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Area-Preserving Diffeomorphisms and Nonlinear Integrable Systems

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Abstract. Present state of the study of nonlinear "integrable" systems related to the group of area-preserving diffeomorphisms on various surfaces is overviewed. Roles of area-preserving diffeomorphisms in 4-d self-dual gravity are reviewed. Recent progress in new members of this family, the SDiff(2) KP and Toda hierarchies, is reported. The group of area-preserving diffeomorphisms on a cylinder plays a key role just as the infinite matrix group GL(∞) does in the ordinary KP and Toda lattice hierarchies. The notion of tau functions is also shown to persist in these hierarchies, and gives rise to a central extension of the corresponding Lie algebra.
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1. Introduction

The groups of area-preserving diffeomorphisms on various surfaces, which we call SDiff(2) rather symbolically, give a natural extension of the notion the group of circle diffeomorphisms Diff(S^1). This type of groups are known to emerge in a wide area of theoretical and mathematical physics ranging from fluid mechanics[1] and dynamical systems[2] to topics of high energy physics such as W_∞ algebras.[3]

The notion of W_∞ algebra is also intriguing from the standpoint of the study of nonlinear integrable systems. We already know that self-dual gravity is characterized by an underlying SDiff(2) group structure. This should be by no means an isolated example; looking for this type of nonlinear "integrable" systems (the "SDiff(2) family," so to speak) is a challenging issue. We expect to find some other examples in a variety of models of field theories related to W_∞ algebras (or, more precisely, their"quasi-classical" counterpart, i.e., w_∞ algebras) such as: w_∞-gravity,[4] large-N limit of nonlinear sigma models,[5] $N = 2$ strings,[6] self-dual quantum gravity,[7] etc.

It is now widely recognized that various infinite dimensional Lie algebras (and associated groups) play a key role in understanding nonlinear integrables systems.[8] This fact has been well established for equations of KdV type; associated Lie algebras are Kac-Moody algebras. The KP hierarchy as well as its Toda lattice version (Toda lattice hierarchy) are known to be characterized by the gl(∞) algebra of $\infty \times \infty$ matrices. These equations are called "soliton equations."

Similar structures have been observed for the 4-d self-dual Yang-Mills equation and their dimensional reductions, the Bogomolny equation (3-d), the principal chiral models and the Ernst equation. These gauge field equations, too, are shown to have underlying infinite dimensional algebras similar to Kac-Moody algebras.[9] (In fact, it is natural to enlarge these algebras by adding derivation operators.[10,11]) The situation is, however, somewhat different from soliton equations; more than one variables are allowed to take place in its loop algebra structure. In 4-d, there are indeed two extra variables along with a "spectral parameter" that also arises in soliton equations. These variables may be thought of as local coordinates of a three-dimensional complex manifold called "twistor space."[12] In 3-d reductions, the twistor space is reduced to a two-dimensional complex manifold called a "minitwistor space"; this is the stage where the Bogomolny equation is treated as a preliminary step towards the principal chiral models and the Ernst equation.[13] In 2-d reductions, one will have a Riemann surface (mostly, a Riemann sphere) that should be called a "miniminitwistor space"; soliton equations of the KdV type are shown to fall into this class.[14]

The situation further drastically changes in the case of self-dual gravity (the vacuum Einstein equation). The role of algebras of Kac-Moody type (with extra variables, if necessary) is now played by algebras of diffeomorphisms. According to the nonlinear graviton construction of Penrose,[15] any self-dual vacuum Einstein space-time has a one-to-one correspondence with a three dimensional complex manifold ("curved twistor space"). This curved twistor space has a projection (fibering) over a Riemann sphere, and each fiber is given a symplectic structure that deforms as the point of the Riemann sphere moves. This is exactly the place where a group of area-preserving diffeomorphisms emerges; those area-preserving diffeomorphisms may also depend on a parameter λ that takes values in the Riemann sphere. It is indeed shown that a loop algebra of SDiff(2), like Kac-Moody-like algebras, gives rise to deformations of curved twistor spaces, hence symmetries on the space of solutions of self-dual gravity.[16,17] We shall first review these stories.

One may naturally ask if the pattern of dimensional reductions in gauge field equations persists in this case. In 3-d, this is indeed the case as pointed out by Park.[5] Self-dual gravity has a 3-d reduction that allows SDiff(2) on a cylinder to act as symmetries. This significant observation of Park has been extended to an SDiff(2) version of both the KP hierarchy and the Toda lattice hierarchy, and yielded several remarkable results. These results will be reported in the latter half of this article. It seems likely that 2-d reductions of these equations will fall into special (solvable) cases of the classical theory of 2-d Monge-Ampère equation, hence less exciting if compared with diverse 2-d reductions of the self-dual Yang-Mills equation. Nevertheless they will still have rich contents in the context of w_∞-gravity.[18]

