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Initial Value Problem for the Toda Lattice Hierarchy

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§ 1. Introduction

A few years ago there appeared several interesting attempts [1–4] to apply some group theoretical point of view to the explicit integration of the Toda lattice. In this direction, however, no result seems to have been established for the infinite lattice without free ends or any periodicity.

This paper presents an algebraic approach toward the integration of the infinite Toda lattice which, in general, does not fall into the cases discussed in [1–4]. As a result, the initial value problem for the Toda lattice hierarchy [5] is explicitly solved.

To set up the initial value problem, let us briefly review the Toda lattice hierarchy [5]:

Let $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ be independent variables with infinite many components, and L, M matrices of size $Z\times Z$ (Z denotes the totality of integers) of the form

$$(1.1)\qquad \begin{aligned} &L=(b_{j-i}(i, x, y))_{ij\in Z}, \quad b_j=0\ (j>1), \quad b_1=1, \\ &M=(c_{j-i}(i, x, y))_{ij\in Z}, \quad c_j=0\ (j<-1), \quad c_{-1}\neq 0. \end{aligned}$$

b_j and c_j serve as the unknown functions of the nonlinear differential equations describing the Toda lattice hierarchy. Auxiliary matrices B_μ, C_μ, $\mu=1, 2, \cdots$, are introduced by

$$(1.2)\qquad B_\mu=(L^\mu)_+, \qquad C_\mu=(M^\mu)_-,$$

where the symbols $(A)_\pm$ denote for a matrix $A=(a_{ij})_{ij\in Z}$ of size $Z\times Z$ the triangular matrices $(a_{ij}Y^\pm_{j-i})_{ij\in Z}$, respectively, with $Y^+_s=0$ $(s<0)$, $=1$ $(s\geq 0)$, $Y^-_s=1$ $(s<0)$, $=0$ $(s\geq 0)$.

The Toda lattice hierarchy is defined by the system of the Lax type

$$(1.3)\qquad \begin{aligned} &\partial_{x_\mu}L=[B_\mu, L], \quad \partial_{y_\mu}L=[C_\mu, M], \\ &\partial_{x_\mu}M=[B_\mu, M], \quad \partial_{y_\mu}M=[C_\mu, M], \quad \mu=1, 2, \cdots, \end{aligned}$$

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or, equivalently, by the system of the Zakharov-Shabat type

$$\text{(1.4)}\quad \begin{aligned} &\partial_{x_\nu}B_\mu-\partial_{x_\mu}B_\nu+[B_\mu, B_\nu]=0, \qquad \partial_{y_\nu}C_\mu-\partial_{y_\mu}C_\nu+[C_\mu, C_\nu]=0,\\ &\partial_{y_\nu}B_\mu-\partial_{x_\mu}C_\nu+[B_\mu, C_\nu]=0, \qquad \mu, \nu=1, 2, \cdots, \end{aligned}$$

where $\partial_{x_\mu}=\partial/\partial x_\mu$, $\partial_{y_\mu}=\partial/\partial y_\mu$. The third equation in (1.4) with $\mu=\nu=1$ corresponds to the original (two dimensional) Toda lattice

$$\text{(1.4)}\qquad \partial_{x_1}\partial_{y_1}u(s)=e^{u(s)-u(s-1)}-e^{u(s+1)-u(s)},$$

with the parametrization $B_1=(\delta_{i,j-1})_{i,j\in Z}+(\partial_{x_1}u(i)\delta_{i,j})_{i,j\in Z}$, $C_1=(e^{u(i)-u(i-1)}\delta_{i,j+1})_{i,j\in Z}$. Here we abbreviated $u(s, x, y)$ to $u(s)$.

Hereafter we shall consider the initial value problem for this hierarchy, i.e. the problem of solving (1.3) and (1.4) under the initial conditions

$$\text{(1.6)}\qquad L|_{x=y=0}=L_0, \qquad M|_{x=y=0}=M_0.$$

The initial values L_0 and M_0 are constant matrices of size $Z\times Z$, and of course they are assumed to take the same form as L and M. Note that the initial values of B_μ and C_μ are determined by (1.6) and (1.2).

Our main theorems, which will be stated in Section 2, show that once the problem is interpreted in the general machinery of the linearization and the τ function, an explicit description of the solution in terms of the initial values can be immediately obtained. The corresponding τ function is shown to have a remarkable structure closely related to the τ functions of the *KP* hierarchy [6–8].

In order to establish these results we shall proceed as follows, inspired by the argument of [6, 7]: In Section 3, we shall investigate a finite lattice model, which plays an important role in Section 4 in constructing the solution to the original Toda lattice hierarchy. It should be noticed that a parametrization of this model is obtained by factorizing a matrix of finite size into the product of a lower triangular matrix and an upper triangular one. Such an idea was also used in [1–4] in some more complicated framework. In Section 4 the proof of our main theorems is completed. First a series of finite lattice models related to the initial values is introduced. Then the solution to the original initial value problem is achieved by a "limit" of these finite lattice models. Using the τ functions, we can visualize that passage from the finite lattices to the infinite one.

In Section 5 supplementary remarks are added.

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§2. Main results

In this section the main theorems are stated. We need some preparations to give their precise statements.

Let us begin with a brief review of the linearization and the τ function of the Toda lattice hierarchy [5]:

Let us consider the following linear system for unknown matrices $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$.

$$\text{(2.1)}\qquad L\hat{W}^{(\infty)}=\hat{W}^{(\infty)}\Lambda, \qquad M\hat{W}^{(0)}=\hat{W}^{(0)}\Lambda^{-1},$$

$$\text{(2.2)}\quad \begin{aligned} &\partial_{x_\mu}\hat{W}^{(\infty)}=B_\mu\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda^\mu, \qquad \partial_{y_\mu}\hat{W}^{(\infty)}=C_\mu\hat{W}^{(\infty)},\\ &\partial_{x_\mu}\hat{W}^{(0)}=B_\mu\hat{W}^{(0)}, \quad \partial_{y_\mu}\hat{W}^{(0)}=C_\mu\hat{W}^{(0)}-\hat{W}^{(0)}\Lambda^{-\mu}, \quad \mu=1, 2, \cdots, \end{aligned}$$

where Λ^μ denotes the μ-th shift matrix, $\Lambda^\mu=(\delta_{i,j-\mu})_{i,j\in Z}$, and $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ are assumed to have the form

$$\text{(2.3)}\quad \begin{aligned} &\hat{W}^{(\infty)}=(\hat{w}^{(\infty)}_{i-j}(i, x, y))_{i,j\in Z}, \quad \hat{w}^{(\infty)}_j=0\ (j<0), \quad \hat{w}^{(\infty)}_0=1,\\ &\hat{W}^{(0)}=(\hat{w}^{(0)}_{j-i}(i, x, y))_{i,j\in Z}, \quad \hat{w}^{(0)}_j=0\ (j<0), \quad \hat{w}^{(0)}_0\neq 0. \end{aligned}$$

The linearization is achieved by (2.1) and (2.2) in the following sense: If (1.3) and (1.4) are fulfilled there exist some solutions $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ of (2.1) and (2.1) as in (2.3), and they are unique up to the arbitrariness $\hat{W}^{(\infty)}\to\hat{W}^{(\infty)}\sum_{n=0}^\infty f_n\Lambda^{-n}$, $\hat{W}^{(0)}\to\hat{W}^{(0)}\sum_{n=0}^\infty g_n\Lambda^n$ with f_n, $g_n\in C$, $f_0=1$, $g_0\neq 0$. Conversely, if some matrices $\hat{W}^{(\infty)}$and $\hat{W}^{(0)}$ of the form like (2.3) solve (2.1) and (2.2) for certain matrices L, M, B_μ and C_μ, then (1.3) and (1.4) are fulfilled.

Note that $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ are invertible, so that L, M, B_μ and C_μ are recovered from $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ by algebraic equations (2.1) and (1.2). Also note that if we introduce

$$\text{(2.4)}\qquad W^{(\infty)}=\hat{W}^{(\infty)}\exp\sum_{\mu=1}^\infty x_\mu\Lambda^\mu, \qquad W^{(0)}=\hat{W}^{(0)}\exp\sum_{\mu=1}^\infty y_\mu\Lambda^{-\mu},$$

(2.2) is rewritten into the system

$$\text{(2.5)}\qquad \partial_{x_\mu}W=B_\mu W, \quad \partial_{y_\mu}W=C_\mu W, \quad W=W^{(\infty)}, W^{(0)}, \quad \mu=1, 2, \cdots.$$

The τ function $\tau(s, x, y)$ is consistently, and uniquely up to constant multipliers, introduced by the equations

$$\text{(2.6)}\quad \begin{aligned} &\hat{w}^{(\infty)}_j(s, x, y)=p_j(-\tilde{\partial}_x)\tau(s, x, y)/\tau(s, x, y), \qquad j=0, 1, \cdots,\\ &\hat{w}^{(0)}_j(s, x, y)=p_j(-\tilde{\partial}_y)\tau(s+1, x, y)/\tau(s, x, y), \qquad j=0, 1, \cdots,\\ &\tilde{\partial}_x=(\partial_{x_1}, \partial_{x_2}/2, \partial_{x_3}/3, \cdots), \qquad \tilde{\partial}_y=(\partial_{y_1}, \partial_{y_2}/2, \partial_{y_3}/3, \cdots), \end{aligned}$$

where p_j, $j=0, 1, 2, \cdots$, are the polynomials defined by

$$\exp \sum_{\mu=1}^{\infty} x_\mu \lambda^\mu = \sum_{j=0}^{\infty} p_j(x)\lambda^j \qquad \text{(cf. [6, 7]).} \tag{2.7}$$

These polynomials will later play an important role in the expression of the solution to the initial value problem.

