right)} ,$$



$$\kappa(\lambda) = 2\sum_{(i,j)\in\lambda}(i-j) = \sum_{i\geq 1}\lambda_i(\lambda_i - 2i+1)$$

$$h(i,j) = (\lambda_i - j) + ({}^t\lambda_j - i) + 1 \quad \text{(hook length)}$$

This is a key to thermodynamic limit. (Also relevant to interpretation by 5D SUSY U(1) gauge theory)

No.
Date 6.

Modified model

$s_\lambda(q^{-\rho})^2 \to s_\lambda(q^{-\rho})\, s_{{}^t\lambda}(q^{-\rho})$

$$Z' = \sum_{\lambda\in P} s_\lambda(q^{-\rho})\, s_{{}^t\lambda}(q^{-\rho})$$ → relates to top. string (on resolved conifold)

By Cauchy identity

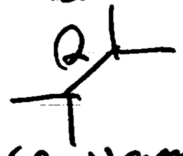
$$Z' = \prod_{i,j=1}^{\infty}\left(1+q^{i-1/2}\cdot q^{j-1/2}\right) = \prod_{k=1}^{\infty}(1+q^k)^k$$

Rem Introduce a parameter Q.

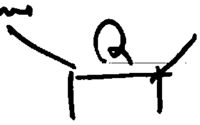
$$Z = \sum_{\lambda\in P} Q^{|\lambda|} s_\lambda(q^{-\rho})^2 = \sum_{\pi\in PP} q^{|\pi|} Q^{|\pi(0)|}$$
$$= \prod_{k=1}^{\infty}(1-Qq^k)^{-k}$$

$$Z' = \sum_{\lambda\in P} Q^{|\lambda|} s_\lambda(q^{-\rho})\, s_{{}^t\lambda}(q^{-\rho}) = \prod_{k=1}^{\infty}(1+Qq^k)^k$$

($Z'|_{Q\to -Q}$ is the amplitude func. of top. string theory on resolved conifold)
local $(-1,-1)$ curve



(Z corresponds to $(0,-2)$ curve



)

Rem Further Deformation by external potential

$Q^{s(s+1)/2} e^{\Phi(\lambda,s,t)}$, $\Phi(\lambda,s,t) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda,s)$ ← discrete parameter (related to charge of fermion)

originate in fermionic formula of partition function

complex conjugated

$Q^{s(s+1)/2} e^{\Phi(\lambda,s,t,\bar t)}$,

$$\Phi(\lambda,s,t,\bar t) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda,s) + \sum_{k=1}^{\infty} \bar t_k \Phi_{-k}(\lambda,s)$$

time variables

eventually identified with lattice coord. of Toda lattice

No. 2
Date

Fermionic representation of partition function

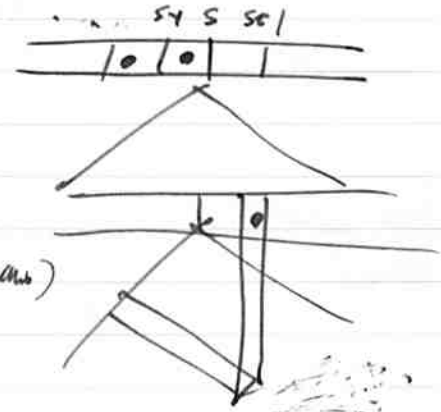
2D fermions usually $-n-\frac{1}{2}$ $(n \in \mathbb{Z}+\frac{1}{2})$

$$\psi_n, \psi_n^* \ (n \in \mathbb{Z}) \qquad \psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum \psi_n^* z^{-n}$$

$$\psi_m\psi_n^* + \psi_n^*\psi_m = \delta_{m+n,0}, \quad \psi_m\psi_n + \psi_n\psi_m = 0, \quad \psi_m^*\psi_n^* + \psi_n^*\psi_m^* = 0$$

Fock space $\langle s|, |s\rangle$, ground state of charge s sector

$$|s\rangle = \psi_{-s}\psi_{-s+1}\cdots|-\infty\rangle$$
$$\langle s| = \langle -\infty|\cdots\psi_{s-1}^*\psi_s^*$$



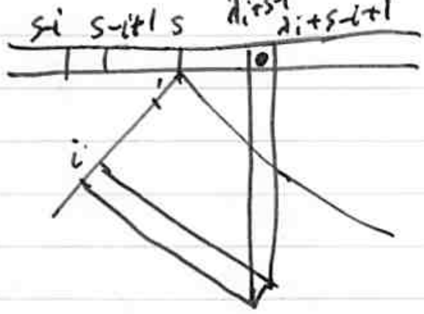
excited states labelled by partitions (or Young diagrams)

~~$|\lambda, s\rangle = \psi_{-\lambda_1 \ldots}$~~

$$|\lambda, s\rangle = \psi_{-\lambda_1-s}\psi_{-\lambda_2-s+1}\cdots|-\infty\rangle$$
$$\langle\lambda, s| = \langle-\infty|\cdots\psi^*_{\lambda_2+s-1}\psi^*_{\lambda_1+s}$$
$$\langle\lambda, s|\mu, s\rangle = \delta_{\lambda\mu}\delta_{rs}$$

$$\phi \mapsto \{-i+1+s\}_{i=1}^{\infty} = \{s, s-1, s-2, \cdots\}$$
$$\lambda \mapsto \{\lambda_i - i + 1 + s\}_{i=1}^{\infty} = \{\lambda_1+s, \lambda_2+s-1, \cdots\}$$

Maya diagram (of charge s)



$\Delta = \sum n\psi_{nn}$

fermion bilinears $\quad :\psi_{-i}\psi_j^*: \longleftrightarrow E_{ij}$

$$L_0 = \sum_{n\in\mathbb{Z}} n :\psi_{-n}\psi_n^*:, \quad J_m = \sum_{n\in\mathbb{Z}} :\psi_{m-n}\psi_n^*: \Longleftrightarrow \Lambda^m = \sum_{n\in\mathbb{Z}} E_{n-m\,n}$$

~~$H_k = \sum_{n\in\mathbb{Z}} q^{kn} :\psi_{-n}\psi_n^*: \longleftrightarrow \sum_n q^{kn} E_{nn}$~~

$$\left(= \oint z^m :\psi(z)\psi^*(z): \frac{dz}{2\pi i}\right.$$
$$\psi_{-n+m} z^{-n+m-1} \psi_n z^{-n}$$