2. Self-dual gravity and nonlinear graviton construction

We start with a brief review of the situation in 4-d self-dual gravity. In 4-d, any vacuum Einstein metric has (locally) a complex Kähler structure, and becomes a hyper-Kähler manifold. In particular, there are three independent Kähler structures and they can be mixed by the unit quaternion group SU(2).[19] This family of Kähler structures and associated Kähler forms play a key role in the nonlinear graviton construction of Penrose.[15]

A more down-to-earth formulation of this basic observation is presented by Plebanski[20] and summarized in a review by Boyer.[21] (Essentially the same formulation is independently discovered by Gindikin[22] from the standpoint of integral geometry.) In the formulation of Plebanski (as well as of Penrose), one starts from a (complexified) metric of the form

$$ds^2 = \det \begin{pmatrix} e^{11} & e^{12} \\ e^{21} & e^{22} \end{pmatrix} = e^{11}e^{22} - e^{12}e^{21} \tag{1}$$

where e^{ab} are linearly independent differential 1-forms. This induces new local SL(2)×SL(2) gauge symmetries that act on both sides of the above 2×2 matrix of 1-forms. With this gauge freedom, one can reduce the Ricci-flatness condition to the closedness

$$d\omega^{cd} = 0 \tag{2}$$

of the exterior 2-forms

$$\omega^{cd} = \omega^{dc} = \frac{1}{2} \in_{ab} e^{ac} \wedge e^{bd} \tag{3}$$

where $\in_{ab}$ is the symplectic form normalized as

$$(\in_{ab}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

and the Einstein summation convention is applied to the symplectic indices $a, b, \ldots$.

These equations can be cast into a single equation if one introduces a complex parameter λ and makes the linear combination

$$\omega(\lambda) = \omega^{11} + 2\omega^{12}\lambda + \omega^{22}\lambda^2. \tag{5}$$

This is a realization of the aforementioned SU(2) mixing of different Kähler forms, and λ plays the role of a mixing parameter ranging over a Riemann sphere (which is locally diffeomorphic to the group manifold of SU(2)). Note that $\omega(\lambda)$ can also be written

$$\omega(\lambda) = \frac{1}{2} \in_{ab} (e^{a1} + e^{a2}\lambda) \wedge (e^{b1} + e^{b2}\lambda), \tag{7}$$

hence

$$\omega(\lambda) \wedge \omega(\lambda) = 0. \tag{8}$$

In terms of $\omega(\lambda)$, the three closedness equations can be rewritten

$$d'\omega(\lambda) = 0, \tag{9}$$

where "d'" means the total differentiation only with respect to space-time coordinates; λ is considered a constant as $d'\lambda = 0$. Eqs. 7 and 8 mean that there are a pair of "Darboux coordinates" $P(\lambda)$ and $Q(\lambda)$ such that

$$\omega(\lambda) = d'P \wedge d'Q. \tag{10}$$

Of course these Darboux coordinates are by no means unique; further, they exist only locally, in particular with respect to λ. As Penrose[15] first pointed out (in the form of an inverse construction), these Darboux coordinates may be thought of as a local expression of sections of the fibering

$$\pi : \ \mathcal{T} \to \mathbf{P}^1 \tag{11}$$

where $\mathcal{T}$ is the corresponding curved twistor space.

To see this correspondence more explicitly,[16,21] let us assume that there are two sets of Darboux coordinates $(u(\lambda), v(\lambda))$ and $(\hat{u}(\lambda), \hat{v}(\lambda))$ with the following Laurent expansion in a neighborhood of a circle $|\lambda| = \rho$ on the complex λ plane.

$$u(\lambda) = p\lambda + x + \sum_{n\le -1} u_n\lambda^n, \quad v(\lambda) = q\lambda + y + \sum_{n\le -1} v_n\lambda^n,$$
$$\hat{u}(\lambda) = \hat{p} + \hat{x}\lambda + \sum_{n\ge 2} \hat{u}_n\lambda^n, \quad \hat{v}(\lambda) = \hat{q} + \hat{y}\lambda + \sum_{n\ge 2} \hat{v}_n\lambda^n. \tag{12}$$

[The first two Laurent coefficients of these four functions are given special status, because they are exactly the space-time coordinates that arise in Plebanski's "heavenly equations."[20] There are three different choices, $(p, q, \hat{p}, \hat{q})$, (x, y, p, q) and $(\hat{x}, \hat{y}, \hat{p}, \hat{q})$, that fit into the first heavenly equation, the second heavenly equation, and the "dual" second heavenly equation, respectively.] The above situation is indeed the case illustrated by Penrose.[15] Note that due to the special form of the Laurent series, $u(\lambda)$ and $v(\lambda)$ can be analytically continued to the outside of the circle and have first order poles at $\lambda = \infty$, whereas $\hat{u}(\lambda)$ and $\hat{v}(\lambda)$ to the inside. Meanwhile, these two pairs of Darboux coordinates should be related by a canonical (i.e., area-preserving) diffeomorphism as