The general framework stated above implies that the initial value problem can be converted into the problem of solving (2.2)—though it should be noted that not only $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ but also B_μ and C_μ are unknown here to be determined together with the former—under the initial conditions

$$\hat{W}^{(\infty)}|_{x=y=0}=A^{(\infty)}, \qquad \hat{W}^{(0)}|_{x=y=0}=A^{(0)}, \tag{2.8}$$

where $A^{(\infty)}$ and $A^{(0)}$ are constant triangular matrices of the form

$$\begin{aligned} A^{(\infty)}&=(a_{ij}^{(\infty)})_{ij\in Z}, \quad a_{ij}^{(\infty)}=0\ (i<j), \quad a_{ii}^{(\infty)}=1, \\ A^{(0)}&=(a_{ij}^{(0)})_{ij\in Z}, \quad a_{ij}^{(0)}=0\ (i>j), \quad a_{ii}^{(0)}\neq 0, \end{aligned} \tag{2.9}$$

and connected with L_0 and M_0 by

$$L_0A^{(\infty)}=A^{(\infty)}\Lambda, \qquad M_0A^{(0)}=A^{(0)}\Lambda^{-1}. \tag{2.10}$$

Note that for any initial values L_0 and M_0 we can find such matrices $A^{(\infty)}$ and $A^{(0)}$, solving linear algebraic equations. $A^{(\infty)}$ and $A^{(0)}$ are then unique up to the arbitrariness $A^{(\infty)}\to A^{(\infty)}\sum_{n=0}^{\infty} f_n\Lambda^{-n}$, $A^{(0)}\to A^{(0)}\sum_{n=0}^{\infty} g_n\Lambda^n$ ($f_n, g_n \in C, f_0=1, g_0\neq 0$), which exactly corresponds to that of $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ mentioned before.

Our main theorems show how $A^{(\infty)}$ and $A^{(0)}$ explicitly parametrize the corresponding solution of the initial value problem. In order to state them we must prepare some more notations:

A Young diagram can be indicated by a sequence of strictly increasing integers as explained in [6]. Modifying it slightly, let us use a sequence $(l_m, l_{m+1}, \cdots, l_{s-1})$ $(m<s, m\le l_m<l_{m+1}<\cdots<l_{s-1})$ to indicate the pair (Ys) of a Young diagram Y with signature $(l_{s-1}-s+1, l_{s-2}-s+2, \cdots, l_m-m)$ and an integer s $(s>m)$ (cf. Fig. 1).

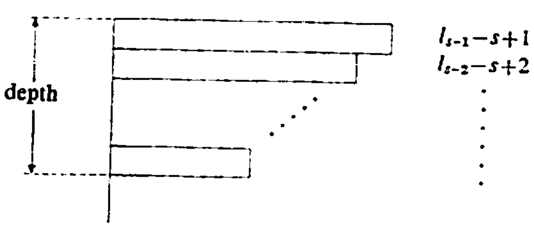

Fig. 1.

Notice that two sequences $(l_m, \cdots, l_{s-1})$ and $(l_{m'}, \cdots, l_{s-1})$ with $m'\le m$, $l_{m'}=m'$, $l_{m'+1}=m'+1, \cdots$, and $l_{m-1}=m-1$ indicate the same pair (Ys). Hereafter we shall obey this convention.

The character polynomial (the Schur function) χ_Y (cf. [7, 8]) is defined, for $(Ys)=(l_m, \cdots, l_{s-1})$, by the formula

$$\chi_Y(x)=\det (p_{l_i-j}(x))_{ij=m\cdots s-1}, \tag{2.11}$$

which is independent of s, and only depends on Y. Furthermore, let us choose a sequence $(a_i)_{i\in Z}$, $a_i\neq 0$, such that

$$a_{ii}^{(0)}=a_{i+1}/a_i \qquad \text{for any } i\in Z, \tag{2.12}$$

and define, for the triplet $A=(A^{(\infty)}, A^{(0)}, (a_i))$ and the pairs $(Ys)=(l_m, \cdots, l_{s-1})$ and $(Y's)=(l'_m, \cdots, l'_{s-1})$, the quantity $A_{(Ys)(Y's)}$ by

$$A_{(Ys)(Y's)}=\lim_{-m,n\to\infty} a_m \det \left(\sum_{k=m}^{n-1} \tilde{a}_{l_i,k}^{(\infty)} a_{k,l'_j}^{(0)}\right)_{ij=m\cdots s-1}, \tag{2.13}$$

where $\tilde{a}_{ij}^{(\infty)}$ denotes the (i,j) component of $A^{(\infty)-1}$, and $(l_m, \cdots, l_{s-1})$, $(l'_m, \cdots, l'_{s-1})$ indicate (Ys), $(Y's)$, respectively, according to the convention mentioned above as m decreases. The right hand side of (2.13) is a stable limit in the sense that $a_m \det(\cdots)$ is independent of m and n when $-m$ and n are sufficiently large (cf. Proposition 4.1).

Now we can state

Theorem 1. *The τ function of the solution to the initial value problem is given by*

$$\tau(s, x, y)=\sum_{Y,Y'} \chi_Y(x)\chi_{Y'}(-y)A_{(Ys)(Y's)}, \tag{2.14}$$

which does not identically vanish (e.g. $\tau(s, 0, 0)=a_s$). *Namely, the matrices L, M, B_μ and M_μ, defined by* (1.2), (2.1), (2.3), (2.6) *and* (2.14), *solve the Toda lattice hierarchy under initial conditions* (1.6). *Here $\sum_{Y,Y'}$ stands for the summation over all pairs of Young diagrams Y and Y'.*

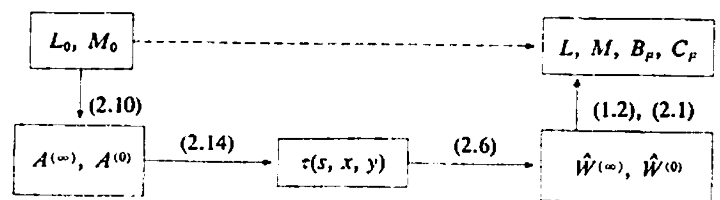

Fig. 2. The scheme solving the initial value problem.

The infinite series in (2.14) can be dealt with in the "formal" sense, i.e. in the ring $C[[x, y]]$ of formal series of weighted homogeneous polynomials in (x, y) (cf. § 5, paragraph 3). On the other hand it is also possible to discuss its "analytic" justification:

Theorem 2. *Suppose that there are constants C, a and b such that*

$$|\tilde{a}_{ij}^{(\infty)}| \leq C a^{|i-j|}, \quad |a_{ij}^{(0)}| \leq C b^{|i-j|}, \quad |a_i| \leq C, \quad i, j \in Z, \tag{2.15}$$

where $A^{(\infty)-1} = (\tilde{a}_{ij}^{(\infty)})_{ij \in Z}$ and $A^{(0)} = (a_{ij}^{(0)})_{ij \in Z}$. Then the right hand side of (2.14) *converges absolutely (with a certain uniformity—cf. Proposition* 4.2) *in the domain*

$$\begin{aligned} \{(x, y) \in C^\infty \times C^\infty;\ \overline{\lim_{n\to\infty}} |x_n|^{1/n} < 1/a,\ \overline{\lim_{n\to\infty}} |y_n|^{1/n} < 1/b, \\ \overline{\lim_{n\to\infty}} |x_n|^{1/n} \cdot \overline{\lim_{n\to\infty}} |y_n|^{1/n} < 1\}. \end{aligned} \tag{2.16}$$

Thus $\tau(s, x, y)$ is a holomorphic function in domain (2.16) in the sense of Section 5, paragraph 3.

The periodic cases are characterized as follows:

Theorem 3. *Under the conditions $[A^{(\infty)}, \Lambda^l] = [A^{(0)}, \Lambda^l] = 0$ the corresponding solution describes an l periodic Toda lattice. More precisely; there exists a constant a $(\neq 0)$ such that*

$$\begin{aligned} \tau(s+l, x, y) &= a\tau(s, x, y), \\ b_j(s+l, x, y) &= b_j(s, x, y), \\ c_j(s+l, x, y) &= c_j(s, x, y) \qquad \textit{for any } j, s \in Z, \end{aligned} \tag{2.17}$$

These theorems present us a complete description of the solution to the initial value problem for the Toda lattice hierarchy.

It is remarkable that the structure of $\tau(s, x, y)$ is very similar to that of the τ function of the KP hierarchy [6–8]. In fact, if we regard $\tau(s, x, y)$ as a function of only x (or y), (2.14) coincides essentially with the description of the τ function of the KP hierarchy presented in [6, 7]. Furthermore, as will be pointed out in Section 5, paragraph 1, $\tau(s, x, y)$ has in itself the structure of the τ function of the "two component" KP hierarchy [6–8].

The proof of Theorems 1–3 will be completed in Section 4.

§ 3. Finite lattice model

In this section a class of finite lattice model is introduced and investigated as an analogue of the Toda lattice hierarchy mentioned in Section 1. In this model the matrices L, M, B_μ, C_μ etc. $\cdots$ of size $Z \times Z$ are replaced

by the corresponding matrices of finite size whose rows and columns are indexed by a sequence of integers $m, m+1, \cdots, n-1$ with $m<n$. This model will play an essential role in Section 4.

As we shall see later, compared with the usual finite, nonperiodic Toda lattice whose corresponding objects of the τ functions are certain linear combinations of exponential functions (cf. [1–3]), our model appears to be more "degenerate" in the sense that its τ functions are polynomials in the independent variables $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$.

Let us begin with the following "factorization" problem for a matrix $A = (a_{ij})_{ij=m\cdots n-1}$ with $\det (a_{ij})_{ij=m\cdots s-1} \neq 0$ for $m < s \leq n$, i.e. the problem of finding some matrices $W^{(\infty)}$ and $W^{(0)}$ such that

$$W^{(0)} = W^{(\infty)} A, \tag{3.1}$$

where $W^{(\infty)}$ and $W^{(0)}$ are assumed to have the form

$$\begin{aligned} & W^{(\infty)} = \hat{W}^{(\infty)} \exp \sum_{\mu=1}^{\infty} x_\mu \Lambda_{[mn)}^\mu, \qquad W^{(0)} = \hat{W}^{(0)} \exp \sum_{\mu=1}^{\infty} y_\mu {}^t\Lambda_{[mn)}^\mu, \\ & \hat{W}^{(\infty)} = (\hat{w}_{i-j}^{(\infty)}(i, x, y))_{ij=m\cdots n-1}, \quad \hat{w}_j^{(\infty)} = 0\ (j<0), \quad \hat{w}_0^{(\infty)} = 1, \\ & \hat{W}^{(0)} = (\hat{w}_{j-i}^{(0)}(i, x, y))_{ij=m\cdots n-1}, \quad \hat{w}_j^{(0)} = 0\ (j<0), \quad \hat{w}_0^{(0)} \neq 0, \\ & \Lambda_{[mn)} = (\delta_{i,j-1})_{ij=m\cdots n-1}, \quad {}^t\Lambda_{[mn)} = (\delta_{i,j+1})_{ij=m\cdots n-1}. \end{aligned} \tag{3.2}$$

Remark. For simplicity of notations the same notations, $W^{(\infty)}$, $W^{(0)}$, etc., as those used in Section 2 are used here. In Section 4 we shall change the notations to distinguish them from those used in Section 1 and Section 2.

