Proceedings of Symposia in Pure Mathematics
Volume 49 (1989), Part 1

Integrable Systems as Deformations of $\mathscr{D}$-Modules

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1. Introduction. Recently M. Sato and his collaborators including the author found, through several case studies, that the theory of nonlinear integrable systems can be reconstructed on the basis of the notion of "deformations of $\mathscr{D}$-modules." This article is an exposition of this point of view. In §2 we illustrate major ideas with a simple model, which is related to $\mathscr{D}$-modules in one-dimension, or, in a more familiar language, linear ordinary differential operators. This model, though it looks too simplified, includes almost all the points of our approach; we discuss this case in every detail. In §3 we deal with the KP hierarchy. The KP hierarchy occupies a "universal" position among other soliton equations in the sense that a number of soliton equations can be derived from it (and its multicomponent version) as "specializations." We refer for the detail of the KP hierarchy to **[Sat-Sat, Jim-Miw, Seg-Wil, Mul, Shi]** and references cited therein, and focus on the relation to $\mathscr{D}$-modules. §4 is devoted to the issue of multidimensional generalization. An ultimate goal of our approach is to find a unified theory of integrable systems on the basis of the notion of $\mathscr{D}$-modules. It would be fair to say that we are now just at the beginning of the whole program. An overview on this program is summarized in §4.5 and §5.

The author wishes to express his sincere gratitude to M. Sato, Y. Sato, and M. Sato's students, N. Suzuki, Y. Ohyama, and A. Nakayashiki. This article owes much to a longstanding seminar with these people.

Before going forward to the text, we give a brief account of several basic notions that will be used throughout the text. A *differential ring* is a ring $\mathscr{R}$ with a set of *derivations* ∂_σ, $0 \le \sigma < s$, i.e., additive maps $\partial_\sigma : \mathscr{R} \to \mathscr{R}$ with the Leibniz rule

$$\partial_\sigma(fg) = \partial_\sigma(f) \cdot g + f \cdot \partial_\sigma(g) \quad \forall f, g \in \mathscr{R}.$$

An element annihilated by all the derivations is called a *constant* of $\mathscr{R}$. The set of all constants $\{c \in \mathscr{R};\ \partial_\sigma(c) = 0 \text{ for } 0 \le \sigma < s\}$ is a subring of

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F07, 35Q20.

$\mathscr{R}$. A *differential operator with coefficients in* $\mathscr{R}$ is a linear combination of higher-order derivations of the form: $P = \sum a_\nu \partial^\nu$ (finite sum), where $\nu = (\nu_0, \ldots, \nu_{s-1})$, a "multi-index," ranges over s-tuples of nonnegative integers, and ∂^ν denote the higher derivations

$$\partial^\nu := (\partial_0)^{\nu_0}(\partial_1)^{\nu_1} \cdots (\partial_{s-1})^{\nu_{s-1}}.$$

The use of such multi-index notations greatly simplifies the presentation. We also use the notation

$$\partial^\nu := (\partial_0)^{\nu_0}(\partial_1)^{\nu_1} \cdots (\partial_{s-1})^{\nu_{s-1}}, \qquad |\nu| := \nu_0 + \cdots + \nu_{s-1},$$
$$\binom{\nu}{\kappa} := \binom{\nu_0}{\kappa_0} \cdots \binom{\nu_{s-1}}{\kappa_{s-1}}, \qquad \binom{\nu_\sigma}{\kappa_\sigma} := \frac{\nu_\sigma(\nu_\sigma - 1) \cdots (\nu_\sigma - \kappa_\sigma + 1)}{\kappa_\sigma!},$$
$$\nu + \kappa := (\nu_0 + \kappa_0, \ldots, \nu_{s-1} + \kappa_{s-1})$$

for $\nu = (\nu_0, \nu_1, \ldots, \nu_{s-1})$ and $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_{s-1})$. The set $\mathscr{D} = \mathscr{D}_\mathscr{R}$ of all such differential operators forms a noncommutative ring with the rules of summation and multiplication

$$\sum a_\nu \partial^\nu + \sum b_\nu \partial^\nu = \sum (a_\nu + b_\nu)\partial^\nu,$$
$$\sum a_\nu \partial^\nu \cdot \sum b_\nu \partial^\nu = \sum c_\nu \partial^\nu, \qquad c_\nu := \sum \binom{\mu}{\kappa} a_\mu \partial^\kappa (b_{\nu+\kappa-\mu}),$$

where the summation in the definition of c_ν ranges over all values of the multi-indices μ and κ with nonnegative integer components. The *order* of a nonzero differential operator $P = \sum a_\nu \partial^\nu$ is the maximal degree of derivations in P.