No.
Date 2

vertex operators

commutator

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right)$$

J_k preserves charge

$$\Gamma_\pm(x_1, x_2, \cdots) = \prod_{i \geq 1} \Gamma_\pm(x_i),$$

$|\lambda|-|\mu|$

fundamental relation (cf. Okounkov Reshetikhin) Macdonald

$$\boxed{\begin{aligned} \Gamma_-(z)|\mu\rangle &= \sum_{\lambda \succ \mu} z^{|\lambda/\mu|} |\lambda\rangle \\ \langle\mu|\Gamma_+(z) &= \sum_{\lambda \succ \mu} z^{|\lambda/\mu|} \langle\lambda| \end{aligned}}$$

$|\mu\rangle \to |\mu, s\rangle$

$|\lambda\rangle \to |\lambda, s\rangle$

$\langle\mu|\Gamma_-(z)|\lambda\rangle = \begin{cases} z^{|\lambda/\mu|} & (\lambda \succ \mu) \\ 0 & (\text{otherwise}) \end{cases}$

$\lambda \succ \mu \Leftrightarrow \lambda/\mu$ is horizontal strip

bosonization formula

$J_{\pm k} \longleftrightarrow \overbrace{\boxed{1 \cdots 1}}^{k}$

$\longleftrightarrow h_k$ (complete symmetric functions)

Pieri formula $\quad h_k s_\mu(x_1, x_2, \cdots) = \sum_{\substack{\lambda \succ \mu \\ |\lambda/\mu| = k}} s_\lambda(x_1, x_2, \cdots)$

$$\Gamma_-(x_1, x_2, \cdots x_N)|\mu\rangle = \sum_{\lambda = \lambda^{(N)} \succ \lambda^{(N-1)} \succ \cdots \succ \lambda^{(0)} = \mu} x_N^{|\lambda^{(N)}/\lambda^{(N-1)}|} x_{N-1}^{|\lambda^{(N-1)}/\lambda^{(N-2)}|} \cdots x_1^{|\lambda^{(1)}/\lambda^{(0)}|} |\lambda\rangle$$

$$= \sum_{\lambda \supset \mu} \sum_{T \in \mathrm{SSTab}(\{1, \cdots, n\}, \lambda/\mu)} \underbrace{(x_1, \cdots, x_n)^T}_{= w(T)} |\lambda\rangle$$

$$= \sum_{\lambda \supset \mu} s_{\lambda/\mu}(x_1, \cdots x_N) |\lambda\rangle$$

$N \to \infty$

$$\boxed{\Gamma_-(x_1, x_2, \cdots)|\mu\rangle = \sum_{\lambda \supset \mu} s_{\lambda/\mu}(x_1, x_2, \cdots)|\lambda\rangle}$$

$s_{\lambda/\mu}(x_1, x_2, \cdots) = \langle\mu|\Gamma_-(x_1, x_2 \cdots)|\lambda\rangle = \langle\lambda|\Gamma_+(x_1 \cdots)|\mu\rangle$

similarly,

$$\boxed{\langle\mu|\Gamma_+(x_1, x_2, \cdots) = \sum_{\lambda \supset \mu} s_{\lambda/\mu}(x_1, x_2 \cdots)|\lambda\rangle}$$

$(x_1, x_2 \cdots) = q^{-\rho} = (q^{1/2}, q^{3/2}, \cdots)$

$$s_\lambda(q^{-\rho}) = \langle\lambda|\Gamma_-(q^{-\rho})|\phi\rangle = \langle\phi|\Gamma_+(q^{-\rho})|\lambda\rangle$$

$$= \langle\lambda, s|\Gamma_-(q^{-\rho})|s\rangle = \langle s|\Gamma_+(q^{-\rho})|\lambda, s\rangle$$

Fermionic representation of partition function

$$Z = \sum_\lambda s_\lambda(q^{-\rho})^2 Q^{|\lambda|}$$

$|\lambda| = \langle\lambda|L_0|\lambda\rangle$

$Q^{|\lambda|} = \langle\lambda|Q^{L_0}|\lambda\rangle$

$\langle\lambda|Q^{L_0}|\mu\rangle = Q^{|\lambda|}\delta_{\lambda\mu}$

$\langle\lambda|\Gamma_-(q^{-\rho})|\phi\rangle = s_\lambda(q^{-\rho})$

$\parallel$

$\langle\phi|\Gamma_+(q^{-\rho})|\lambda\rangle$

$$= \sum_\lambda \langle\phi|\Gamma_+(q^{-\rho})|\lambda\rangle\langle\lambda|Q^{L_0}|\lambda\rangle\langle\lambda|\Gamma_-(q^{-\rho})|\phi\rangle$$

$$= \langle\phi|\Gamma_+(q^{-\rho})Q^{L_0}\Gamma_-(q^{-\rho})|\phi\rangle$$

$\Gamma'_\pm(z) = \exp\left(-\sum_{h=1}^{\infty}\frac{(-z)^h}{h}J_{\pm h}\right) = \Gamma_\pm(-z)^{-1}$, $\Gamma'_\pm(x_1, x_2, \cdots) = \prod_{i\geq 1}\Gamma'_\pm(x_i)$

$\langle\lambda|\Gamma'_-(z)|\mu\rangle = \sum_{{}^t\lambda \succ {}^t\mu} z^{|\lambda/\mu|}$,

$\parallel$

$\langle\mu|\Gamma'_-(z)|\lambda\rangle$

$$\langle\lambda|\Gamma'_-(x_1, x_2, \cdots)|\mu\rangle = \langle\mu|\Gamma'_+(x_1, x_2, \cdots)|\lambda\rangle = s_{{}^t\lambda/{}^t\mu}(x_1, x_2, \cdots)$$

$$Z' = \sum_\lambda s_\lambda(q^{-\rho})s_{{}^t\lambda}(q^{-\rho})Q^{|\lambda|}$$

$$= \langle\phi|\Gamma_+(q^{-\rho})Q^{L_0}\Gamma'_-(q^{-\rho})|\phi\rangle$$

8/8 L $\left(= \langle\phi|\Gamma'_+(q^{-\rho})Q^{L_0}\Gamma_-(q^{-\rho})|\phi\rangle\right)$