(3.1) is explicitly solved as follows.

Proposition 3.1. *$W^{(\infty)}$ and $W^{(0)}$ are uniquely determined, and given by the formulas*

$$\hat{w}_k^{(\infty)}(s, x, y) = \frac{(-)^k \det (a_{ij}(x, y))_{\substack{i=m\cdots s\hat{-}k\cdots s \\ j=m\cdots s-1}}}{\det (a_{ij}(x, y))_{ij=m\cdots s-1}}, \tag{3.3}$$

$$\hat{w}_k^{(0)}(s, x, y) = \frac{\det (a_{ij}(x, y))_{\substack{i=m\cdots s \\ j=m\cdots s-1, s+k}}}{\det (a_{ij}(x, y))_{ij=m\cdots s-1}}, \tag{3.4}$$

where we set

$$(a_{ij}(x, y))_{ij=m\cdots n-1} = \exp \left[\sum_{\mu=1}^{\infty} x_\mu \Lambda_{[mn)}^\mu\right] A \exp \left[-\sum_{\mu=1}^{\infty} y_\mu {}^t\Lambda_{[mn)}^\mu\right], \tag{3.5}$$

and $s \hat{-} k$ in (3.3) *is used to show that $s-k$ is removed from the indices of rows in the determinant.*

Remark. $a_{ij}(x, y)$, $i, j=m, \cdots, n-1$, are polynomials, since $\Lambda_{[mn)}$ and ${}^t\Lambda_{[mn)}$ are nilpotent. Furthermore the assumption $\det(a_{ij})_{ij=m\cdots s-1} \neq 0$ for $m<s\leq n$ implies that $\det(a_{ij}(x, y))_{ij=m\cdots s-1}$ does not identically vanish for $m<s\leq n$. Thus (3.3) and (3.4) make sense.

Proof of Proposition 3.1. Equation (3.1) implies

$$(3.6)\qquad (\hat{w}^{(\infty)}_{s-m}(s), \hat{w}^{(\infty)}_{s-m-1}(s), \cdots, \hat{w}^{(\infty)}_0(s))(a_{ij}(x, y))_{\substack{i=m\cdots s\\ j=m\cdots s-1}}=0,$$

$$(3.7)\qquad \begin{aligned}&(\hat{w}^{(0)}_0(s), \hat{w}^{(0)}_1(s), \cdots, \hat{w}^{(0)}_{n-s+1}(s))\\ &=(\hat{w}^{(\infty)}_{s-m}(s), \hat{w}^{(\infty)}_{s-m-1}(s), \cdots, \hat{w}^{(\infty)}_0(s))(a_{ij}(x, y))_{\substack{i=m\cdots s\\ j=s\cdots n-1}}.\end{aligned}$$

Here we used the simplified notations $\hat{w}^{(\infty)}_k(s)$ and $\hat{w}^{(0)}_k(s)$ for $\hat{w}^{(\infty)}_k(s, x, y)$ and $\hat{w}^{(0)}_k(s, x, y)$. Since we assumed the condition $\hat{w}^{(\infty)}_0=1$, Cramer's formula and (3.6) immediately lead us to (3.3).

Then by virtue of (3.3) and (3.7) we have

$$\begin{aligned}\hat{w}^{(0)}_k(s)&=\sum_{\mu=0}^{s-m}\frac{(-)^{\mu}\det(a_{ij}(x, y))_{\substack{i=m\cdots \hat{s-\mu}\cdots s\\ j=m\cdots s-1}}a_{s-\mu, s+k}(x, y)}{\det(a_{ij}(x, y))_{ij=m\cdots s-1}}\\ &=\frac{\det(a_{ij}(x, y))_{\substack{i=m\cdots s\\ j=m\cdots s-1, s+k}}}{\det(a_{ij}(x, y))_{ij=m\cdots s-1}}.\end{aligned}$$

This is nothing but (3.4). Q.E.D.

Our finite lattice model is derived as follows.

Proposition 3.2. *Set*

$$(3.8)\qquad L=W^{(\infty)}\Lambda_{[mn)}W^{(\infty)-1}, \qquad M=W^{(0)}\,{}^t\Lambda_{[mn)}W^{(0)-1},$$

$$(3.9)\qquad B_\mu=(L^\mu)_+, \quad C_\mu=(M^\mu)_-, \quad \mu=1, 2, \cdots,$$

where the symbols $(\ )_\pm$ *denote the same operations as those used for* $Z\times Z$ *matrices except that in the present case the indices of the rows and columns are restricted to the integers* $m, m+1, \cdots, n-1$. *Then we have the following analogues of the Lax type system, the Zakharov-Shabat type system and the linearization* (cf. (1.3), (1.4), (2.2)).

$$(3.10)\qquad \begin{aligned}&\partial_{x_\mu}L=[B_\mu, L], \quad \partial_{y_\mu}L=[C_\mu, L],\\ &\partial_{x_\mu}M=[B_\mu, M], \quad \partial_{y_\mu}M=[C_\mu, M], \quad \mu=1, 2, \cdots,\end{aligned}$$

$$(3.11)\qquad \begin{aligned}&\partial_{x_\nu}B_\mu-\partial_{x_\mu}B_\nu+[B_\mu, B_\nu]=0, \quad \partial_{y_\nu}C_\mu-\partial_{y_\mu}C_\nu+[C_\mu, C_\nu]=0,\\ &\partial_{y_\nu}B_\mu-\partial_{x_\mu}C_\nu+[B_\mu, C_\nu]=0, \quad \mu, \nu=1, 2, \cdots,\end{aligned}$$

$$(3.12)\qquad \begin{aligned}&\partial_{x_\mu}\hat{W}^{(\infty)}=B_\mu\hat{W}^{(\infty)}-\hat{W}^{(\infty)}\Lambda^\mu_{[mn)}, \quad \partial_{y_\mu}\hat{W}^{(\infty)}=C_\mu\hat{W}^{(\infty)},\\ &\partial_{y_\mu}\hat{W}^{(0)}=B_\mu\hat{W}^{(0)}, \quad \partial_{y_\mu}\hat{W}^{(0)}=C_\mu\hat{W}^{(0)}-\hat{W}^{(0)}\,{}^t\Lambda^\mu_{[mn)}, \quad \mu=1, 2, \cdots.\end{aligned}$$

Remark. We used also here, for simplicity of notations, the same notations L, M, etc. as those used in Section 1 and Section 2, though of course they denote different objects.

Proof of Proposition 3.2. Notice, at first, that (3.1) implies

$$\partial_{x_\mu}W^{(\infty)}\cdot W^{(\infty)-1}=\partial_{x_\mu}W^{(0)}\cdot W^{(0)-1}, \quad \partial_{y_\mu}W^{(\infty)}\cdot W^{(\infty)-1}=\partial_{y_\mu}W^{(0)}\cdot W^{(0)-1}.$$

Then, by virtue of (3.2), we have

$$(3.13)\qquad \begin{aligned}&\partial_{x_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1}+\hat{W}^{(\infty)}\Lambda^\mu_{[mn)}\hat{W}^{(\infty)-1}=\partial_{x_\mu}\hat{W}^{(0)}\cdot\hat{W}^{(0)-1},\\ &\partial_{y_\mu}\hat{W}^{(0)}\cdot\hat{W}^{(0)-1}+\hat{W}^{(0)}\,{}^t\Lambda^\mu_{[mn)}\hat{W}^{(0)-1}=\partial_{y_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1}.\end{aligned}$$

Furthermore (3.2) implies

$$\begin{aligned}&(\partial_{x_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1})_+=0, \quad (\partial_{x_\mu}\hat{W}^{(0)}\cdot\hat{W}^{(0)-1})_+=\partial_{x_\mu}\hat{W}^{(0)}\cdot\hat{W}^{(0)-1},\\ &(\partial_{y_\mu}\hat{W}^{(0)}\cdot\hat{W}^{(0)-1})_-=0, \quad (\partial_{y_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1})_-=\partial_{y_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1}.\end{aligned}$$

Hence (3.13) leads us to

$$(3.14)\qquad \begin{aligned}&B_\mu=(L^\mu)_+=\partial_{x_\mu}\hat{W}^{(0)}\cdot\hat{W}^{(0)-1},\\ &C_\mu=(M^\mu)_-=\partial_{y_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1}.\end{aligned}$$

(3.12) is an immediate consequence of (3.13) and (3.14). Also some simple calculation leads us to (3.10). For example,

$$\begin{aligned}\partial_{x_\mu}L&=\partial_{x_\mu}\hat{W}^{(\infty)}\cdot\Lambda^\mu_{[mn)}\hat{W}^{(\infty)-1}-\hat{W}^{(\infty)}\Lambda^\mu_{[mn)}\hat{W}^{(\infty)-1}\cdot\partial_{x_\mu}\hat{W}^{(\infty)}\cdot\hat{W}^{(\infty)-1}\\ &=[B_\mu-\hat{W}^{(\infty)}\Lambda^\mu_{[mn)}\hat{W}^{(\infty)-1}, \quad \hat{W}^{(\infty)}\Lambda_{[mn)}\hat{W}^{(\infty)-1}]\\ &=[B_\mu, L].\end{aligned}$$

The other equalities in (3.10) can be similarly verified.

Differentiating (3.12) and comparing the cross-derivatives $\partial_{x_\mu}\partial_{x_\nu}W^{(\infty)}=\partial_{x_\nu}\partial_{x_\mu}W^{(\infty)}$, etc., we have

$$(\partial_{x_\nu}B_\mu-\partial_{x_\mu}B_\nu+[B_\mu, B_\nu])W^{(\infty)}=0, \text{ etc.}$$

Since $W^{(\infty)}$ and $W^{(0)}$ are invertible matrices for the generic values of (x, y) (cf. (3.2)–(3.4)), (3.11) follows. Q.E.D.