Such a notation as $\partial^\nu f$ is occasionally confusing, because it is not clear whether this represents the operator product of ∂^ν and f, the latter being considered an operator of order zero, or the operation of ∂^ν on the operand f. In case there is such confusion, let us use the following notations to distinguish them:

$$\partial^\nu \cdot f := \text{operator product } \in \mathscr{D};$$
$$\partial^\nu(f) := \text{operation on } f \in \mathscr{R}.$$

A *left* $\mathscr{D}$*-module* is an additive group $\mathscr{M}$ equipped with an action of $\mathscr{D}$ from the left side. We write the sum of u, $v \in \mathscr{M}$ as $u + v$, and the action of an element P of $\mathscr{D}$ on u as Pu. A *left* $\mathscr{D}$*-submodule* of $\mathscr{M}$ is a subset $\mathscr{N}$ that is a left $\mathscr{D}$-module with regard to the summation and action rules of $\mathscr{M}$. The notion of *right* $\mathscr{D}$-(sub)*modules* is defined in a fully parallel way.

2. Toy model in one dimension.

2.1. *Setup—Rings and modules.* Throughout this section $\mathscr{R}$ denotes a differential ring with a single derivation ∂, $\mathscr{C}$ the ring of constants in $\mathscr{R}$, and $\mathscr{D} = \mathscr{D}_\mathscr{R} = \mathscr{R}[\partial]$ the ring of linear ordinary differential operators with coefficients in $\mathscr{R}$. We assume that all the rational numbers Q are included in $\mathscr{C}$. The left $\mathscr{R}$-submodules

(2.1.1) $$\mathscr{D}^{(i)} := \mathscr{R} + \mathscr{R}\partial + \cdots + \mathscr{R}\partial^i, \qquad i \geq 0,$$

incorporates into $\mathscr{D}$ a natural filtration. In the following we focus on all left $\mathscr{D}$-submodules $\{\mathscr{J}\}$ of $\mathscr{D}$ that satisfy the splitting condition

(2.1.2) $$\mathscr{D} = \mathscr{J} \oplus \mathscr{D}^{(m-1)} \quad \text{(direct sum of left } \mathscr{R}\text{-modules)},$$

where m is a fixed positive integer. In other words, $\mathscr{D} = \mathscr{J} + \mathscr{D}^{(m-1)}$, $\mathscr{J} \cap \mathscr{D}^{(m-1)} = 0$.

2.2. *Generators of $\mathscr{D}$-modules.* An immediate consequence of the splitting condition above is the existence of a distinguished $\mathscr{R}$-generator system.

PROPOSITION. *Under splitting condition* (2.1.2), *there is a unique $\mathscr{R}$-generator system $\{W_i;\, i \geq m\}$ of $\mathscr{J}$, $\mathscr{J} = \sum_{i\geq m} \mathscr{R} W_i$, of the form*

(2.2.1) $$W_i = \partial^i - \sum_{j=0}^{m-1} w_{ij}\partial^j$$

that satisfies the relations ("structure equations")

(2.2.2) $$W_{i+1} - \partial \cdot W_i - w_{i,m-1} W_m = 0.$$

The existence of such an $\mathscr{R}$-generator system, conversely, characterizes left $\mathscr{D}$-submodules of $\mathscr{D}$ with the splitting property.

PROOF. One can indeed obtain such generators by decomposing the monomial ∂^i into the sum of an element of $\mathscr{J}$ and an element of $\mathscr{D}^{(m-1)}$ according to the splitting $\mathscr{D} = \mathscr{J} \oplus \mathscr{D}^{(m-1)}$, the first component then giving W_i as required. From this argument the uniqueness of such a generator system is also evident. Further, the left side of (2.2.2) is chosen so as to lie in the intersection $\mathscr{J} \cap \mathscr{D}^{(m-1)}$, hence vanishes due to the assumption. The former half of the proposition thereby follows. The latter half can be checked by simply reversing the present argument. Q.E.D.