Rem

$\Gamma_+(z)\Gamma_-(w) = (1-zw)^{-1}\cdot\Gamma_-(w)\Gamma_+(z)$

$\Gamma_+(z)\Gamma'_-(w) = (1+zw)^{-1}\cdot\Gamma'_-(w)\Gamma_+(z)$

$\Gamma_+(z)|\phi\rangle = |\phi\rangle$, etc

$\Gamma'_+(z)|\phi\rangle = |\phi\rangle$

→ same result from commutator

$\cancel{[L_0, J_{\pm k}] = \mp J_{\pm k}}$ ~~in general~~

$\cancel{\Gamma'_\pm(z)Q^{L_0} = Q^{L_0}\Gamma'_\pm(Q^{\pm 1}z)}$

$\cancel{\Gamma_+(z)\Gamma_-(w) = \exp\left(\sum_{h=1} \right) \neq 1}$

$\cancel{[L_0, J_m] = -J_m}$

$[L_0, J_m] = -mJ_m$

$Q^{-L_0}J_mQ^{L_0} = Q^m J_m$

$[J_m, J_n] = m\delta_{m+n,0}$

No.

Date 4

$$[L_0, J_n] = -nJ_n$$

$$[J_m, J_n] = m\,\delta_{m+n,0}$$

$$Q^{-L_0} J_n Q^{L_0} = Q^n$$

$$\Gamma_\pm(q^{-\rho}) Q^{L_0} = Q^{L_0} \Gamma_\pm(Q^{\pm 1} q^{-\rho})$$

$$\Gamma_+(z)\Gamma_-(w) = \exp\Big(\sum_{h=1}^{\infty} \frac{(zw)^h}{h}\Big) \Gamma_-(w)\Gamma_+(z)$$

$$= \frac{1}{1-zw} \Gamma_-(w)\Gamma_+(z)$$

$$\Gamma'_\pm(q^{-\rho}) Q^{L_0} = Q^{L_0} \Gamma'_\pm(Q^{\pm 1} q^{-\rho})$$

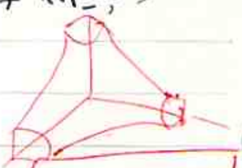
$$\Gamma'_+(z)\Gamma'_-(w) = \frac{1}{1-zw} \Gamma'_-(w)\Gamma'_+(z)$$

$$\Gamma_+(z)\Gamma'_-(w) = \exp\Big(-\sum_{h=1}^{\infty} \frac{(-zw)^h}{h}\Big) \Gamma'_-(w)\Gamma_+(z) = (1+zw)\Gamma'_-(w)\Gamma_+(z)$$

$$\Gamma'_+(z)\Gamma_-(w) = (1+zw)\Gamma_-(w)\Gamma'_+(z)$$

$$\Gamma_+(x_1, x_2, \dots)\Gamma_-(y_1, y_2, \dots) = \prod_{i,j\geq 1} (1 - z_i w_j)^{-1} \Gamma_-(y_1, y_2, \dots)\Gamma_+(z_1, z_2, \dots)$$

$$\Gamma_+(x_1, x_2, \dots)\Gamma'_-(y_1, y_2, \dots) = \prod_{i,j\geq 1} (1 + z_i w_j) \Gamma'_-(y_1, y_2, \dots)\Gamma_+(z_1, z_2, \dots)$$



Cauchy identity)

$\longrightarrow$ same result for Z and Z' and generalization (ORV)

(MacMahon functions)

Deformed models

$$H(t, \bar{t}) = \sum_{h\geq 1} t_h H_h + \sum_{h\geq 1} \bar{t}_h H_{h}$$

$$H_h = \sum_{n\in\mathbb{Z}} q^{hn} :\psi_{-n}\psi_n^*: , \quad H(t) = \sum_{h\geq 1} t_h H_h ,$$

$$Z(s, t) = \langle s | \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(t)} \Gamma_-(q^{-\rho}) | s \rangle$$

$$Z'(s, t, \bar{t}) = \langle s | \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(t, \bar{t})} \Gamma_-(q^{-\rho}) | s \rangle$$

$$Z(s, t) = \sum_{\lambda} \langle s | \Gamma_+(q^{-\rho}) | \lambda, s \rangle \underbrace{\langle \lambda, s | Q^{L_0} e^{H(t)} | \lambda, s \rangle}_{\text{diagonal.}}$$

$$\times \langle \lambda, s | \Gamma_-(q^{-\rho}) | s \rangle$$

No. 2
Date 5

$$\langle \lambda s | L_0 | \mu, r\rangle = \delta_{\lambda\mu}\delta_{sr}(|\lambda| + s(s+1)/2)$$

$$\langle \lambda, s | Q^{L_0} | \mu, r\rangle = \delta_{\lambda\mu}\delta_{sr}\, Q^{|\lambda| + s(s+1)/2}$$

$$\cancel{\langle \lambda, s | H_k | \mu, r\rangle = \delta_{\lambda\mu}\delta_{sr}\,\Phi_k(\lambda, s)}$$

$$\cancel{\Phi_k(\lambda, s) = \sum_{k\geq 1}(\lambda_i - i + 1 + s)^k - \sum_{k\geq 1}(-i+1)^k} \quad \text{(formal expression)}$$

$$\cancel{= \sum_{k\geq 1}\Big(\underbrace{(\lambda_i - i + 1 + s)^k - (-i+1+s)^k}_{\text{finite sum}}\Big)}$$

$$\cancel{+ \underbrace{\sum_{k\geq 1}\underbrace{(-i+1+s)^k}_{s,\, s-1,\, s-2,} - \sum_{k\geq 1}\underbrace{(-i+1)^k}_{0,\, -1,\, -2 \cdots}}_{\text{finite sum}}}$$

(normal ordering)

$$\langle \lambda, s | e^{H(t)} | \lambda, s\rangle = e^{\Phi(\lambda, s, t)},$$

$$\Phi(s, \lambda, t) = \sum_{k\geq 1} t_k \Phi_k(s, \lambda),$$

$$\Phi_k(s, \lambda) = \sum_{i\geq 1} q^{k(\lambda_i - i + 1 + s)} - \sum_{i\geq 1} q^{k(-i+1)} \quad \text{(formal expression)}$$

$$= \sum_{i\geq 1}\left(q^{k(\lambda_i - i + 1 + s)} - q^{k(-i+1+s)}\right) \longleftarrow \text{finite sum}$$

$$+ \underbrace{\sum_{i\geq 1} q^{k\overbrace{(-i+1+s)}^{s,\, s-1,\, \cdots}} - \sum_{i\geq 1} q^{k\overbrace{(-i+1)}^{0,\, -1,\, \cdots}}}$$