As an analogue of the initial value problem we have

Proposition 3.3. *Suppose that the matrix* A *is factorized as follows:*

$$A = A^{(\infty)-1}A^{(0)},$$
(3.15) $$A^{(\infty)} = (a_{ij}^{(\infty)})_{ij=m\cdots n-1}, \quad a_{ij}^{(\infty)} = 0 \ (i<j), \quad a_{ii}^{(\infty)} = 1,$$
$$A^{(0)} = (a_{ij}^{(0)})_{ij=m\cdots n-1}, \quad a_{ij}^{(0)} = 0 \ (i>j), \quad a_{ij}^{(0)} \neq 0.$$

Then $W^{(\infty)}$ and $W^{(0)}$ satisfy the initial conditions

(3.16) $$W^{(\infty)}|_{x=y=0} = A^{(\infty)}, \qquad W^{(0)}|_{x=y=0} = A^{(0)}.$$

Proof. (3.1) implies

$$W^{(\infty)}|_{x=y=0}A^{(\infty)-1} = W^{(0)}|_{x=y=0}A^{(0)-1}.$$

Notice that the left hand side is a lower triangular matrix with the diagonal part$=1$, while the right hand side is upper triangular. Hence we conclude that both hand sides coinside with the unit matrix. Q.E.D.

An analogue of the τ function is derived as follows.

Proposition 3.4. *The function*

(3.17) $$\tau(s, x, y) = \det(a_{ij}(x, y))_{ij=m\cdots s-1}$$

satisfies the equations

(3.18) $$\hat{w}_k^{(\infty)}(s, x, y) = p_k(-\tilde{\partial}_x)\tau(s, x, y)/\tau(s, x, y), \quad k=0, 1, \cdots,$$
$$\hat{w}_k^{(0)}(s, x, y) = p_k(-\tilde{\partial}_y)\tau(s+1, x, y)/\tau(s, x, y), \quad k=0, 1, \cdots,$$

where $\tilde{\partial}_x$ and $\tilde{\partial}_y$ are the same as those in (2.6).

Proof. Since

$$\tau(s, x-\varepsilon(\lambda), y) = \sum_{j=0}^{\infty} p_j(-\tilde{\partial}_x)\tau(s, x, y)\lambda^j,$$
$$\tau(s+1, x, y-\varepsilon(\lambda)) = \sum_{j=0}^{\infty} p_j(-\tilde{\partial}_y)\tau(s+1, x, y)\lambda^j,$$
$$\varepsilon(\lambda) = (\lambda, \lambda^2/2, \lambda^3/3, \cdots),$$

equations (3.18) are equivalent to

(3.19) $$\sum_{k=0}^{s-m} \hat{w}_k^{(\infty)}(s, x, y)\lambda^k = \tau(s, x-\varepsilon(\lambda), y)/\tau(s, x, y),$$
$$\sum_{k=0}^{n-1-s} \hat{w}_k^{(0)}(s, x, y)\lambda^k = \tau(s+1, x, y-\varepsilon(\lambda))/\tau(s, x, y).$$

Let us prove (3.19) instead of (3.18).

We have the following formulas

$$\sum_{\mu=1}^{\infty} \lambda^\mu \Lambda_{[mn)}^\mu/\mu = -\log(1-\lambda\Lambda_{[mn)}),$$
$$\exp\left[-\sum_{\mu=1}^{\infty} \lambda^\mu \Lambda_{[mn)}^\mu/\mu\right] = 1-\lambda\Lambda_{[mn)},$$
$$\exp\left[\sum_{\mu=1}^{\infty} \lambda^\mu\, {}^t\Lambda_{[mn)}^\mu/\mu\right] = (1-\lambda\,{}^t\Lambda_{[mn)})^{-1}$$
$$= 1+\lambda\,{}^t\Lambda_{[mn)} + \cdots + (\lambda\,{}^t\Lambda_{[mn)})^{n-m-1},$$

which imply

(3.20) $$a_{i,j}(x-\varepsilon(\lambda), y) = a_{i,j}(x, y) - a_{i+1,j}(x, y),$$
$$a_{i,j}(x, y-\varepsilon(\lambda)) = \sum_{k=0}^{n-1-j} a_{i,j+k}(x, y)\lambda^k.$$

Furthermore, in order to calculate the right hand side of (3.19), we shall use the following formula of linear algebra [9].

(3.21) $$\det\left(\sum_{l=1}^{p+q} g_{il}h_{lj}\right)_{ij=1\cdots p} = \sum_{1\le l_1<\cdots<l_p\le p+q} \det(g_{il_j})_{ij=1\cdots p} \cdot \det(h_{l_ij})_{ij=1\cdots p}.$$

By virtue of (3.20) and (3.21),

$$\tau(s, x-\varepsilon(\lambda), y) = \det\left(\underbrace{\begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix}}_{\longleftarrow s-m+1 \longrightarrow} \cdot [a_{ij}(x, y)]_{\substack{i=m\cdots s \\ j=m\cdots s-1}}\right)$$

$$= \sum_{k=0}^{s-m} \det\left[\begin{array}{cccc|cccc} 1 & -\lambda & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & -\lambda & & & & \\ & & & 1 & & & & \\ \hline & & & & -\lambda & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -\lambda \end{array}\right] \det(a_{ij}(x, y))_{\substack{i=m\cdots \hat{s-k}\cdots s \\ j=m\cdots s-1}}$$
$$\longleftarrow s-m+k \longrightarrow \longleftarrow k \longrightarrow$$

$$= \sum_{k=0}^{s-m} (-)^k \det(a_{ij}(x, y))_{\substack{i=m\cdots \hat{s-k}\cdots s \\ j=m\cdots s-1}} \lambda^k.$$

Similarly,

$$\tau(s+1, x, y-\varepsilon(\lambda)) = \det \left([a_{ij}(x, y)]_{\substack{i=m\cdots s \\ j=m\cdots n-1}} \cdot \begin{bmatrix} 1 & & & & \\ \lambda & 1 & & & \\ \lambda^2 & \lambda & 1 & & \\ \cdot\cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & 1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \lambda^{n-m-1} & \cdots & & & \lambda^{n-s-2} \end{bmatrix} \right)$$

$$\longleftarrow s-m+1 \longrightarrow$$

$$= \sum_{k=0}^{n-s-1} \det (a_{ij}(x, y))_{\substack{i=m\cdots s \\ j=m\cdots s-1,\, s+k}} \det (\lambda^{i-j})_{\substack{i=m\cdots s-1,\, s+k \\ j=m\cdots s}}$$

$$= \sum_{k=0}^{n-s-1} \det (a_{ij}(x, y))_{\substack{i=m\cdots s \\ j=m\cdots s-1,\, s+k}} \lambda^k.$$

A comparison with (3.3) and (3.4) leads us to (3.18). Q.E.D.

$\tau(s, x, y)$ has another expression:

Using the formulas

$$\exp\left[\sum_{\mu=1}^{\infty} x_\mu \Lambda^\mu_{[mn)}\right] = \sum_{j=0}^{\infty} p_j(x) \Lambda^j_{[mn)} = (p_{j-i}(x))_{ij=m\cdots n-1},$$

$$\exp\left[-\sum_{\mu=1}^{\infty} y_\mu {}^t\Lambda^\mu_{[mn)}\right] = \sum_{j=0}^{\infty} p_j(-y) {}^t\Lambda^j_{[mn)} = (p_{i-j}(-y))_{ij=m\cdots n-1},$$

we can rewrite (3.17) into

$$\tau(s, x, y) = \det \left(\sum_{k,l=m}^{n-1} p_{k-i}(x) a_{kl} p_{l-j}(-y) \right)_{ij=m\cdots s-1}. \tag{3.22}$$

Applying (3.21) to this formula repeatedly, we obtain

$$\tau(s, x, y) = \sum_{Y, Y' \subset \square^{s-m}_{n-s}} \chi_Y(x) \chi_{Y'}(-y) A_{(Ys)(Y's)}, \tag{3.23}$$

where $\square^{s-m}_{n-s}$ denotes the rectangular Young diagram of size $(s-m) \times (n-s)$,

$$A_{(Ys)(Y's)} = \det (a_{l_i l'_j})_{ij=m\cdots s-1}, \tag{3.24}$$
$$(Ys) = (l_m, \cdots, l_{s-1}), \qquad (Y's) = (l'_m, \cdots, l'_{s-1}),$$

and the summation is taken over all the pairs (Y, Y') with $Y, Y' \subset \square^{s-m}_{n-s}$.

In the case when A is factorized as in (3.15), formula (3.21) is once more applied to yield

$$A_{(Ys)(Y's)} = \sum_{Y'' \subset Y, Y'} (A^{(\infty)-1})_{(Ys)(Y''s)} A^{(0)}_{(Y''s)(Y's)}, \tag{3.25}$$

where the symbols $A^{(0)}_{(Y''s)(Y's)}$ etc. are used in the same sense as in (3.24), and the relations $Y'' \subset Y, Y'$ are understood to show the natural inclusion as geometric objects on the plane. Note that (3.15) implies

$$(A^{(\infty)-1})_{(Ys)(Y''s)} = 0 \quad \text{unless } Y'' \subset Y, \tag{3.26}$$
$$A^{(0)}_{(Y''s)(Y's)} = 0 \quad \text{unless } Y'' \subset Y'.$$

It should be noticed that formulas (3.22) and (3.23) reveal a remarkable structure of $\tau(s, x, y)$, which is closely related with that of the polynomial τ functions of the *KP* hierarchy. It will be shown later in Section 5 that $\tau(s, x, y)$ has also the structure of the τ function of the two component *KP* hierarchy [6–8].

Formulas (3.22)–(3.25) will play an essential role in the discussion of the next section. The content of this section, however, seems to provide in itself some interesting problems independent of the main purpose of the present paper.