$$f(\lambda, u(\lambda), v(\lambda)) = \hat{u}(\lambda), \quad g(\lambda, u(\lambda), v(\lambda)) = \hat{v}(\lambda), \tag{13}$$

where f and g are holomorphic functions of three variables, say (λ, u, v), and satisfy the area-preserving condition

$$\frac{\partial f(\lambda, u, v)}{\partial u} \frac{\partial g(\lambda, u, v)}{\partial v} - \frac{\partial f(\lambda, u, v)}{\partial v} \frac{\partial g(\lambda, u, v)}{\partial u} = 1. \tag{14}$$

Geometrically, the pair of functions f and g (after appropriate "twisting" by λ) give patching functions of local coordinates on the twistor space $\mathcal{T}$, and the given Ricci-flat Kähler metric is (at least locally) now encoded into this data. Penrose[15] further argues that this is a one-to-one correspondence, ensuring the converse construction with the aid of the Kodaira-Spencer deformation theory of complex manifolds. Analytically, this amounts to a kind of "Riemann-Hilbert problem" now setup in the group of area-preserving diffeomorphisms.[14,19] (Actually, here is a technical subtlety that should be taken into account to establish a true one-to-one correspondence; let us postpone it to the next section.) The variable λ now plays the role of a loop parameter, hence the fundamental group structure is not of SDiff(2) but of the loop group of SDiff(2) (on a plane).

3. Origin of SDiff(2) symmetries

SDiff(2) symmetries originate in the group structure of the data (f, g) (due to composition of diffeomorphisms). More precisely, one starts from left and right translations on the SDiff(2) group manifold of the form

$$(f, g) \longrightarrow \exp\left(\epsilon\{\hat{F}, \cdot\}\right) \circ (f, g) \circ \exp\left(-\epsilon\{F, \cdot\}\right) \tag{15}$$

generated by the Hamiltonian vector fields

$$\{F, \cdot\} = \frac{\partial F}{\partial u}\frac{\partial}{\partial v} - \frac{\partial F}{\partial v}\frac{\partial}{\partial u}, \quad \{\hat{F}, \cdot\} = \frac{\partial \hat{F}}{\partial \hat{u}}\frac{\partial}{\partial \hat{v}} - \frac{\partial \hat{F}}{\partial \hat{v}}\frac{\partial}{\partial \hat{u}}, \tag{16}$$

where $F = F(\lambda, u, v)$ and $\hat{F} = \hat{F}(\lambda, \hat{u}, \hat{v})$ are functions of three variables and assumed to have the same analyticity properties as f and g. This should give rise to a one-parameter family of transformations for solutions of the vacuum Einstein equation via the fundamental relation

$$\omega(\lambda) = d'u(\lambda) \wedge d'v(\lambda) = d'\hat{u}(\lambda) \wedge d'\hat{v}(\lambda). \tag{17}$$

It is now convenient (and even essential) to understand the self-dual vacuum Einstein equation as an enlarged system[23] with auxiliary unknown functions $u(\lambda)$, $v(\lambda)$, $\hat{u}(\lambda)$, $\hat{v}(\lambda)$ obeying the above equations. SDiff(2) symmetries, in fact, act on this system rather than the original self-dual vacuum Einstein equation.

An infinitesimal form of these SDiff(2) symmetries will be obtained by calculating the transformations to the first order of ϵ as

$$\begin{aligned} u &\longrightarrow u + \epsilon\delta u + O(\epsilon^2), \quad v \longrightarrow v + \epsilon\delta v + O(\epsilon^2), \\ \hat{u} &\longrightarrow \hat{u} + \epsilon\delta\hat{u} + O(\epsilon^2), \quad \hat{v} \longrightarrow \hat{v} + \epsilon\delta\hat{v} + O(\epsilon^2). \end{aligned} \tag{18}$$

The coefficients of ϵ then defines a linear map $\delta = \delta_{F,\hat{F}}$ that represents an infinitesimal SDiff(2) symmetries. We now have to mention a technical remark announced in the last section: The transformations of the Darboux coordinates are to be determined only after selecting a reference space-time coordinate system. Such a choice of coordinates remains arbitrary in the nonlinear graviton construction and gives residual "gauge" freedom; we have to fix it. Our prescription of this "gauge fixing" is to select one of the aforementioned Plebanski coordinate systems and to require that the transformations of $(u, v, \hat{u}, \hat{v})$ leave invariant these coordinates, i.e.,

$$\delta_{F,\hat{F}} z = 0 \quad \text{for all } z \text{ in reference coordinate system.} \tag{19}$$