(also valid for $k<0$)

$$\hookrightarrow \begin{cases} q^{ks} + q^{k(s-1)} + \cdots + q^{k} & (s>0) \\ -1 - q^{-k} - \cdots - q^{k(s+1)} & (s<0) \end{cases} = \frac{1-q^{ks}}{1-q^{k}}\, q^{k}$$

$$Z(s, t) = \sum_\lambda s_\lambda(q^{-\rho})^2\, Q^{|\lambda| + s(s+1)/2}\, e^{\Phi(s, t)}$$

$$Z'(s, t, \bar t) = \sum_\lambda s_\lambda(q^{-\rho})\, s_{t\lambda}(q^{-\rho})\, Q^{|\lambda| + s(s+1)/2}\, e^{\Phi(s, t, \bar t)}$$

(heuristic) other motivation from cf. 5d SUSY YM on $\mathbb{R}^4 \times S^1$ (Wilson-loop-like) observables $\mathcal{O}_k$ (Nakatsu-Noma-T.) equivariant Chern character (for $SU(N)$)

$$Z = \sum_{n=0}^{\infty} Q^{nN} \langle \exp \sum_k t_k \int_{\mathbb{R}^4} \mathcal{O}_k \rangle_{\text{D-instanton}}$$

for $SU(N)$

No 6

Relation to Toda τ function : Result

References
- T. Nakatsu & K.T. CMP 285 (2009), 445-468 arXiv:0710.5339
- arXiv:0807.4970 (Toric)
- K.T. J Phys A 46 (2013) 245202 arXiv:1302.6129

$$Z(s,t) = \exp\left(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\right) q^{-s(s+1)(2s+1)/6}\, \tau(s,\iota(t))$$

$$\tau(s,t) = \langle s|\exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g|s\rangle = \langle s|g\exp\left(-\sum_{k=1}^{\infty} t_k J_{-k}\right)|s\rangle$$

(1D Toda τ function)

$$\iota(t) = (-t_1, t_2, -t_3, \cdots, (-1)^k t_k, \cdots)$$

$$g = q^{W_0/2}\,\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\,Q^{L_0}\,\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\,q^{W_0/2}$$

$$W_0 = \sum_{n\in\mathbb{Z}} n^2 :\psi_{-n}\psi_n^*:$$

$J_k g = g J_{-k}$ $(k=1,2,\cdots)$ $\longrightarrow$ 1D reduction
$\tau(s,t,\bar{t}) = \tau(s,t-\bar{t})$

$$Z'(s,t,\bar{t}) = \exp\left(\sum_{k=1}^{\infty}\frac{q^k t_k - \bar{t}_k}{1-q^k}\right)\tau'(s,\iota(t),-\bar{t}),$$

$$\tau'(s,t,\bar{t}) = \langle s|\exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g' \exp\left(-\sum_{k=1}^{\infty}\bar{t}_k J_{-k}\right)|s\rangle,$$

$$g' = q^{W_0/2}\,\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\,Q^{L_0}\,\Gamma_-'(q^{-\rho})\Gamma_+'(q^{-\rho})\,q^{-W_0/2}$$

τ' is a solution of Ablowitz-Ladik (rel. Toda);
embedded in 2D Toda hierarchy
(Lax operators are special)
"Toeplitz reduction" (Brini et al)

(implications)

$s_\lambda(-t) = (-1)^{|\lambda|} s_{{}^t\lambda}(t)$

$\tau(s,t) = \sum_\lambda s_\lambda(t)\langle \lambda,s|g|\phi\rangle$

$(-1)^{|\lambda|}\langle s|g|{}^t\lambda,s\rangle$

$\sim$ λ, ϕ, ϕ, ϕ

$\tau(s,t,\bar{t}) = \sum_{\lambda,\mu} s_\lambda(t) s_\mu(-\bar{t}) \times\langle\lambda,s|g'|\mu,s\rangle$

$\sim$ λ, ϕ, μ, ϕ

Clue : $\{H_k\}\subset$ quantum torus algebra $\{V_m^{(k)}\}_{m,k\in\mathbb{Z}}$ ($\{J_k\}$)

use "shift symmetries" to move $e^{H(t)}$, $e^{\bar{H}(\bar{t})}$

Quantum torus algebra and Shift symmetries

$$V^{(k)}_m = q^{-km/2} \sum_{n\in\mathbb{Z}} q^{kn} :\psi_{m-n} \psi^*_n: \quad \longleftrightarrow q^{-km/2} \Lambda^m q^{k\Delta} = v^{(k)}_m$$

$$= q^{k/2} \oint z^m :\psi(q^{k/2} z) \psi^*(q^{-k/2} z) \frac{dz}{2\pi i}$$

$$\begin{pmatrix} \Lambda = \sum E_{i,i+1} \\ \Delta = \sum i E_{ii} \end{pmatrix}$$

$V^{(0)}_m = J_m, \quad V^{(k)}_0 = H_k$

$$[V^{(k)}_m, V^{(l)}_n] = \left(q^{(lm-kn)/2} - q^{(kn-lm)/2}\right)\left(V^{(k+l)}_{m+n} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}}\right)$$

central extension (second quantization)

meaningfull when $k+l=0$ $\longrightarrow$ $(*)$

$$\Delta \sim z\frac{\partial}{\partial z} \qquad q^{k\Delta} \sim q^{kz\partial/\partial z} = q^{-ki\partial/\partial\theta}$$

$$\Lambda^m \sim z^m \sim e^{mi\theta}$$

$$z = e^{i\theta}$$

$$\log z = i\theta$$

$$z\frac{\partial}{\partial z} = \frac{\partial}{\partial \log z} = -i\frac{\partial}{\partial\theta}$$

$$\boxed{\begin{aligned} &v^{(k)}_m \\ &= q^{-km/2} \Lambda^m q^{k\Delta} \end{aligned}}$$

(no c-number term)