§ 4. Approach to the infinite lattice

In this section the proof of Theorems 1–3 is completed. We shall proceed as follows:

We shall first introduce a series of finite lattice models, each of which corresponds to a pair of integers m, n with $m<n$ and a matrix $A_{[mn)}$ of size $(n-m)\times(n-m)$ related to the initial values $A^{(\infty)}$ and $A^{(0)}$ given in Section 2. At each (m, n) stage, (3.17) defines the corresponding τ function with $A=A_{[mn)}$. Its limit, as $-m, n\to\infty$, might not make sense in itself. If, however, the τ function at the (m, n) stage is multiplied by the factor a_m, we can establish its limit as $-m, n\to\infty$ so that (2.14) is obtained. (1.3), (1.4) and (1.6) are then immediately verified. The convergence of the infinite series in (2.14) is proved by making use of Hadamard's inequality which was also effectively used in the estimate of convergence of the τ function of the *KP* hierarchy [7]. The characterization of the periodic cases follows immediately from the construction.

To begin with, let us introduce a series of finite lattice models:

Suppose that $A^{(\infty)}=(a_{ij}^{(\infty)})_{ij\in Z}$, $A^{(\infty)-1}=(\tilde{a}_{ij}^{(\infty)})_{ij\in Z}$, $A^{(0)}=(a_{ij}^{(0)})_{ij\in Z}$ and $(a_i)_{i\in Z}$ are given as in Section 2. Let us set

$$A_{[mn)}=(a_{ij}^{(\infty)})_{ij=m\cdots n-1}(a_{ij}^{(0)})_{ij=m\cdots n-1},\qquad m<n, \tag{4.1}$$

and consider the "factorization problem" (3.1) for $A_{[mn)}$, i.e. the problem of finding some matrices $W_{[mn)}^{(\infty)}$ and $W_{[mn)}^{(0)}$ such that

$$W_{[mn)}^{(0)}=W_{[mn)}^{(\infty)}A_{[mn)}, \tag{4.2}$$

where $W_{[mn)}^{(\infty)}$ and $W_{[mn)}^{(0)}$ are assumed to have the same form as (3.2), i.e.

$$\begin{aligned} &W_{[mn)}^{(\infty)}=\hat{W}_{[mn)}^{(\infty)}\exp\sum_{\mu=1}^{\infty}x_\mu\Lambda_{[mn)}^{\mu},\quad W_{[mn)}^{(0)}=\hat{W}_{[mn)}^{(0)}\exp\sum_{\mu=1}^{\infty}y_\mu{}^t\Lambda_{[mn)}^{\mu},\\ &\hat{W}_{[mn)}^{(\infty)}=(\hat{w}_{[mn)i-j}^{(\infty)}(i,x,y))_{ij=m\cdots n-1},\quad \hat{w}_{[mn)j}^{(\infty)}=0\ (j<0),\quad \hat{w}_{[mn)0}^{(\infty)}=1,\\ &\hat{W}_{[mn)}^{(0)}=(\hat{w}_{[mn)j-i}^{(0)}(i,x,y))_{ij=m\cdots n-1},\quad \hat{w}_{[mn)j}^{(0)}=0\ (j<0),\quad \hat{w}_{[mn)0}^{(0)}\neq 0.\end{aligned} \tag{4.3}$$

As we have seen in Section 3, $W_{[mn)}^{(\infty)}$ and $W_{[mn)}^{(0)}$ are uniquely determined, and all the results of Section 3 follow.

Let us denote by $L_{[mn)}$, $M_{[mn)}$, $B_{[mn)\mu}$, $C_{[mn)\mu}$, $\tau_{[mn)}(s,x,y)$ the corresponding objects of L, M, B_μ, C_μ, $\tau(s,x,y)$ for $A_{[mn)}$ so that we can clearly distinguish the finite lattice models we are working with from the hierarchy for the infinite Toda lattice.

Thus we have

$$\tau_{[mn)}(s,x,y)=\sum_{Y,Y'\subset\square_{n-s}^{s-m}}\chi_Y(x)\chi_{Y'}(-y)A_{[mn)(Ys)(Y's)}, \tag{4.4}$$

$$\begin{aligned} A_{[mn)(Ys)(Y's)}&=\det\left(\sum_{k=m}^{n-1}\tilde{a}_{l_ik}^{(\infty)}a_{kl_j'}^{(0)}\right)_{ij=m\cdots s-1}\\ &=\sum_{m\le l_m''<\cdots<l_{s-1}''<n}\det(\tilde{a}_{l_il_j''}^{(\infty)})_{ij=m\cdots s-1}\det(a_{l_i''l_j'}^{(0)})_{ij=m\cdots s-1},\end{aligned} \tag{4.5}$$

with $(Ys)=(l_m,\cdots,l_{s-1})$, $(Y's)=(l_m',\cdots,l_{s-1}')$.

Our first step toward the infinite lattice is

Proposition 4.1. *$a_mA_{[mn)(Ys)(Y's)}$ is independent of m and n as far as Y, $Y'\subset\square_{n-s}^{s-m}$.*

Proof. Recall that two sequences $(l_m,\cdots,l_{s-1})$ and $(l_{m'},\cdots,l_{s-1})$, with $m'\le m$, $l_{m'}=m'$, $\cdots$, $l_{m-1}=m-1$, indicate the same pair (Ys) of a Young diagram Y and an integer s, as we noticed in Section 2. Let us utilize the convention noticed there, in order to prove Proposition 4.1.

Suppose Y, $Y'\subset\square_{n-s}^{s-m}$, $m'\le m$ and $n'\ge n$. By virtue of (2.9) and (2.12),

$$\begin{aligned}&a_{ij}^{(\infty)}=\tilde{a}_{ij}^{(\infty)}=0\quad\text{for } i<j,\quad a_{ii}^{(\infty)}=\tilde{a}_{ii}^{(\infty)}=1,\\ &a_{ij}^{(0)}=0\quad\text{for } i>j,\quad a_{ii}^{(0)}=a_{i+1}/a_i.\end{aligned}$$

Hence we have

$$\begin{aligned}\det(\tilde{a}_{l_il_j''}^{(\infty)})_{ij=m'\cdots s-1}&=\det\left[\begin{array}{ccc|c} 1 & & & \\ * & \ddots & & 0\\ * & & 1 & \\ \hline * & * & & \\ * & * & & (\tilde{a}_{l_il_j''}^{(\infty)})_{ij=m\cdots s-1}\end{array}\right]\\ &=\det(\tilde{a}_{l_il_j''}^{(\infty)})_{ij=m\cdots s-1},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\det(a_{l_i''l_j'}^{(0)})_{ij=m'\cdots s-1}&=\det\left[\begin{array}{ccc|cc} a_{m',m'}^{(0)} & * & * & * & *\\ & \ddots & & & \\ & & a_{m-1,m-1}^{(0)} & * & *\\ \hline & 0 & & & (a_{l_i''l_j'}^{(0)})_{ij=m\cdots s-1}\end{array}\right]\\ &=(a_m/a_{m'})\det(a_{l_i''l_j'}^{(0)})_{ij=m\cdots s-1},\end{aligned} \tag{4.7}$$

where, according to the remark mentioned above, we prolonged the sequences $(l_m,\cdots,l_{s-1})$ etc. $\cdots$ into $(l_{m'},\cdots,l_{s-1})$ etc. $\cdots$ with $l_{m'}=m'$, $\cdots$, $l_{m-1}=m-1$, $l_{m'}'=m'$, $\cdots$, $l_{m-1}'=m-1$ and $l_{m'}''=m'$, $\cdots$, $l_{m-1}''=m-1$. (4.6) and (4.7), compared with (4.5), imply $a_{m'}A_{[m'n')(Ys)(Y's)}=a_mA_{[mn)(Ys)(Y's)}$. This proves proposition 4.1. Q.E.D.

Proposition 4.1 shows that $A_{(Ys)(Y's)}$ is well defined by (2.13). Furthermore (4.4) yields

$$a_m\tau_{[mn)}(s,x,y)=\sum_{Y,Y'\subset\square_{n-s}^{s-m}}\chi_Y(x)\chi_{Y'}(-y)A_{(Ys)(Y's)}. \tag{4.8}$$

Note that the right hand side is a partial sum of that of (2.14) whose terms are exhausted by these partial sums as $-m,n\to\infty$. Thus

$$\tau(s,x,y)=\lim_{-m,n\to\infty}a_m\tau_{[mn)}(s,x,y). \tag{4.9}$$

The limit in (4.9) makes sense in the ring $C[\![x,y]\!]$ (cf. Section 5, paragraph 3). This is a "formal" justification of the series in (2.14). Its "analytic" justification is given by

Proposition 4.2. *Suppose that inequalities* (2.15) *are fulfilled. Then, for any positive constants a', b', C_1 and C_2 with*

(4.10) $$a'>a,\ b'>b \quad and \quad a'b'>1,$$

the right hand side of (2.14) *converges absolutely and uniformly in the domain*

(4.11) $$\{(x, y) \in C^\infty \times C^\infty;\ |x_\mu| \le C_1/a'^\mu,\ |y_\mu| \le C_2/b'^\mu,\ \mu=1, 2, \cdots\}.$$

Proof. The right hand side of (2.14) may be rearranged with respect to the depth (cf. Fig. 1) of Y and Y' as follows:

(4.12)
$$\begin{aligned}\tau(s, x, y) &= \sum_{m,m'=-\infty}^{s} \sum{}^{(1)} \det (p_{l_i-j}(x))_{ij=m\cdots s-1} \det (p_{l'_i-j}(-y))_{ij=m'\cdots s-1} \\ &\quad \cdot a_m \det \Big(\sum_{k=\tilde m}^{\tilde n-1} \tilde a^{(\infty)}_{l_i k} a^{(0)}_{k l'_j}\Big)_{ij=\tilde m\cdots s-1} \\ &= \sum_{m,m'=-\infty}^{s} \frac{1}{(s-m)!\,(s-m')!} \sum{}^{(2)} \det (p_{l_i-j}(x))_{ij=m\cdots s-1} \\ &\quad \cdot \det (p_{l'_i-j}(-y))_{ij=m'\cdots s-1} a_m \det \Big(\sum_{k=\tilde m}^{\tilde n-1} \tilde a^{(\infty)}_{l_i k} a^{(0)}_{k l'_j}\Big)_{ij=\tilde m\cdots s-1},\end{aligned}$$

where $\tilde m=\min(m, m')$, $\tilde n=\max(l_{s-1}, l'_{s-1})$, $l_i=i\ (\tilde m \le i<m)$, $l'_i=i\ (\tilde m\le i<m')$, and $\sum^{(1)}$, $\sum^{(2)}$ denote the summation over the totality of integers $l_m, \cdots, l_{s-1}$ and $l'_{m'}, \cdots, l'_{s-1}$ with

$$\sum{}^{(1)}:\ m<l_m<\cdots<l_{s-1}, \quad m'<l'_{m'}<\cdots<l'_{s-1},$$
$$\sum{}^{(2)}:\ m<l_i\ (m\le i<s), \quad l_i\ne l_j\ (i\ne j), \quad m'<l'_i\ (m'<i<s), \quad l'_i\ne l'_j\ (i\ne j),$$

respectively.