If, for example, (x, y, p, q) are adopted as a coordinate system, we have[17]:

$$\delta_{F,\hat{F}} w(\lambda) = \Big\{ F(\lambda, u(\lambda), v(\lambda))_{\le -1} - \hat{F}(\lambda, \hat{u}(\lambda), \hat{v}(\lambda))_{\le -1}, w(\lambda) \Big\}_{x,y} \tag{20}$$

for $w(\lambda) = u(\lambda), v(\lambda)$ and

$$\delta_{F,\hat{F}} \hat{w}(\lambda) = \Big\{ \hat{F}(\lambda, \hat{u}(\lambda), \hat{v}(\lambda))_{\ge 0}, -F(\lambda, u(\lambda), v(\lambda))_{\ge 0}, \hat{w}(\lambda) \Big\}_{x,y} \tag{21}$$

for $\hat{w}(\lambda) = \hat{u}(\lambda), \hat{v}(\lambda)$, where $(\quad)_{\ge 0}$ and $(\quad)_{\le -1}$ stand for the projection operators acting on Laurent series of λ as

$$\Big(\sum a_n \lambda^n\Big)_{\ge 0} = \sum_{n \ge 0} a_n \lambda^n, \quad \Big(\sum a_n \lambda^n\Big)_{\le -1} = \sum_{n \le -1} a_n \lambda^n, \tag{22}$$

and $\{\quad,\quad\}_{x,y}$ a Poisson bracket in (x, y),

$$\{F, G\}_{x,y} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}. \tag{23}$$

In general, these symmetries satisfy the commutation relations

$$\left[\delta_{F_1,\hat{F}_1},\delta_{F_2,\hat{F}_2}\right] = \delta_{\{F_1,F_2\},\{\hat{F}_1,\hat{F}_2\}}, \tag{24}$$

thereby respect the Lie algebraic structure of SDiff(2). It is shown[17] that these symmetries can be further extended to the Plebanski "key functions," and shown to obey the same commutation relations. This is far from obvious because these key functions are obtained as "potentials," hence determined only up to an integration constant.

It should be noted that the above construction is still mathematically ambiguous; we have taken a circle $|\lambda| = \rho$ in an ad hoc way, and assumed a priori that two distinct Darboux coordinates should live on each side of this circle. This is obviously not very elegant. A more elegant formulation will be achieved in terms of sheaf cohomology as Park[10] pointed out.

4. SDiff(2) KP hierarchy

The SDiff(2) version of the KP hierarchy that we now consider is due to Krichever.[24] Instead of a pseudo-differential operator L in the KP hierarchy, one starts from a Laurent series $\mathcal{L}$ of λ,

$$\mathcal{L} = \lambda + \sum_{i=1}^{\infty} u_{i+1}\lambda^{-i}, \tag{25}$$

where the Laurent coefficients are unknown functions of an infinite number of variables $t = (t_1, t_2, \ldots)$, $t_1 = x$. The first variable t_1 is identified with a 1-d space variable x as in the ordinary KP hierarchy. Krichever's hierarchy consists of the evolution equations ("dispersionless Lax equations")

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}_{\lambda,x}, \quad n = 1, 2, \ldots, \tag{26}$$

where $\mathcal{B}_n$ are polynomials of λ given by

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \tag{27}$$

$(\quad)_{\geq 0}$, as already mentioned, stands for the polynomial part of a Laurent series of λ; $\{\quad,\quad\}_{\lambda,x}$ now denotes a Poisson bracket with respect to (λ, x),

$$\{F, G\}_{\lambda,x} = \frac{\partial F}{\partial \lambda}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial \lambda}. \tag{28}$$

Obviously, this hierarchy is a kind of "quasi-classical" version to be obtained from the ordinary KP hierarchy by replacing

$$\begin{aligned} \partial/\partial x &\longrightarrow \lambda, \\ [\quad,\quad] &\longrightarrow \{\quad,\quad\}_{\lambda,x}. \end{aligned} \tag{29}$$

The term "SDiff(2)" now refers to the group of symplectic diffeomorphisms that leave invariant the above Poisson bracket. This is the group of area-preserving diffeomorphisms on a cylinder on which $(\arg\lambda, x)$ give a coordinate system. Krichever seems to have been led to this hierarchy from two routes: One is the route from the study of "hydrodynamic" Hamiltonian structures and the "averaging method" in soliton theory.[25] The other is the route from exact solutions of "topological minimal models."[26] The above hierarchy contains the so called "dispersionless KP equation" in the 3-d sector $(x, y, t) = (t_1, t_2, t_3)$,[25] which is the same as the ordinary KP (2-d KdV) equation except that a dispersion term is dropped.