(2.2.2) recursively determines $W_{m+1}, W_{m+2}, \ldots$, in such a form as: (differential operator)$\cdot W_m$. Thereby follows

COROLLARY. *Under splitting condition* (2.1.2), $\mathscr{J}$ *is generated, over* $\mathscr{D}$, *by a single element as*

(2.2.3) $$\mathscr{J} = \mathscr{D} W_m.$$

This conversely characterizes left $\mathscr{D}$-submodules of $\mathscr{D}$, the generator W_m being an arbitrary monic operator of order m.

2.3. *Principle of deformations.* Having obtained a family of left $\mathscr{D}$-modules of $\mathscr{D}$, we now consider a particular type of deformations (which we call *time evolutions*) $\mathscr{J}(0) = \mathscr{J} \to \mathscr{J}(t)$, t being some time parameter(s). The argument presented below is of heuristic nature, indicating a general scheme of how to incorporate time evolutions of such $\mathscr{D}$-modules, which can also apply to all other cases of deformations of $\mathscr{D}$-modules. More rigorous treatments are discussed later.

An "informal definition" of the time evolution reads:

$$\text{“}\mathscr{J}(t) = \mathscr{J}e^{-tF},\text{”} \tag{2.3.1}$$

where F is an element of $\mathscr{D} = \mathscr{C}[\partial]$, i.e., a differential operator with constant coefficients. (One may also consider the case where F is a general element of $\mathscr{R}[\partial]$, but we focus on the above case for simplifying the analysis.)

The "definition" above inherits a subtlety, because the boost operator e^{-tF} of the time evolution is a differential operator of *infinite order*. Accordingly $\mathscr{J}e^{-tF}$ does not lie in the ring $\mathscr{D}$ of operators of *finite* order. One thereby cannot adopt (2.3.1) as a rigorous definition of $\mathscr{J}(t)$.

A remedy to overcome this (rather technical) difficulty is to re-interpret (2.3.1) as a relation in an adequate extension $\hat{\mathscr{D}}$ of $\mathscr{D}$ admitting operators of infinite order. As such an extension one may take, for example, the following.

$$\hat{\mathscr{D}} := \mathscr{D}[[t]] = \left\{ \sum_{n=0}^{\infty} t^n A_n;\ A_n \in \mathscr{D} \right\}. \tag{2.3.2}$$

A precise definition of the time evolution $\mathscr{J} \to \mathscr{J}(t)$ then reads

$$\hat{\mathscr{D}}\mathscr{J}(t) = \hat{\mathscr{D}}\mathscr{J}e^{-tF}. \tag{2.3.3}$$

2.4. *Evolution equations of generators.* Exploiting generators of $\mathscr{D}$-modules as discussed in §2.2, one can give a more explicit form to such a time evolution. Let us start once again with making clear the setting.

We formulate everything within the framework of formal power series in t. The basic differential ring is therefore the ring $\mathscr{R}[[t]]$ of formal power series with the derivation ∂ uniquely extended from $\mathscr{R}$ onto $\mathscr{R}[[t]]$ by the rule

$$\partial(t) = 0.$$

The basic ring of differential operators is then not $\mathscr{D}_{\mathscr{R}} = \mathscr{R}[\partial]$, but its extension $\mathscr{D}_{\mathscr{R}[[t]]} = \mathscr{R}[[t]][\partial]$. In this setting we now consider $\mathscr{D}_{\mathscr{R}[[t]]}$-submodules $\{\mathscr{J}(t)\}$ of $\mathscr{D}_{\mathscr{R}[[t]]}$ that satisfy splitting condition (2.1.2) with $\mathscr{D}$ etc. replaced by $\mathscr{D}_{\mathscr{R}[[t]]}$.

Such a $\mathscr{D}_{\mathscr{R}[[t]]}$-submodule $\mathscr{J}(t)$ has a unique system of $\mathscr{R}$-generators

$$W_i(t) = \partial^i - \sum_{j=0}^{m-1} w_{ij}(t)\partial^j, \qquad i \geq m,$$

with the coefficients $w_{ij}(t)$ lying in $\mathscr{R}[[t]]$. As a $\mathscr{D}_{\mathscr{R}[[t]]}$-module $\mathscr{J}(t)$ is generated by a single element, $W_m(t)$.