$\Lambda = e^{\partial/\partial s}$

$\Delta = s.$

$$e^{i\theta} \cdot \underbrace{q^{-i\partial/\partial\theta}}_{\exp(-i\log q \frac{\partial}{\partial\theta})} = \underbrace{q}_{e^{\log q}} \; q^{-i\partial/\partial\theta} \cdot e^{i\theta}$$

$$\Lambda \cdot q^{\Delta} = q \cdot q^{\Delta} \Lambda$$

quantization $\uparrow$ $\downarrow$ $q\to 1$ classical limit

$$\boxed{\begin{aligned} &\text{Torus algebra:} \quad z_1^m z_2^n \quad \{z_1, z_2\} = z_1 z_2, \ \text{Poisson bracket} \\ &\{z_1^m z_2^k, z_1^n z_2^l\} = (ml - kn) z_1^{m+n} z_1^{k+l} \end{aligned}}$$

$(*)$ $k+l=0$: $\quad [V^{(k)}_m, V^{(-k)}_n] = \left(q^{-k(m+n)} - q^{k(m+n)}\right) V^{(0)}_{m+n} + m\delta_{m+n,0}$

No.

Date 2

Shift symmetries

(i) $$\Gamma_+(q^{-\rho})\left(V_m^{(k)} - \frac{q^k}{1-q^k}\delta_{m,0}\right)\Gamma_+(q^{-\rho})^{-1}$$
$$= (-1)^k \Gamma_-(q^{-\rho})^{-1}\left(V_{m+k}^{(k)} - \frac{q^k}{1-q^k}\delta_{m+k,0}\right)\Gamma_-(q^{-\rho})$$
for $k>0$, $m\in\mathbb{Z}$

(ii) $$\Gamma_+'(q^{-\rho})\left(V_m^{(-k)} + \frac{1}{1-q^k}\delta_{m,0}\right)\Gamma_+'(q^{-\rho})^{-1}$$
$$= \Gamma_-'(q^{-\rho})^{-1}\left(V_{m+k}^{(-k)} + \frac{1}{1-q^k}\delta_{m+k,0}\right)\Gamma_-'(q^{-\rho})$$
for $k>0$, $m\in\mathbb{Z}$

(iii) $$q^{W_0/2}V_m^{(k)}q^{-W_0/2} = V_m^{(k-m)} \quad \text{for } k,m\in\mathbb{Z}$$

References

Derivation $$V_m^{(k)} = q^{k/2}\oint z^m :\psi(q^{k/2}z)\psi^*(q^{-k/2}z): \frac{dz}{2\pi i}$$

$$\cancel{:\psi(z)\psi^*(w): = \psi(z)\psi^*(w) - \frac{1}{z-w} \quad (|z|>|w|)}$$

(i) $$:\psi(z)\psi^*(w): = -\psi^*(w)\psi(z) - \frac{1}{z-w} \quad (|z|<|w|)$$

$k>0$ $$:\psi(q^{k/2}z)\psi^*(q^{-k/2}z): = -\psi^*(q^{-k/2}z)\psi(q^{k/2}z) + \frac{q^{k/2}}{(1-q^k)z}$$

$$\cancel{\Gamma_\pm(q^{-\rho})\psi(z)\Gamma_\pm(q^{-\rho})^{-1} =}$$
$$\Gamma_\pm(x)\psi(z)\Gamma_\pm(x)^{-1} = (1-xz^{\pm1})^{-1}\psi(z)$$
$$\Gamma_\pm(x)\psi^*(z)\Gamma_\pm(x)^{-1} = (1-xz^{\pm1})\psi^*(z)$$
$|xz^{\pm1}|<1$

$q^{\frac{1-k}{2}}, q^{\frac{3-k}{2}}, \ldots$

$$\Gamma_+(q^{-\rho})\psi^*(q^{-k/2}z)\psi(q^{k/2}z)\Gamma_+(q^{-\rho})^{-1} = \frac{\prod_{i=1}^{\infty}(1-q^{i-1/2}\cdot q^{-k/2}z)}{\prod_{i=1}^{\infty}(1-q^{i-1/2}\cdot q^{k/2}z)}\psi^*(q^{-k/2}z)\psi(q^{k/2}z)$$

$q^{\frac{1+k}{2}}, q^{\frac{3+k}{2}}, \ldots$

$$\frac{q^{k/2}}{(1-q^k)z} - :\psi(q^{k/2}z)\psi^*(q^{-k/2}z): = \prod_{i=1}^{k}\frac{\cancel{(1-q^{i-1/2}q^{-k/2}z)}}{(1-q^{i-\frac{k+1}{2}}z)}\cdot\psi^*(q^{-k/2}z)\psi(q^{k/2}z) \quad \left(|z|<q^{\frac{k+1}{2}}\right)$$

$\downarrow q^{k/2}\oint \frac{dz}{2\pi i} z^m (\quad)$

$$\Gamma_+(q^{-\rho})\left(-V_m^{(k)}+\frac{q^k}{1-q^k}\delta_{m,0}\right)\Gamma_+(q^{-\rho})$$

$$= q^{k/2}\oint_{|z|=R<q^{(k-1)/2}} \frac{dz}{2\pi i}\, z^m \prod_{i=1}^{k}\left(1-q^{i-\frac{k+1}{2}}z\right)\cdot \psi^*(q^{-k/2}z)\psi(q^{k/2}z) \quad \cdots ①$$

$$(1-q^{\frac{1-k}{2}}z)(1-q^{\frac{3-k}{2}}z)\cdots$$

$$\cdots(1-q^{\frac{k-1}{2}}z)$$

$$\frac{q^{k/2}}{(1-q^k)z} - :\psi(q^{k/2}z)\psi^*(q^{k/2}z):$$

$$\Gamma_-(q^{-\rho})^{-1}\,\psi^*(q^{-k/2}z)\psi(q^{k/2}z)\,\Gamma_-(q^{-\rho})$$

$$= \frac{\prod_{i=1}^{\infty}(1-q^{i-1/2}\cdot q^{-k/2}z^{-1})}{\prod_{i=1}^{\infty}(1-q^{i-1/2}\cdot q^{k/2}z^{-1})}\psi^*(q^{-k/2}z)\psi(q^{k/2}z)$$

$$= \boxed{\prod_{i=1}^{k}\left(1-q^{i-\frac{k+1}{2}}z^{-1}\right)}\cdot\psi^*(q^{-k/2}z)\psi(q^{k/2}z) \quad \left(|z|>q^{\frac{k-1}{2}}\right)$$