Our approach to this new hierarchy[27] is more close to the work of the Kyoto group.[28] A goal is to reorganize everything in terms of a "tau function" and an underlying Lie algebra. To this end, however, we have to borrow crucial ideas from the method developed for the self-dual vacuum Einstein equations. We first look for a Kähler-like 2-form; then introduce a pair of "Darboux coordinates"; this leads to a twistor theoretical description (i.e., a nonlinear graviton construction) of general solutions; infinitesimal variations of "patching functions" (which give a SDiff(2) group element) give rise to infinitesimal symmetries of the hierarchy. The use of a pair of "Darboux coordinates," one of which is the above $\mathcal{L}$ itself and the other is written $\mathcal{M}$ below, is a characteristic of our approach. Our definition of the tau function is based upon these two functions.

It deserves to be mentioned that the ordinary KP hierarchy, too, has a counterpart of $\mathcal{M}$ that plays the role of a second Lax operator. Such an improved Lax formalism of the KP hierarchy is introduced by Orlov[29] and later applied to $d = 1$ string theory by Awada and Sin.[30] This seems to suggest a possibility to interpret the well known relation of the KP hierarchy and an infinite dimensional Grassmannian manifold[28,31] as a kind of *noncommutative (mini)twistor theory.*

The rest of this section is devoted to a more detailed account of our approach to the SDiff(2) KP hierarchy. In our definition, the hierarchy consists of the Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}_{\lambda,x}, \quad \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}_{\lambda,x} \tag{30}$$

and the canonical Poisson relation

$$\{\mathcal{L}, \mathcal{M}\}_{\lambda,x} = 1, \tag{31}$$

where $\mathcal{L}$ is the same as explained above, and $\mathcal{M}$ is a Laurent series (now expanded in powers of $\mathcal{L}$ rather than λ for technical reasons) of the form

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + \sum_{i=1}^{\infty} v_{i+1} \mathcal{L}^{-i-1}. \tag{32}$$

Thus there are two series of unknown functions, u_i and v_i. Note that we have excluded u_1 and v_1. In fact, one may include these terms and consider the same equations; u_1 and v_1 then turn out to be constants (independent of t), thereby can be absorbed into redefinition of λ and $\mathcal{M}$ as

$$\lambda \to \lambda - u_1, \quad \mathcal{M} \to \mathcal{M} - v_1 \mathcal{L}^{-1}. \tag{33}$$

One can therefore put $u_1 = 0$ and $v_1 = 0$ in the beginning. It is however sometimes convenient to retain v_1 as a free parameter.

We then introduce a Kähler-like 2-form as

$$\omega = \sum_{n=1}^{\infty} d\mathcal{B}_n \wedge dt_n = d\lambda \wedge dx + \sum_{n=2}^{\infty} d\mathcal{B}_n \wedge dt_n, \tag{34}$$

where "d" now stands for total differentiation with respect to *both* t *and* λ. From the definition, ω is a closed form,

$$d\omega = 0. \tag{35}$$

Meanwhile, as in the case of the ordinary KP hierarchy,[28] the Lax equations for $\mathcal{L}$ turn out to be equivalent to the "zero-curvature equations"

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + \{B_m, B_n\}_{\lambda,x} = 0, \tag{36}$$

and these zero-curvature equations can be cast into the algebraic relation

$$\omega \wedge \omega = 0. \tag{37}$$

Eqs. 35 and 37 ensure the existence of two Darboux coordinates. The fact is that $\mathcal{L}$ and $\mathcal{M}$ are such Darboux coordinates, i.e., they satisfy the fundamental equation

$$\omega = d\mathcal{L} \wedge d\mathcal{M}, \tag{38}$$

and, actually, this equation is an equivalent expression of Eqs. 30 and 31.

Once we have arrived the above situation, it is very natural to take another pair of Darboux coordinates $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ such that

$$\omega = d\hat{\mathcal{L}} \wedge d\hat{\mathcal{M}} \tag{39}$$

but with different Laurent expansion as

$$\hat{\mathcal{L}} = \sum_{n\geq 0} \hat{u}_i \lambda^i, \quad \hat{\mathcal{M}} = \sum_{n\geq 0} \hat{v}_i \hat{\mathcal{L}}^i. \tag{40}$$

They should be linked with $\mathcal{L}$ and $\mathcal{M}$ by a pair of patching functions $f = f(\lambda, x)$ and $g = g(\lambda, x)$

$$f(\mathcal{L}, \mathcal{M}) = \hat{\mathcal{L}}, \quad g(\mathcal{L}, \mathcal{M}) = \hat{\mathcal{M}}. \tag{41}$$

The patching functions should satisfy the SDiff(2) condition

$$\{f(\lambda, x), g(\lambda, x)\}_{\lambda,x} = 1. \tag{42}$$

Actually, $\hat{\mathcal{L}}$ and $\hat{\mathcal{M}}$ are redundant variables and one may characterize f and g as functions for which

$$f(\mathcal{L}, \mathcal{M})_{\leq -1} = 0, \quad g(\mathcal{L}, \mathcal{M})_{\leq -1} = 0. \tag{43}$$