In order to estimate the determinants in (4.12), we shall use Hadamard's inequality [9], which asserts that, if a matrix $(g_{ij})_{ij=1\ldots p}$ satisfies, for some constants r_i, s_i $(i=1\cdots p)$ and K, the inequalities

(4.13) $$|g_{ij}|\le K r_i s_j, \qquad i, j=1, \cdots, p,$$

we have

(4.14) $$|\det(g_{ij})_{ij=1\ldots p}| \le K^p p^{p/2} \prod_{i=1}^{p} (r_i s_i).$$

Let us proceed to the proof that (4.12) actually converges in (4.11). For some technical reasons we must choose here some positive constants a_1, a_2, b_1 and b_2 with

(4.15) $$a'>a_2>a_1>a, \quad b'>b_2>b_1>b \quad and \quad a_1b_1>1.$$

$\det(\sum_{k=\tilde m}^{\tilde n-1} \tilde a^{(\infty)}_{l_i k} a^{(0)}_{k l'_j})_{ij=\tilde m\cdots s-1}$ can be estimated as follows: Formula (3.21), applied to this determinant, yields

(4.16)
$$\begin{aligned}&\det \Big(\sum_{k=\tilde m}^{\tilde n-1} \tilde a^{(\infty)}_{l_i k} a^{(0)}_{k l'_j}\Big)_{ij=\tilde m\cdots s-1} \\ &= \sum{}^{(3)} \det (\tilde a^{(\infty)}_{l_i l''_j})_{ij=\tilde m\cdots s-1} \det (a^{(0)}_{l''_i l'_j})_{ij=\tilde m\cdots s-1} \\ &= \sum{}^{(4)} \frac{1}{(s-\tilde m)!} \det (\tilde a^{(\infty)}_{l_i l''_j})_{ij=\tilde m\cdots s-1} \det (a^{(0)}_{l''_i l'_j})_{ij=\tilde m\cdots s-1},\end{aligned}$$

where $\sum^{(3)}$ and $\sum^{(4)}$ denotes the summation over the totality of integers $l''_{\tilde m}, \cdots, l''_{s-1}$ with

$$\sum{}^{(3)};\ \tilde m<l''_{\tilde m}<\cdots<l''_{s-1},$$
$$\sum{}^{(4)};\ \tilde m<l''_i\ (\tilde m\le i<s), \qquad l''_i\ne l''_j\ (i\ne j),$$

respectively. By virtue of (2.9), (2.15) and (4.15) we have

$$|\tilde a^{(\infty)}_{ij}|\le C a_1^{i-j}, \quad |a^{(0)}_{ij}|\le C b_1^{j-i}, \quad |a_i|\le C \quad \text{for } i, j \in Z,$$

(we may also suppose $C\ge 1$), so that Hadamard's inequality implies

(4.17)
$$\begin{aligned}|\det (\tilde a^{(\infty)}_{l_i l''_j})_{ij=\tilde m\cdots s-1}| &\le C^{s-\tilde m}(s-\tilde m)^{(s-\tilde m)/2} a_1^{|Y|-|Y''|}, \\ |\det (a^{(0)}_{l''_i l'_j})_{ij=\tilde m\cdots s-1}| &\le C^{s-\tilde m}(s-\tilde m)^{(s-\tilde m)/2} b_1^{|Y'|-|Y''|},\end{aligned}$$

where

(4.18) $$|Y|=\sum_{i=\tilde m}^{s-1} (l_i-i), \quad |Y'|=\sum_{i=\tilde m}^{s-1} (l'_i-i), \quad |Y''|=\sum_{i=\tilde m}^{s-1} (l''_i-i).$$

Thus, using a rough estimate for $p!$,

(4.19) $$p^{p/2}\le C_0^p p!, \qquad p=0, 1, 2, \cdots.$$

(C_0 is a constant with $C_0\ge 1$), and noting the inequalities $s-\tilde m\le 2s-m-m'$, $C_0\ge 1$, $C\ge 1$ and $a_1b_1>1$, we have

(4.20)
$$\begin{aligned}\det \Big(\sum_{k=\tilde m}^{\tilde n-1} \tilde a^{(\infty)}_{l_i k} a^{(0)}_{k l'_j}\Big)_{ij=\tilde m\cdots s-1} &\le (C^2C_0)^{s-\tilde m} a_1^{|Y|} b_1^{|Y'|} \sum{}^{(4)} (a_1b_1)^{-|Y''|} \\ &\le (C^2C_0)^{2s-m-m'}\Big(1-\frac{1}{a_1b_1}\Big)^{m+m'-2s} a_1^{|Y|} b_1^{|Y'|}.\end{aligned}$$

$\det(p_{l_i-j}(x))_{ij=m\cdots s-1}$ and $\det(p_{l'_i-j}(-y))_{ij=m'\cdots s-1}$ can be estimated as follows:

If (x, y) is in domain (4.11), $\exp(\sum_{\mu=1}^\infty x_\mu \lambda^\mu)$ and $\exp(-\sum_{\mu=1}^\infty y_\mu\lambda^\mu)$ are holomorphic functions of λ in $\{|\lambda|<a'\}$ and $\{|\lambda|<b'\}$, respectively. $p_k(x)$ and $p_k(-y)$ are recovered by

$$p_k(x)=\oint_{|\lambda|=a_2}\exp\left(\sum_{\mu=1}^{\infty}x_\mu\lambda^\mu\right)\lambda^{-k-1}d\lambda/(2\pi\sqrt{-1}),$$

$$p_k(-y)=\oint_{|\lambda|=b_2}\exp\left(-\sum_{\mu=1}^{\infty}y_\mu\lambda^\mu\right)\lambda^{-k-1}d\lambda/(2\pi\sqrt{-1})\qquad\text{(cf. (2.7))}.$$

Hence we can show that there are some constants C_3 and C_4 such that, if (x, y) is in domain (4.11),

$$|p_k(x)|\leq C_3a_2^{-k},\quad |p_k(-y)|\leq C_4b_2^{-k},\quad k=0, 1, 2, \cdots. \tag{4.21}$$

Then, applying Hadamard's inequality once more, we have

$$\begin{aligned}&|\det(p_{l_i-j}(x))_{ij=m\cdots s-1}|\leq C_3^{s-m}(s-m)^{(s-m)/2}a_2^{-|Y|},\\ &|\det(p_{l_i'-j}(-y))_{ij=m'\cdots s-1}|\leq C_4^{s-m'}(s-m')^{(s-m')/2}b_2^{-|Y'|},\\ &\text{for any } (x, y) \text{ in domain (4.11)}.\end{aligned}\tag{4.22}$$

Now, let us go back to (4.12).

By virtue of (4.19) and (4.22) the right hand side of (4.12) has a convergent "majorant" series as follows:

$$\begin{aligned}&[\text{The r.h.s. of (4.12)}]\\ &\ll C\sum_{m,m'=-\infty}^{s}\sum^{(2)}\frac{(C^2C_3C_0)^{s-m}(C^2C_4C_0)^{s-m'}}{(s-m)!\,(s-m')!}\\ &\quad\cdot\left(1-\frac{1}{a_1b_1}\right)^{m+m'-2s}(s-m)^{(s-m)/2}(s-m')^{(s-m')/2}(a_1/a_2)^{|Y|}(b_1/b_2)^{|Y'|}\\ &\ll C\sum_{m,m'=-\infty}^{s}(s-m)^{-(s-m)/2}(s-m')^{-(s-m')/2}C_5^{s-m}C_6^{s-m'}\\ &\text{for any } (x, y) \text{ in domain (4.11)},\end{aligned}\tag{4.23}$$

where we set $C_5=C^2C_3C_0^2(1-a_1/a_2)^{-1}(1-1/(a_1b_1))^{-1}$, $C_6=C^2C_4C_0^2(1-b_1/b_2)^{-1}\cdot(1-1/(a_1b_1))^{-1}$, and the symbol $\ll$ denotes the "majorant" relation, i.e. $\sum_{m,m'=-\infty}^{s}g_{m,m'}\ll\sum_{m,m'=-\infty}^{s}h_{m,m'}$ means, by definition, $|g_{m,m'}|\leq h_{m,m'}$ for any $m, m'=s, s-1, s-2, \cdots$.

The last series in (4.23) converges for any values of C_5 and C_6. This proves Proposition 4.2. Q.E.D.

Proposition 4.2 provides a detailed formulation of Theorem 2. Thus the proof of Theorem 2 has been completed, and the passage from the finite lattice models to the infinite lattice has been accomplished at least at the level of the τ functions.

Once that passage is achieved for the τ functions, the derivation of

(1.3), (1.4) and (2.2) is a simple (but a little tedious) argument: Let us define L, M, B_μ, C_μ, $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ by (1.2), (2.1), (2.3) and (2.6), using $\tau(s, x, y)$ defined by (2.14).

First, we notice

Proposition 4.3. *The matrices, $\hat{W}^{(\infty)}_{[mn)}$, $\hat{W}^{(0)}_{[mn)}$, $L_{[mn)}$, $M_{[mn)}$, $B_{[mn)\mu}$ and $C_{[mn)\mu}$ converge "component-wise", as $-m$, $n\to\infty$, to the matrices $\hat{W}^{(\infty)}$, $\hat{W}^{(0)}$, L, M, B_μ and C_μ, respectively, where the components of the former are supposed to be arrayed in the corresponding places of the indices of rows and columns in $Z\times Z$.*

Proof. The components of $\hat{W}^{(\infty)}_{[mn)}$ and $\hat{W}^{(0)}_{[mn)}$ are expressed by formula (3.18) with $\tau=\tau_{[mn)}$. If we consider an index fixed in $Z\times Z$, the corresponding components have, for all sufficiently large values of $-m$ and n, common expressions in which the dependence on (m, n) comes only from $\tau_{[mn)}(s, x, y)$. Hence they converge, as $-m$, $n\to\infty$, to the corresponding expressions of the components of $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$ in terms of $\tau(s, x, y)$. Thus the convergence of $\hat{W}^{(\infty)}_{[mn)}$ and $\hat{W}^{(0)}_{[mn)}$ has been verified.