Let us derive an infinitesimal version of (2.3.3). While $\hat{\mathscr{D}}$, defined as in (2.3.2), is made up of differential operators with t-dependence, $\mathscr{J}$ itself is independent of t. $\partial/\partial t$ therefore induces a $\mathscr{C}$-linear map: $\hat{\mathscr{D}}\mathscr{J} \to \hat{\mathscr{D}}\mathscr{J}$.

Twisted by e^{tF}, it gives rise to a $\mathscr{C}$-linear map: $\hat{\mathscr{D}}\mathscr{J}(t) \to \hat{\mathscr{D}}\mathscr{J}(t)$ sending $P \in \hat{\mathscr{D}}\mathscr{J}(t) \to \partial(Pe^{tF})/\partial t \cdot e^{-tF} = \partial P/\partial t + PF$. If P is of finite order (i.e., a member of $\mathscr{J}(t)$), so is the image of this $\mathscr{C}$-linear map. To summarize, one obtains the following *infinitesimal version of the law of time evolution*:

(2.4.1) $$\{\partial P/\partial t + PF;\ P \in \mathscr{J}(t)\} \subset \mathscr{J}(t).$$

Applying (2.4.1) to the aforementioned generators $W_i(t)$, one obtains the evolution equations

(2.4.2) $$\frac{\partial W_i(t)}{\partial t} + W_i(t)F = \sum_{j\geq m} b_{ij}(t)W_j(t),$$

where $b_{ij}(t)$ are elements of $\mathscr{R}[[t]]$, the right side being actually a finite sum. Another equivalent expression of these equations is due to the $\mathscr{D}_{\mathscr{R}[[t]]}$-generator $W_m(t)$ of $\mathscr{J}(t)$, with which the time evolution is governed by the single equation

(2.4.3) $$\frac{\partial W_m(t)}{\partial t} + W_m(t)F = B(t)W_m(t),$$

where $B(t)$ is an element of $\mathscr{D}_{\mathscr{R}[[t]]}$. The coefficients $b_{ij}(t)$ and the operator $B(t)$ are uniquely determined by the equations themselves. For example, comparing the ∂^j-terms in (2.4.2), one finds:

(2.4.4) $$b_{ij}(t) = \sum_{n\geq 0} w_{i,j-n}f_n,$$

where f_n denotes the coefficients of F

(2.4.5) $$F = \sum_{n\geq 0} f_n\partial^n, \qquad f_n \in \mathscr{C}.$$

An explicit formula to $B(t)$ will be given in §3 in a more general context.

2.5. *Matrix form of evolution equations.* The evolution equations above can be written in a more compact matrix form, which is a key to solving them in a closed form as we shall see later. To derive it, we use the following two matrices of infinite size.

(2.5.1) $$\xi = \xi(\mathscr{J}) := (w_{ij})_{0\leq i<\infty, 0\leq j<m},$$

(2.5.2) $$\eta = \eta(\mathscr{J}) := (-w_{ij})_{m\leq i<\infty, 0\leq j<\infty},$$

where i and j denote the indices of rows and columns, respectively, and the w_{ij}'s outside the original range $m \leq i < \infty$, $0 \leq j < m$ are supplemented as

(2.5.3) $$w_{ij} := \delta_{ij} \quad (0 \leq i, j < m-1); \qquad -\delta_{ij} \quad (m \leq i, j < \infty).$$

Note the following simple relation connecting η and $\mathscr{J}$.

(2.5.4) $$\eta(\partial^j)_{0\leq j<\infty} = (W_i)_{m\leq i<\infty}.$$

Our goal below is to show what evolution equations are satisfied by $\xi(t)$ and $\eta(t)$. For simplicity we write ξ and η instead of $\xi(t)$ and $\eta(t)$ as far as there is no fear of confusion. The first step is the following, which is an immediate consequence of the construction of η.