Anyway, we thus have an SDiff(2) group element (f, g), and one can recover $(\mathcal{L}, \mathcal{M})$ as a solution of a Rimann-Hilbert problem as far as (f, g) is sufficiently close to an identity map. The action of a one-parameter group generated by a Hamiltonian vector field

$$\{F(\lambda, x), \cdot\}_{\lambda,x} = \frac{\partial F(\lambda, x)}{\partial \lambda}\frac{\partial}{\partial x} - \frac{\partial F(\lambda, x)}{\partial x}\frac{\partial}{\partial \lambda} \tag{44}$$

gives rise to an infinitesimal symmetry δ_F of the SDiff(2) KP hierarchy. One can indeed obtain an explicit formula that resemble the case of the self-dual vacuum Einstein equation:

$$\begin{aligned} \delta_F \mathcal{L} &= \{F(\mathcal{L}, \mathcal{M})_{\leq -1}, \mathcal{L}\}_{\lambda,x}, \\ \delta_F \mathcal{M} &= \{F(\mathcal{L}, \mathcal{M})_{\leq -1}, \mathcal{M}\}_{\lambda,x}, \end{aligned} \tag{45}$$

and similar commutation relations:

$$[\delta_{F_1}, \delta_{F_2}] = \delta_{\{F_1,F_2\}_{\lambda,x}}. \tag{46}$$

Let us now turn to the problem of the tau function. In our approach, the tau function is defined by the differential equations

$$\frac{\partial \log \tau}{\partial t_n} = v_{n+1}, \quad n = 1, 2, \ldots, \tag{47}$$

or, equivalently, by

$$d \log \tau = \sum_{n=1}^{\infty} v_{n+1} dt_n. \tag{48}$$

The tau function therefore is always accompanied with an integration constant:

$$\tau \longrightarrow c\tau, \ c \neq 0. \tag{49}$$

Of course one has to prove that the right hand side of Eq. 48 is indeed a closed form. This requires considerably technical calculations exploiting the notion of formal residue,

$$\text{res} \sum a_n \lambda^n d\lambda = a_{-1}. \tag{50}$$

With the aid of such formal residue calculus, one can also prove that the infinitesimal symmetries δ_F have a consistent extension to the tau function. "Consistency" means that extended symmetries do not contradict the relation between $\log \tau$ and v_{n+1}. Such an extension is given by

$$\delta_F \log \tau = - \text{ res } F^x(\mathcal{L}, \mathcal{M}) d_\lambda \mathcal{L}, \tag{51}$$

where F^x is a primitive function of $F = F(\lambda, x)$ with respect to x normalized as

$$F^x(\lambda, x) = \int_0^x F(\lambda, y) dy, \tag{52}$$

and "d_λ" stands for total differentiation with respect to λ. A remakable consequence of this construction is that the extended symmetries obey anomalous commutation relations:

$$[\delta_{F_1}, \delta_{F_2}] \log \tau = \delta_{\{F_1, F_2\}_{\lambda,x}} \log \tau + c(F_1, F_2), \tag{53}$$

where

$$c(F_1, F_2) = \text{ res } F_1(\lambda, 0) dF_2(\lambda, 0). \tag{54}$$

The anomalous term $c(F_1, F_2)$ gives a nontrivial cocycle of the SDiff(2) algebra, hence a central extension. This is reminiscent of the case of the ordinary KP hierarchy; its tau function has anomalous gl(∞) symmetries and leads to a central extension of the infinite matrix algebra gl(∞).

Cocycles of SDiff(2) algebras on various surfaces are classified by physicists.[32] According to their observations, there are $2g$ linearly independent cocycles on a genus g surface. Since the present algebra lives on a cylinder $S^1 \times \mathbf{R}^1$, and this cylinder has genus $g = 1/2$, the above cocycle may be thought of as a realization of those predicted cocycles.

5. SDiff(2) Toda hierarchy

The ordinary 2-d Toda field theory on an infinite chain is described by

$$\partial_z \partial_{\bar{z}} \phi_i + \exp(\phi_{i+1} - \phi_i) - \exp(\phi_i - \phi_{i-1}) = 0, \tag{55}$$

or, equivalently, for $\varphi_i = \phi_i - \phi_{i-1}$ by

$$\partial_z \partial_{\bar{z}} \varphi_i + \exp \varphi_{i+1} + \exp \varphi_{i-1} - 2 \exp \varphi_i = 0. \tag{56}$$

In continuum limit as lattice spacing tends to 0, ϕ_i and φ_i will scale to 3-d fields $\phi = \phi(z, \bar{z}, s)$ and $\varphi = \partial \phi(z, \bar{z}, s)/\partial s$, where s is a coordinate on the continuum limit of the infinite chain. Their equations of motions are then given by

$$\partial_z \partial_{\bar{z}} \phi + \partial_s \exp \partial_s \phi = 0 \tag{57}$$

and

$$\partial_z \partial_{\bar{z}} \varphi + \partial_s^2 \exp \varphi = 0. \tag{58}$$

It is for this 3-d nonlinear field theory that Bakas[3] and Park[5] pointed out a w_∞-algebraic structure. Saveliev and his coworkers[33] attempted a different approach that exploits the notion of continual Lie algebras, and presented a construction of solutions. A Lax formalism of the above equation, which contains a Poisson bracket, is proposed in their work. That type of Lax equations are also studied more systematically by Golenisheva-Kutuzova and Reiman[34] in the context of the coadjoint orbit method.