The convergence of their inverse matrices is an immediate consequence. For example, if $-m$ and n are sufficiently large, (4.3), (2.3) and Cramer's formula yield

$$[\text{The } (i, j) \text{ component of } \hat{W}^{(\infty)-1}_{[mn)}]=\det\left[\frac{(\hat{w}^{(\infty)}_{[mn)p-q}(p, x, y))_{\substack{p=j\cdots i-1\\ q=j\cdots i}}}{(\delta_{i,q})_{q=j\cdots i}}\right],$$

$$[\text{the } (i, j) \text{ component of } \hat{W}^{(\infty)-1}]=\det\left[\frac{(\hat{w}^{(\infty)}_{p-q}(p, x, y))_{\substack{p=j\cdots i-1\\ q=j\cdots i}}}{(\delta_{i,q})_{q=j\cdots i}}\right].$$

Hence the former converges to the latter, as $-m$, $n\to\infty$. The limit of $\hat{W}^{(0)-1}_{[mn)}$ can be dealt with in a similar way.

Finally, the convergence of the other matrices also follows immediately. For example, let us consider $L_{[mn)}$, which is given by

$$L_{[mn)}=\hat{W}^{(\infty)}_{[mn)}\Lambda_{[mn)}\hat{W}^{(\infty)-1}_{[mn)}.$$

The right hand side of this formula converges "component-wise", as $-m$, $n\to\infty$, to $\hat{W}^{(\infty)}\Lambda\hat{W}^{(\infty)-1}$, which coincides with L. Q.E.D.

By virtue of Proposition 4.3, we can derive (1.3), (1.4) and (2.2) as the limit, as $-m$, $n\to\infty$, of the corresponding objects (3.10)–(3.12) to the finite lattice model with $\Lambda=\Lambda_{[mn)}$.

Furthermore initial conditions (2.8) are satisfied, since Proposition 3.3 implies

$$\hat{W}^{(\infty)}_{[mn)}|_{x=y=0}=(a^{(\infty)}_{ij})_{ij=m\cdots n-1}, \qquad \hat{W}^{(0)}_{[mn)}|_{x=y=0}=(a^{(0)}_{ij})_{ij=m\cdots n-1}.$$

Hence (1.6) follows.

Thus we have proved Theorem 1.

Let us, finally, prove Theorem 3.

Under the conditions $[A^{(\infty)}, \Lambda^l]=0$ and $[A^{(0)}, \Lambda^l]=0$, we have

$$a^{(\infty)}_{i+l,j+l}=a^{(\infty)}_{i,j}, \quad a^{(0)}_{i+l,j+l}=a^{(0)}_{i,j}, \quad a_{i+l}=aa_i \quad \text{for any } i, j \in Z,$$

where a is a constant with $a\neq 0$. Hence (2.13) and (2.14) imply

$$\tau(s+l, x, y)=a\tau(s, x, y) \qquad \text{for any } s \in Z.$$

Then, noting (1.2), (2.1), (2.3) and (2.6), we conclude that $\hat{W}^{(\infty)}$, $\hat{W}^{(0)}$, L, M, B_n and C_n all commute with Λ^l. This proves Theorem 3.

§5. Supplementary remarks

1. A relation with the two component *KP* hierarchy.

The τ function of the Toda lattice hierarchy is closely related with the two component *KP* hierarchy, as pointed out in [5]. In [5] this fact was derived through rather indirect arguments. In this paragraph we shall describe the relation more concretely, using the expression of the τ functions presented in the preceding sections.

Let us introduce the following matrices consisting of four blocks,

$$(5.1) \qquad f(s)=\begin{bmatrix} (f^{(1)}_{ij}(s))_{\substack{i\in Z\\ j<s}} & (f^{(2)}_{ij}(s))_{\substack{i\in Z\\ j<-s}} \\ (f^{(3)}_{ij}(s))_{\substack{i\in Z\\ j<s}} & (f^{(4)}_{ij}(s))_{\substack{i\in Z\\ j<-s}} \end{bmatrix}=\begin{bmatrix} (\tilde{a}^{(\infty)}_{ij})_{\substack{i\in Z\\ j<s}} & (\tilde{a}^{(\infty)}_{ij})_{\substack{i\in Z\\ j<-s}} \\ (\tilde{a}^{(0)}_{ij})_{\substack{i\in Z\\ j<s}} & (\tilde{a}^{(0)}_{ij})_{\substack{i\in Z\\ j<-s}} \end{bmatrix},$$

$$(5.2) \qquad f_{[mn)}(s)=\begin{bmatrix} (f^{(1)}_{ij}(s))_{\substack{i=m\cdots n-1\\ j=m\cdots s-1}} & (f^{(2)}_{ij}(s))_{\substack{i=m\cdots n-1\\ j=-n\cdots -s-1}} \\ (f^{(3)}_{ij}(s))_{\substack{i=-n\cdots -m-1\\ j=m\cdots s-1}} & (f^{(4)}_{ij}(s))_{\substack{i=-n\cdots -m-1\\ j=-n\cdots -s-1}} \end{bmatrix}, \quad m<n,$$

where $(\tilde{a}^{(\infty)}_{ij})_{ij\in Z}=A^{(\infty)-1}$ and $(\tilde{a}^{(0)}_{ij})_{ij\in Z}=A^{(0)-1}$. $f(s)$ and $f_{[mn)}(s)$ serve as "frames" representing the points of the "two component" version of the Grassmann manifold [6]. Their "Plücker coodinates" are defined, according to [6, 7], by

$$(5.3) \qquad f_{[mn)}(s)_{Y,Y'}=\det\begin{bmatrix} (f^{(1)}_{l_ij}(s))_{ij=m\cdots s-1} & (f^{(2)}_{l_ij}(s))_{\substack{i=m\cdots s-1\\ j=-n\cdots -s-1}} \\ (f^{(3)}_{l'_ij}(s))_{\substack{i=-n\cdots -s-1\\ j=m\cdots s-1}} & (f^{(4)}_{l'_ij}(s))_{ij=-n\cdots -s-1} \end{bmatrix}$$

with $(Ys)=(l_m, \cdots, l_{s-1})$, $(Y's)=(l'_{-n}, \cdots, l'_{-s-1})$, $Y\subset \underset{n-s}{\square}^{s-m}$ and $Y'\subset \underset{s-m}{\square}^{n-s}$, and by

$$(5.4) \qquad f(s)_{Y,Y'}=\lim_{-m,n\to\infty} a_n f_{[mn)}(s)_{Y,Y'}.$$

The right hand side of (5.4) makes sense as a stable limit like that of (2.13), i.e. $a_n f_{[mn)}(s)_{Y,Y'}$ is independent of m and n as far as $Y\subset \underset{n-s}{\square}^{s-m}$ and $Y'\subset \underset{s-m}{\square}^{n-s}$.

Some calculation of linear algebra leads us to

$$(5.5) \qquad A_{[mn)(Ys)(Y's)}=(-)^{|Y'|}(a_n/a_m)f_{[mn)}(s)_{Y,{}^tY'},$$

where ${}^tY'$ denotes the transposition of Y', e.g. ${}^t\square=\square$, ${}^t\square\square=\begin{smallmatrix}\square\\ \square\end{smallmatrix}$, ${}^t\begin{smallmatrix}\square\\ \square\end{smallmatrix}=\square\square$, ${}^t\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}=\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$, etc. $\cdots$, and the other notations are the same as those in Section 4.

Hence noting the formula

$$(5.6) \qquad \chi_{Y'}(-y)=(-)^{|Y'|}\chi_{{}^tY'}(y),$$

we obtain

$$(5.7) \qquad \tau_{[mn)}(s, x, y)=(a_n/a_m)\sum{}^{(s)}\chi_Y(x)\chi_{Y'}(y)f_{[mn)}(s)_{Y,Y'},$$

$$(5.8) \qquad \tau(s, x, y)=\sum_{Y,Y'}\chi_Y(x)\chi_{Y'}(y)f(s)_{Y,Y'},$$

where $\sum^{(s)}$ stands for the summation over the totality of pairs (Y, Y') with $Y\subset \underset{n-s}{\square}^{s-m}$ and $Y'\subset \underset{s-m}{\square}^{n-s}$. The right hand side of these formulas have the same form as τ functions of the "two component" *KP* hierarchy [6–8]. Thus we conclude that the τ functions of our hierarchy—both the finite lattice model and the infinite lattice—have the structure of the τ functions of the two component *KP* hierarchy.

2. The infinite lattice with a free end.

If we consider the case when $n=\infty$ and $-\infty<m<\infty$ for the model discussed in Section 3, an infinite lattice model with a free end is derived. The machinery of Section 3 works the same way for this model.

The τ function is then given by the formula

$$(5.9) \qquad \tau_{[m\infty)}(s, x, y)=\det(a_{ij}(x, y))_{ij=m\cdots s-1},$$

where

$$(5.10)\quad \begin{aligned} &(a_{ij}(x,y))_{ij=m,m+1,\dots}=\exp\Big[\sum_{\mu=1}^{\infty} x_\mu \Lambda^{\mu}_{[m\infty)}\Big] A \exp\Big[-\sum_{\mu=1}^{\infty} y_\mu {}^t\Lambda^{\mu}_{[m\infty)}\Big],\\ &A=(a_{ij})_{ij=m,m+1,\dots},\qquad \Lambda_{[m\infty)}=(\delta_{i,j-1})_{ij=m,m+1,\dots}. \end{aligned}$$

It is remarkable that the determinant in (5.9) has the following Wronskian structure with respect to both rows and columns,

$$(5.10)\qquad \tau_{[m\infty)}(s,x,y)=\det(\partial_{x_1}^{i-m}\partial_{y_1}^{j-m}a_{mm}(x,y))_{ij=m,m+1,\dots,s-1},$$

since we have the formulas

$$(5.11)\qquad \partial_{x_\mu}a_{i,j}(x,y)=a_{i+\mu,j}(x,y),\qquad \partial_{y_\mu}a_{i,j}(x,y)=a_{i,j+\mu}(x,y)$$

which follow immediately from (5.10). Such an expression of the τ functions was already presented in [2], though the main interest of [2] seems to be in how to extract the finite Toda lattices from the above infinite lattice. It could be said that, in contrast to [2], our main subject throughout the discussion in the preceding sections consisted in how to achieve the limit of (5.9) as $m\to-\infty$. Actually such a limit was realized by introducing the factor a_m:

$$(5.12)\qquad \tau(s,x,y)=\lim_{m\to-\infty} a_m\tau_{[m\infty)}(s,x,y).\qquad \text{(Cf. (4.9).)}$$

3. Formal power series and holomorphic functions of infinitely many independent variables.

In this paper we have been working with the hierarchy. There appeared infinitely many independent variables $x_1, x_2, \cdots, y_1, y_2, \cdots$, which entered into, for example, formula (2.14) of the τ function.