$$= (-1)^k z^{-k}\prod_{i=1}^{k}\left(1-q^{-i+\frac{k+1}{2}}z\right)$$

$$= (-1)^k z^{-k}\prod_{i=1}^{k}\left(1-q^{i-\frac{k+1}{2}}z\right)$$

$\oint (-1)^k q^{k/2}\frac{dz}{2\pi i}z^{m+k}(\quad)$

$$(-1)^k\Gamma_-(q^{\rho})^{-1}\left(-V_{m+k}^{(k)}+\frac{q^k}{1-q^k}\delta_{m+k,0}\right)\Gamma_-(q^{-\rho})$$

$$= q^{k/2}\oint_{|z|=R_1>q^{\frac{k-1}{2}}}\frac{dz}{2\pi i}\, z^m\prod_{i=1}^{k}\left(1-q^{i-\frac{k+1}{2}}z\right)\cdot\psi^*(q^{-k/2}z)\psi(q^{k/2}z) \quad --②$$

$k>0$ ならば $R\leq z\leq R_1$ に特異点がないので ①=②

$k<0$ ならば 特異点があるので ①=② とはならない

(ii) $V_m^{(-k)} = q^{k/2}\oint z^m :\psi(q^{-k/2}z)\psi^*(q^{k/2}z): \frac{dz}{2\pi i}$

$$:\psi(q^{-k/2}z)\psi^*(q^{k/2}z): = \psi(q^{-k/2}z)\psi^*(q^{k/2}z) - \frac{q^{k/2}}{(1-q^k)z}$$

として計算を進める。

k<0 ならば成立せず。

No. 4

Date

(iii) $W_0 = \sum_{n\in\mathbb{Z}} n^2 :\psi_{-n}\psi_n^*:$, $J_m = \sum_n :\psi_{m-n}\psi_n^*:$

$$[W_0, \psi_n] = n^2\psi_n,$$

$$[W_0, \psi_n^*] = -n^2\psi_n$$

$$[W_0, :\psi_{m-n}\psi_n^*:] = \left((m-n)^2 - n^2\right)\underbrace{\psi_{m-n}\psi_n^*}_{= :\psi_{m-n}\psi_n^*:\ (m\neq 0)}$$

$m = 0$ or $n = 0$

$$[W_0, :\psi_{m-n}\psi_n^*:] = ((m-n)^2 - n^2) :\psi_{m-n}\psi_n^*: \text{ ですね}$$

$$q^{W_0/2} :\psi_{m-n}\psi_n^*: q^{-W_0/2} = q^{((m-n)^2 - n^2)/2} :\psi_{m-n}\psi_n^*:$$

$$q^{W_0/2} V_m^{(k)} q^{-W_0/2} = \sum_n q^{-km/2} q^{(n-m)^2/2 - n^2/2} q^{kn} :\psi_{m-n}\psi_n^*:$$

$$-\frac{km}{2} - \frac{(n-m)^2}{2} - \frac{n^2}{2} + kn$$

$$= -\frac{km}{2} + nm - \frac{m^2}{2} + kn$$

$$= -\frac{(k+m)m}{2} + (k+m)n$$

$$= V_m^{(k+m)}.$$

Converting partition function to tau functions

$$Z = \langle s| \Gamma_+(q^{-\rho}) \underbrace{Q^{L_0} e^{H(t)}}_{\shortparallel\ e^{H(t)} Q^{L_0}} \Gamma_-(q^{-\rho}) |s\rangle$$

$$H(t) = \sum_{k=1} t_k H_k = \sum_{k\geq 1} t_k V_0^{(k)}$$

$k>0$ $\quad \cancel{\Gamma_+(q^{-\rho}) V_0^{(k)} \Gamma_-(q^{\rho}) = (-1)^k \Gamma_-(q^{-\rho})^{-1} (V_k^{(k)} - \frac{q^k}{1-q^k}}$

$$\Gamma_+(q^{-\rho}) \left(V_0^{(k)} - \frac{q^k}{1-q^k}\right) \Gamma_+(q^{-\rho})^{-1} = (-1)^k \Gamma_-(q^{-\rho})^{-1} V_k^{(k)} \Gamma_-(q^{-\rho})$$

$$\Gamma_+(q^{-\rho}) V_0^{(k)} \Gamma_+(q^{-\rho})^{-1} = \frac{q^k}{1-q^k} + (-1)^k \Gamma_-(q^{-\rho})^{-1} V_k^{(k)} \Gamma_-(q^{-\rho})$$

$$\Gamma_+(q^{-\rho}) \exp\left(\sum_{k\geq 1} t_k V_0^{(k)}\right) \Gamma_+(q^{-\rho})^{-1} = \exp\left(\sum_{k\geq 1} t_k \underbrace{\Gamma_+(q^{-\rho}) V_0^{(k)} \Gamma_+(q^{-\rho})^{-1}}_{\shortparallel\ \frac{q^k}{1-q^k} + (-1)^k \Gamma_-(q^{-\rho})^{-1} q^{-W_0/2} J_k \times q^{W_0/2} \Gamma_-(q^{-\rho})}\right)$$

$$V_k^{(k)} = q^{-W_0/2} V_k^{(0)} q^{+W_0/2} = q^{-W_0/2} J_k q^{W_0/2}$$

$$= \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k}{1-q^k}\right) \Gamma_-(q^{-\rho})^{-1} q^{-W_0/2} \exp\left(\sum (-1)^k t_k J_k\right) q^{W_0/2} \Gamma_-(q^{\rho})$$

$$Z = \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k}{1-q^k}\right) \overbrace{\langle s| \Gamma_-(q^{-\rho})}^{\shortparallel\ \langle s|} q^{-W_0/2} \exp\left(\sum_{k=1}^{\infty} (-1)^k t_k J_k\right)$$

$$\times q^{W_0/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \Gamma_-(q^{-\rho}) |s\rangle$$

$$= \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k}{1-q^k}\right) q^{-s(s+1)(2s+1)/12} \langle s| \exp\left(\sum_{k=1} (-1)^k t_k J_k\right)$$

$$\times q^{W_0/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \boxed{\Gamma_-(q^{-\rho}) |s\rangle}$$