The above equations have two other sources. One is discovered by relativists.[35] They pointed out that self-dual vacuum Einstein space-times ("$\mathcal{H}$-spaces" or "heavens" in their terminology) with a rotational Killing symmetry are described by the above equation. Another source is Einstein-Weyl geometry and associated curved minitwistor spaces. This line is pursued in detail by twistor people.[36]

We call the above equation (for ϕ) the SDiff(2) Toda equation. In this respect, the ordinary Toda equation may be called the GL(∞) Toda equation. The GL(∞) Toda equation can be embedded into a Toda lattice version of the KP hierarchy, i.e., the Toda lattice hierarchy.[37] Remarkably, the SDiff(2) Toda equation, too, has a similar hierarchy, the "SDiff(2) Toda hierarchy."[38] We now show a brief account of the theory of the SDiff(2) Toda hierarchy.

We start from two pairs of Laurent series $(\mathcal{L},\mathcal{M})$ and $(\hat{\mathcal{L}},\hat{\mathcal{M}})$ of the form

$$\begin{aligned} \mathcal{L} &= \lambda + \sum_{n\le 0} u_n\lambda^n, \quad \mathcal{M} = \sum_{n\ge 1} n z_n \mathcal{L}^n + s + \sum_{n\le -1} v_n \mathcal{L}^n, \\ \hat{\mathcal{L}} &= \sum_{n\ge 1} \hat{u}_n \lambda^n, \quad \hat{\mathcal{M}} = -\sum_{n\ge 1} n\hat{z}_n \hat{\mathcal{L}}^{-n} + s + \sum_{n\ge 1} \hat{v}_n \hat{\mathcal{L}}^n, \end{aligned} \tag{59}$$

where z_n and $\hat{z}_n$, $n = 1, 2, \ldots$, now supply an infinite number of independent variables along with s and λ. The hierarchy consists of the Lax equations

$$\frac{\partial K}{\partial z_n} = \{\mathcal{B}_n, K\}_{\lambda,s}, \quad \frac{\partial K}{\partial \bar{z}_n} = \{\hat{\mathcal{B}}_n, K\}_{\lambda,s} \tag{60}$$

for $K = \mathcal{L}, \mathcal{M}, \hat{\mathcal{L}}, \hat{\mathcal{M}}$ and the canonical Poisson relations

$$\{\mathcal{L},\mathcal{M}\}_{\lambda,s} = \mathcal{L}, \quad \{\hat{\mathcal{L}},\hat{\mathcal{M}}\}_{\lambda,s} = \hat{\mathcal{L}}, \tag{61}$$

where $\mathcal{B}_n$ and $\hat{\mathcal{B}}_n$ are given by

$$\mathcal{B}_n = (\mathcal{L}^n)_{\ge 0}, \quad \hat{\mathcal{B}}_n = (\hat{\mathcal{L}}^{-n})_{\le -1}, \tag{62}$$

and the Poisson bracket is different from the SDiff(2) KP hierarchy:

$$\{F, G\}_{\lambda,s} = \lambda\frac{\partial F}{\partial \lambda}\frac{\partial G}{\partial s} - \lambda\frac{\partial G}{\partial \lambda}\frac{\partial F}{\partial s} \tag{63}$$

It is again crucial to introduce a Kähler-like 2-form as

$$\omega = \frac{d\lambda}{\lambda}\wedge ds + \sum_{n\ge 1} d\mathcal{B}_n \wedge dz_n + \sum_{n\ge 1} d\hat{\mathcal{B}}_n \wedge d\hat{z}_n. \tag{64}$$

This satisfies the relations

$$d\omega = 0, \quad \omega \wedge \omega = 0, \tag{65}$$

hence ensures the existence of Darboux coordinates. In fact, $(\mathcal{L}, \mathcal{M})$ and $(\hat{\mathcal{L}}, \hat{\mathcal{M}})$ both give Darboux coodinates of ω,

$$\omega = \frac{d\mathcal{L}}{\mathcal{L}} \wedge d\mathcal{M} = \frac{d\hat{\mathcal{L}}}{\hat{\mathcal{L}}} \wedge d\hat{\mathcal{M}}, \tag{66}$$

and this conversely characterizes the above defining equations of the SDiff(2) KP hierarchy. The ϕ-field can be reproduced from the hierarchy as a potential:

$$d\phi = \sum_{n\geq 1} \text{ res } (\mathcal{L}^n d \log \lambda) dz_n - \sum_{n \geq 1} \text{ res } (\hat{\mathcal{L}}^{-n} d \log \lambda) d\hat{z}_n - \log \hat{u}_1 ds. \tag{67}$$