There are two frameworks in which such objects are dealt with—the "formal" one and the "analytic (or holomorphic)" one.

The "formal" framework is based on the ring

$$(5.13)\qquad \boldsymbol{C}[\![x,y]\!]=\Big\{\sum_{j,k=0}^{\infty} g_{jk}(x,y);\ g_{jk}(x,y)=\sum_{\substack{\|\alpha\|=j\\ \|\beta\|=k}} g^{(\alpha\beta)}x^\alpha y^\beta,\ g^{(\alpha\beta)}\in\boldsymbol{C}\Big\},$$

of formal series of weighted homogeneous polynomials with weight $(x_\mu)=(\mu,0)$, weight $(y_\mu)=(0,\mu)$. Here we used the multi-index notations

$$x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots,\quad y^\beta=y_1^{\beta_1}y_2^{\beta_2}\cdots,\quad \|\alpha\|=\sum_{\mu=1}^{\infty}\mu\alpha_\mu,\quad \|\beta\|=\sum_{\mu=1}^{\infty}\mu\beta_\mu,$$

for $\alpha=(\alpha_1,\alpha_2,\cdots)$ and $\beta=(\beta_1,\beta_2,\cdots)$ with $\alpha_\mu=\beta_\mu=0$ for any sufficiently large value of μ. The product and the sum of any two series are defined by

$$(5.14)\quad \begin{aligned} &\sum_{j,k} g_{jk}+\sum_{j,k} h_{jk}=\sum_{j,k}(g_{jk}+h_{jk}),\qquad \sum_{j,k} g_{jk}\cdot\sum_{j,k} h_{jk}=\sum_{j,k} e_{jk}\\ &\text{with}\quad e_{jk}=\sum_{\substack{j'+j''=j\\ k'+k''=k}} g_{j'k'}h_{j''k''}. \end{aligned}$$

$\tau(s,x,y)$, defined by (2.14), is an element of $\boldsymbol{C}[\![x,y]\!]$, since $\chi_Y(x)$ and $\chi_{Y'}(-y)$ are weighted homogeneous polynomials of weight $(|Y|,0)$ and $(0,|Y'|)$, respectively.

The "analytic (or holomorphic)" framework is based on the following subring of $\boldsymbol{C}[\![x,y]\!]$,

$$(5.15)\quad \begin{aligned} \boldsymbol{C}\{x,y\}=\Big\{&\sum_{j,k=0}^{\infty} g_{jk}(x,y);\ g_{jk}(x,y)=\sum_{\|\alpha\|=j,\|\beta\|=k} g^{(\alpha\beta)}x^\alpha y^\beta,\\ &\exists a>0,\ \exists b>0,\ \exists C>0,\ \forall\alpha,\ |g^{(\alpha\beta)}|\le Ca^{-\|\alpha\|}b^{-\|\beta\|}\Big\}. \end{aligned}$$

Any element $g(x,y)=\sum_{j,k} g_{j,k}(x,y)$ of $\boldsymbol{C}\{x,y\}$ has a convergent "majorant" series

$$(5.16)\qquad \sum_{j,k=0}^{\infty}\sum_{\substack{\|\alpha\|=j\\ \|\beta\|=k}} Ca^{-\|\alpha\|}b^{-\|\beta\|}x^\alpha y^\beta=C\Big/\prod_{\mu=1}^{\infty}(1-a^{-\mu}x_\mu)(1-b^{-\mu}y_\mu),$$

where C, a and b are some constants depending on $g(x,y)$ as appeared in (5.15). Hence $g(x,y)$ converges absolutely and uniformly in the domain $\{(x,y)\in\boldsymbol{C}^\infty\times\boldsymbol{C}^\infty;|x_\mu|\le a'^\mu,|y_\mu|\le b'^\mu,\mu=1,2,\cdots\}$ for any positive constants a' and b' with $a'<a$ and $b'<b$. Thus $g(x,y)$ defines a function in the domain $\{(x,y)\in\boldsymbol{C}^\infty\times\boldsymbol{C}^\infty;|x_\mu|<a^\mu,|y_\mu|<b^\mu,\mu=1,2,\cdots\}$, bounded in the above subdomain for any a' and b' with $a'<a$, $b'<b$, and holomorphic with respect to each of x_μ, y_μ, $\mu=1,2,\cdots$, separately.

Conversely, any function $g(x,y)$ with these properties is identified with an element of $\boldsymbol{C}\{x,y\}$, i.e. its "Taylor expansion"

$$\sum_{j,k=1}^{\infty}\sum_{\|\alpha\|=j,\|\beta\|=k} g^{(\alpha\beta)}x^\alpha y^\beta$$

defined by

$$(5.17)\qquad g^{(\alpha\beta)}=(\alpha!\,\beta!)^{-1}\partial_x^\alpha\partial_y^\beta g(x,y)|_{x=y=0},$$

with $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots$, $\alpha!=\alpha_1!\,\alpha_2!\cdots$, etc.. Note that the usual Cauchy inequality can be used to estimate $g^{(\alpha\beta)}$, since the derivative in (5.17) is related with only a finite number of variables.

Thus we have arrived at the concept of holomorphic functions defined in a "neighborhood" of the origin of $\boldsymbol{C}^\infty\times\boldsymbol{C}^\infty$, where the fundamental neighborhood system of the origin is, by definition, given by the domains

$\{(x, y) \in C^\infty \times C^\infty; |x_\mu| < a^\mu, |y_\mu| < b^\mu, \mu = 1, 2, \cdots\}$, $a > 0$, $b > 0$.

The above consideration is similarly applied to any point $(\dot{x}, \dot{y}) \in C^\infty \times C^\infty$. Let us define a uniform topology of $C^\infty \times C^\infty$, attaching the parallel transform of the fundamental neighborhood system of the origin to each point $(\dot{x}, \dot{y})$ of $C^\infty \times C^\infty$. Then the concept of analytic continuation makes sense, since any holomorphic function defined in a neighborhood of a point in $C^\infty \times C^\infty$ has its unique Taylor expansion at any point of the domain where it is defined. Thus the concept of holomorphic functions can be established in $C^\infty \times C^\infty$.

$\tau(s, x, y)$, defined by (2.14), is a holomorphic function in this sense (cf. Theorem 2 and Proposition 4.2).

4. The hierarchies of B type, C type and the multi-component hierarchy (cf. [5]).

In [5] three types of versions of the Toda lattice hierarchy mentioned in Section 1 were discussed. The method of the present paper can be also applied to them with some slight modification.

In order to deal with the hierarchies of B and C types, we have only to impose the additional conditions

$$A^{(\infty)}, A^{(0)} \in Sp(\infty) \ (B \text{ type}), \quad A^{(\infty)}, A^{(0)} \in O(\infty) \ (C \text{ type}).$$

(As for the definition of $Sp(\infty)$ and $O(\infty)$, see [5]).

The initial value problem for the multi-component Toda lattice hierarchy is solved the same way as the argument of the preceding sections, though we need more complicated discussions: We must replace each component of $A^{(\infty)}$, $A^{(0)}$, etc.$\cdots$ by the corresponding matrix of size $r \times r$, if we consider the r-component hierarchy. The factor a_i is then introduced by

$$\det(a_{ii}^{(0)}) = a_{i+1}/a_i, \qquad i \in Z,$$

where $a_{ii}^{(0)}$ is the (i, i) block of $A^{(0)} = (a_{ij}^{(0)})_{i,j \in Z}$ with $a_{ij}^{(0)}$ being a matrix of size $r \times r$. Furthermore, in order to describe the solution, we need several τ functions, just as we need for the multi-component KP hierarchy [7, 8]. Their expressions like (2.14) become more complicated.

References

[1] B. Kostant, The solution to the generalized Toda lattice and representation theory, Adv. in Math., **34** (1979), 155–338.

[2] A. N. Leznov and M. V. Saveliev, Representation of zero curvature for the system of non-linear partial differential equations $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ and its integrability, Lett. Math. Phys., **3** (1979), 498–494.

A. N. Leznov, On the complete integrability of a nonlinear system of partial differential equations in two-dimensional space, Theoret. and Math. Phys., **42** (1980), 225–229.

A. N. Leznov, M. V. Saveliev and V. G. Smirnov, Theory of group representation and integration of nonlinear dynamical systems, Theoret. and Math. Phys., **48** (1982), 565–571.

[3] M. A. Olshanetsky and A. M. Perelomov, Explicit solutions of classical generalized Toda models, Invent. Math., **54** (1979), 261–269.

[4] A. G. Reyman and M. A. Semenov-Tian-Shansky, Reduction of Hamiltonian systems, affine Lie algebras and Lax equation, Invent. Math., **54** (1979), 81–100; ditto. II, ibid., **63** (1981), 423–432.

[5] K. Ueno and K. Takasaki, Toda lattice hierarchy, in this volume, 1–95.

[6] M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, RIMS Kôkyûroku, **439** (1981), 30–46, Kyoto Univ.

[7] ——, Lectures delivered at the University of Tokyo, 1981–1983.

[8] M. Kashiwara and T. Miwa, Transformation groups for soliton equations I, Proc. Japan Acad., **57A** (1981), 342–347.

E. Date, M. Jimbo, M. Kashiwara and T. Miwa, ditto. III, J. Phys. Soc. Japan, **40** (1981), 3806–3812.

[9] F. R. Gantmacher, The theory of matrices, Chelsea, 1959.

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