$\Gamma_-(q^{-\rho})^{-1} e^{H(t)} \Gamma_-(q^{\rho})$ の方からの計算と合わせなきゃ。 $\longrightarrow$ $\shortparallel$ $\Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{W_0/2} \times |s\rangle q^{-s(s+1)(2s+1)/6}$ $\longleftarrow$ to be consisted ~~with another~~ exponent

2013.12.3?

$$= \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k}{1-q^k}\right) q^{-s(s+1)(2s+1)\cdots} \langle s| \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k t_k J_k}{k}\right) g |s\rangle$$

$$g = q^{W_0/2} \Gamma_-(q^{\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{W_0/2}$$

6

$J_k g = g J_{-k}$ ← shift symmetry

(g is Hankel?)

$$Z' = \langle s| \Gamma_+(q^{-\rho})\, \mathfrak{Q}^{L_0} e^{H(t,\bar{t})}\, \Gamma'_-(q^{-\rho})|s\rangle$$

$$e^{H(t)}\mathfrak{Q}^{L_0}e^{\bar{H}(\bar{t})}$$

$$\Gamma_+(q^{-\rho})e^{H(t)}\Gamma_+(q^{-\rho})^{-1} = \exp\left(\sum_{k=1}^{\infty}\frac{q^k t_k}{1-q^k}\right)\Gamma_-(q^{-\rho})^{-1}q^{W_0/2}$$

$$\times \exp\left(\sum_{k=1}^{\infty}(-1)^k t_k J_k\right) q^{-W_0/2}\Gamma_-(q^{-\rho})$$

$$\Gamma'_-(q^{-\rho})^{-1}e^{\bar{H}(\bar{t})}\Gamma'_-(q^{-\rho}) = \exp\left(-\sum_{k=1}^{\infty}\frac{\bar{t}_k}{1-q^k}\right)\Gamma'_+(q^{-\rho})q^{W_0/2}$$

$$\boxed{J_k^{(k_0)} = q^{W_0/2}J_k^{(0)}q^{-W_0/2}} \qquad \times \exp\left(\sum_{k=1}^{\infty}\bar{t}_k J_{-k}\right)q^{-W_0/2}\Gamma'_+(q^{-\rho})^{-1}$$

$$Z' = \exp\left(\sum_{k=1}^{\infty}\frac{q^k t_k - \bar{t}_k}{1-q^k}\right)\langle s|\exp\left(\sum_{k=1}^{\infty}(-1)^k t_k J_k\right)$$

$$\times \left(q^{-W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\mathfrak{Q}^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{W_0/2}\right)$$

$$\times \exp\left(\sum_{k=1}^{\infty}\bar{t}_k J_{-k}\right)|s\rangle$$

$$\| \quad g'$$

$J_k g' \neq g' J_k$ (g' is not Toeplitz)

Matrix representation

1

$\psi_m \psi_n^* \longleftrightarrow E_{mn}$ $\quad \mathbb{Z}\times\mathbb{Z}$ matrix

$= (\delta_{im}\delta_{jn})$

$$J_m \longleftrightarrow \Lambda^m = \sum_{n\in\mathbb{Z}} E_{n-m,n}$$

$n-m$ diagonal

$$H_k \longleftrightarrow q^{k\Delta} = \sum_{n\in\mathbb{Z}} q^{kn} E_{nn}$$

$$V_m^{(k)} \longleftrightarrow q^{-km/2} \sum_{n\in\mathbb{Z}} q^{kn} E_{n-m,n}$$

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \Lambda^{\pm k}\right) = (1 - z\Lambda^{\pm 1})^{-1}$$

$\Gamma_+(z)$ $\quad$ $\Gamma_-(z)$

$$\Gamma_\pm(q^{-\rho}) = \prod_{i=1}^{\infty} (1-q^{i-1/2}\Lambda^{\pm 1})^{-1}$$

cf. $\prod_{i=1}^{\infty} (1-q^{i-1} z)^{-1}$: quantum dilog

$= (q, z)_\infty^{-1}$

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) = \prod_{i=1}^{\infty}(1-q^{i-1/2}\Lambda^{-1})^{-1} \cdot \prod_{i=1}^{\infty}(1-q^{i-1/2}\Lambda)^{-1}$$

$$\theta(z) = \sum_{n\in\mathbb{Z}} q^{inz+n^2/2} = \prod_{m=1}^{\infty}(1-q^{m}) \times \prod_{m=1}^{\infty}(1+q^{n-1/2}z) \times \prod_{n=1}^{\infty}(1+q^{n-1/2}z^{-1})$$

~~= theta function !!~~ $\sim$ theta function !! (Jacobi triple product)

cf. Okuda hep-th/0409270 Ooguri–Sulkowski–Yamazaki matrix model for top string amplitudes (resolved conifold ...)

~~Origin of shift symmetry~~

$$[\Lambda, \Delta] = \Lambda$$

$$V_m^{(k)} = q^{-km/2}\Lambda^m q^{k\Delta}$$

$$q^{-k\Delta}\Gamma_+(q^{-\rho})q^{k\Delta} = \prod_{i=1}^{\infty}(1-q^{i-1/2}q^{-k\Delta}\Lambda q^{+k\Delta})^{-1}$$

$$q^{-k\Delta}\Lambda q^{k\Delta} = \sum_n q^{-k(n+1)} E_{n+1,n}\, q^{kn} = q^k\Lambda \quad \left(\Lambda = \sum E_{n+1,n}\right)$$

$$= \prod_{i=1}^{\infty}(1-q^{i+k-1/2}\Lambda)^{-1}$$

$$= (1-q^{1/2}\Lambda)(1-q^{3/2}\Lambda)\cdots(1-q^{k-1/2}\Lambda)\,\Gamma_+(q^{-\rho})$$