(This resembles the Plebanski key functions.) The tau function τ is also defined as a potential:

$$d \log \tau = \sum_{n\geq 1} v_{-n} dz_n - \sum_{n\geq 1} \hat{v}_n d\hat{z}_n + \phi ds. \tag{68}$$

The nonlinear graviton construction now takes the following form. The two pairs $(\mathcal{L}, \mathcal{M})$ and $(\hat{\mathcal{L}}, \hat{\mathcal{M}})$ of Darboux coordinates are connected by patching functions $f = f(\lambda, s)$ and $g = g(\lambda, s)$ as

$$f(\mathcal{L}, \mathcal{M}) = \hat{\mathcal{L}}, \quad g(\mathcal{L}, \mathcal{M}) = \hat{\mathcal{M}}, \tag{69}$$

and the patching functions obey the SDiff(2) condition

$$\{f(\lambda, s), g(\lambda, s)\}_{\lambda, s} = f(\lambda, s). \tag{70}$$

The situation is thus almost parallel to the case of the self-dual vacuum Einstein equation; one can obtain an infinitesimal symmetry operator $\delta_{F,\hat{F}}$ for a pair of generating functions $F = F(\lambda, s)$ and $\hat{F} = \hat{F}(\lambda, s)$. Its action on $\mathcal{L}$, $\mathcal{M}$, $\hat{\mathcal{L}}$, $\hat{\mathcal{M}}$ and ϕ can be calculated explicitly:

$$\begin{aligned}
\delta_{F,\hat{F}} K &= \{F(\mathcal{L}, \mathcal{M})_{\leq -1} - \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}})_{\leq -1}, K\}_{\lambda, s} \quad \text{for } K = \mathcal{L}, \mathcal{M}, \\
\delta_{F,\hat{F}} K &= \{\hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}})_{\geq 0} - F(\mathcal{L}, \mathcal{M})_{\geq 0}, K\}_{\lambda, s} \quad \text{for } K = \hat{\mathcal{L}}, \hat{\mathcal{M}}. \\
\delta_{F,\hat{F}} \phi &= - \text{ res } F(\mathcal{L}, \mathcal{M}) d \log \lambda + \text{ res } \hat{F}(\hat{\mathcal{L}}, \hat{\mathcal{M}}) d \log \lambda.
\end{aligned} \tag{71}$$

A consistent extension of these symmetries for the tau function can be found:

$$\delta_{F,\hat{F}} \log \tau = - \text{ res } F^s(\mathcal{L}, \mathcal{M}) d_\lambda \log \mathcal{L} + \text{ res } \hat{F}^s(\hat{\mathcal{L}}, \hat{\mathcal{M}}) d_\lambda \log \hat{\mathcal{L}}, \tag{72}$$

where

$$F^s(\lambda, s) = \int_0^s F(\lambda, t) dt, \quad \hat{F}^s(\lambda, s) = \int_0^s \hat{F}(\lambda, t) dt. \tag{73}$$

The symmetries at the level of the tau function, again, obey anomalous commutation relations as

$$\left[\delta_{F_1,\hat{F}_1}, \delta_{F_2,\hat{F}_2}\right] \log\tau = \delta_{\{F_1,F_2\}_{\lambda,s},\{\hat{F}_1,\hat{F}_2\}_{\lambda,s}} \log\tau + c(F_1,F_2) + \hat{c}(\hat{F}_1,\hat{F}_2), \tag{74}$$

where c and $\hat{c}$ are cocycles of the SDiff(2) algebra given by

$$\begin{aligned} c(F_1,F_2) &= -\ \mathrm{res}\ F_2(\lambda,0)dF_1(\lambda,0), \\ \hat{c}(\hat{F}_1,\hat{F}_2) &= \ \mathrm{res}\ \hat{F}_2(\lambda,0)d\hat{F}_1(\lambda,0). \end{aligned} \tag{75}$$

This gives a central extension of the direct sum of two SDiff(2) algebras.

6. Concluding remark

A problem left open is to find a geometric structure that should correspond to the infinite dimensional Grassmannian manifold.[28,31,37] This should lead to a practical formula for the tau function like the determinant formula and the field theoretical formula already known for the KP and Toda lattice hierarchies. Pursuing Orlov's approach[29,30] to the KP hierarchy is also an intriguing problem. As mentioned in Section 4, Orlov's improved Lax formalism may be thought of as a candidate of non-commutative (mini)twistor theory. This might be related to geometric quantization of w_∞ gravity.

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