No. 2

Date

$$\cancel{\Gamma_+(q^{-\rho})q^{k\Delta}\Gamma_+(q^{-\rho})^{-1} = q^{k\Delta}(1-q^{1/2}\Lambda)(1-q^{3/2}\Lambda)\cdots(1-q^{k-1/2}\Lambda)}$$

$$= (1-q^{1/2-k}\Lambda)(1-q^{3/2-k}\Lambda)\cdots(1-q^{-1/2}\Lambda)q^{k\Delta}$$

$$q^{k\Delta}\Gamma_-(q^{-\rho})q^{-k\Delta} = \prod_{i=1}^{\infty}(1-q^{i-1/2}\cdot q^{k\Delta}\Lambda^{-1}q^{-k\Delta})^{-1} \quad \left(q^{k(n+1)-kn} = q^k\right.$$

$$= \prod_{i=1}^{\infty}(1-q^{i-1/2}\cdot q^{k}\Lambda^{-1})^{-1}$$

$$= \Gamma_-(q^{-\rho})(1-q^{1/2}\Lambda^{-1})(1-q^{3/2}\Lambda^{-1})\cdots(1-q^{k-1/2}\Lambda^{-1})\cancel{\Gamma_-(q^{-\rho})}$$

$$\Gamma_-(q^{-\rho})^{-1}q^{k\Delta}\Gamma_-(q^{-\rho}) = (1-q^{1/2}\Lambda^{-1})\cdots(1-q^{k-1/2}\Lambda^{-1})q^{k\Delta}$$

$$q^{\frac12}q^{\frac32}\cdots q^{k-\frac12}$$
$$= q^{\frac{k}{2}}\cdot q^{\frac{k(k-1)}{2}}$$
$$= q^{k^2/2}$$

$$\cancel{= (-1)^k\Lambda^{-k}(1-q}$$

$$\cancel{= q^{k\Delta}(1-q^{1/2-k}\Lambda^{-1})\cdots(1-q^{-1/2}\Lambda^{-1})}$$

$$= \underbrace{(-1)^k\Lambda^{-k}}_{q^{k^2/2}}(1-q^{-1/2}\Lambda)\cdots(1-q^{1/2-k}\Lambda)q^{k\Delta}$$

$$\therefore\ \Gamma_+(q^{-\rho})\underbrace{\Lambda^m}_{q^{-km/2}}q^{k\Delta}\Gamma_+(q^{-\rho})^{-1} = (-1)^k\Gamma_-(q^{-\rho})^{-1}\underbrace{\Lambda^{m+k}}_{q^{-k(m+k)/2}}q^{k\Delta}\Gamma_-(q^{-\rho})$$

($\longrightarrow$ shift symmetry)

$$\Gamma'_\pm(q^{-\rho}) = \prod_{i=1}^{\infty}(1+q^{i-1/2}\Lambda^{\pm1})$$

$$q^{k\Delta}\Gamma'_+(q^{-\rho})q^{-k\Delta} = \prod_{i=1}^{\infty}(1+q^{i-1/2}\cdot \underbrace{q^{k\Delta}\Lambda q^{-k\Delta}}_{q^{-k}\Lambda})$$

$$= (1+q^{\frac12-k}\Lambda)\cdots(1+q^{-1/2}\Lambda)\cdot\Gamma'_+(q^{-\rho})$$

$$\Gamma'_+(q^{-\rho})q^{-k\Delta}\Gamma'_+(q^{-\rho})^{-1} = q^{-k\Delta}(1+q^{\frac12-k}\Lambda)\cdots(1+q^{-1/2}\Lambda)$$

$$= (1+q^{k-1/2}\Lambda)\cdots(1+q^{1/2}\Lambda)q^{-k\Delta}$$

$$q^{-k\Delta}\Gamma'_-(q^{-\rho})q^{k\Delta} = \prod_{i=1}^{\infty}(1+q^{i-1/2}\, q^{-k\Delta}\Lambda^{-1}q^{k\Delta} = q^{-k}\Lambda^{-1})$$

$$= \Gamma'_-(q^{-\rho})(1+q^{\frac12-k}\Lambda^{-1})\cdots(1+q^{-1/2}\Lambda^{-1})$$

$$\Gamma'_-(q^{-\rho})^{-1}q^{k\Delta}\Gamma'_-(q^{-\rho}) = (1+q^{1/2-k}\Lambda^{-1})\cdots(1+q^{-1/2}\Lambda^{-1})q^{-k\Delta} \quad \cdots$$

No. 4

Date 3

matrix representation of g and g'

$$g' = q^{\Delta/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\,Q^{\Delta}\,\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\,q^{\Delta/2}$$

$$g' = q^{\Delta/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\,Q^{\Delta}\,\Gamma_-'(q^{-\rho})\Gamma_+'(q^{-\rho})\,q^{-\Delta/2}$$

solution of 2DToda hierarchy is ~~determined by~~ obtained by (characterized)

Factorization problem ($\to$ Lax-Sato formalism)

$$\exp\Big(\sum_{k=1}^{\infty} t_k\Lambda^k\Big)\,U\,\exp\Big(-\sum_{k=1}^{\infty}\bar{t}_k\Lambda^{-k}\Big) = W^{-1}\bar{W}$$

Equivalent to fermionic contraction of two fields $(g \leftrightarrow U)$

$$W = \begin{pmatrix} \ddots & & 0 \\ & 1 & \\ /// & & 1 \end{pmatrix}, \quad \bar{W} = \begin{pmatrix} \diagdown & /// \\ 0 & \diagdown \end{pmatrix}$$

$\leftrightarrow$ difference operators (Sato-points)

Lax operators $L = W\Lambda W^{-1}$ $\quad (\leftrightarrow\ W e^{\partial s} W^{-1})$

$\bar{L} = \bar{W}\Lambda\bar{W}^{-1}$ $\quad (\leftrightarrow\ \bar{W} e^{\partial s}\bar{W}^{-1})$

$$\frac{\partial W}{\partial t_k} = B_k W - W\Lambda^k, \qquad \frac{\partial W}{\partial \bar{t}_k} = \bar{B}_k W,$$

$$\frac{\partial \bar{W}}{\partial t_k} = B_k \bar{W}, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_k} = \bar{B}_k\bar{W} - \bar{W}\Lambda^{-k}.$$

$$B_k = \begin{pmatrix} \diagdown & /// \\ 0 & \diagdown \end{pmatrix}_{k,0} = (W\Lambda^k W^{-1})_{\geq 0} \qquad \bar{B}_k = \begin{pmatrix} \ddots & 0 \\ /// & \ddots \end{pmatrix}_{\infty} = (\bar{W}\Lambda^{-k}\bar{W}^{-1})_{<0}$$

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$

$$\frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}], \quad \frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}]$$

---

g and $\bar{g}$ are factorizable explicitly.

$\to$ initial values of L and $\bar{L}$ can be read off.

$\to$ $\boxed{\text{Factorized form}}$ $\quad +$ factorized form is preserved under time evolution