PROPOSITION. *Evolution equations* (2.4.2) *are equivalent to the matrix system*

$$\partial\eta/\partial t = B_F\eta - \eta F(\Lambda), \tag{2.5.5}$$

where B_F *and* $F(\Lambda)$ *denote the matrices*

$$B_F := \left(\sum_{n\geq 0} w_{i,j-n} f_n\right)_{m\leq i,j<\infty}, \tag{2.5.6}$$
$$F(\Lambda) := \sum_{n\geq 0} f_n\Lambda^n, \qquad \Lambda^n := (\delta_{i,j-n})_{0\leq i,j<\infty}.$$

In order to transfer this matrix system for η into one for ξ, we note that ξ and η are connected by the relation

$$\eta\xi = 0. \tag{2.5.7}$$

What we need here is the following

LEMMA. *Suppose that a matrix* $\xi' = (\xi'_{ij})$ *of the same size as* ξ *satisfies the relation* $\eta\xi' = 0$. *Then there is a unique matrix* $A = (a_{ij})_{0\leq i,j<m}$ *with which* ξ' *is written* $\xi' = \xi A$.

PROOF. Divide the matrices ξ, η, ξ' into blocks as

$$\xi = \begin{pmatrix} 1 \\ W \end{pmatrix}, \quad \eta = (-W, 1), \quad \xi' = \begin{pmatrix} \xi'_{--} \\ \xi'_{+-} \end{pmatrix},$$

where $W := (w_{ij})_{m\leq i<\infty, 0\leq j<m}$, $\xi'_{+-(--)} := (\xi'_{ij})_{m\leq i<\infty(0\leq i<m), 0\leq j<m}$, and 1 represents unit matrices of various sizes. The relation $\eta\xi' = 0$ then reads

$$W\xi'_{+-} - \xi'_{--} = 0.$$

Therefore $\xi' = \xi B$ with $B = \xi'_{--}$, which is evidently unique. Q.E.D.

REMARK. The roles of ξ and η may be interchanged. I.e., *if a matrix* η' *of the same size as* η *satisfies* $\eta'\xi = 0$, *then there is a unique matrix* B *such that* $\eta' = B\eta$.

NOTATION. From here throughout we use, as in the above proof, the signatures $++$, $+-$, $-+$, $--$ to indicate the four blocks of an $\infty\times\infty$ matrix $A = (a_{ij})_{0\leq i,j<\infty}$. Thus

$$A = \begin{pmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{pmatrix}, \qquad A_{--} = (a_{ij})_{0\leq i,j<m}, \quad \text{etc.} \tag{2.5.8}$$

We now apply this lemma to prove

PROPOSITION. (2.5.5) *is equivalent to the matrix system*

$$\partial\xi/\partial t = F(\Lambda)\xi - \xi A_F, \tag{2.5.9}$$

where

$$A_F = \left(\sum_{n\geq 0} f_n w_{i+n,j}\right)_{0\leq i,j<m}, \tag{2.5.10}$$

the other notation being the same as in the previous proposition.

PROOF. Suppose (2.5.5) is satisfied. Differentiating (2.5.7) with regard to t and using (2.5.5), one has

$$\eta(\partial\xi/\partial t - F(\Lambda)\xi) = 0.$$

The lemma above then ensures that (2.2.9) is satisfied for some A_F. To see that A_F is indeed given by (2.5.10), one divides each matrix in (2.5.9) into "blocks" as in the proof of the lemma, and compares the "upper-half part." Then

$$0 = (F(\Lambda)\xi)_{--} - A_F.$$

This exactly implies (2.5.10). One can thus derive (2.5.9) from (2.5.5). A precisely parallel reasoning deduces the converse. Q.E.D.

REMARKS. (i) The argument above is also applicable to (2.2.2), the structure equations of the $\mathscr{D}$-module $\mathscr{F}$. In terms of ξ and η they read

(2.5.11) $\quad \partial(\xi) = \Lambda\xi - \xi A_\partial, \quad A_\partial := (w_{i+1,j})_{0\leq i,j<m},$

(2.5.12) $\quad \partial(\eta) = B_\partial\eta - \eta\Lambda, \quad B_\partial := (w_{i,j-1})_{m\leq i,j<\infty}.$

(ii) As we have seen in the proof of the last proposition, the equations to ξ and η are made up of two distinct parts, one of which may be interpreted as algebraic equations defining the matrices A_F, B_F, A_∂, and B_∂ in terms of the entries of ξ and η. The $(--)$ block of (2.5.9) and (2.5.11) and the $(++)$ block of (2.5.5) and (2.5.12) are of that nature. They can be readily solved to reproduce (2.5.6), (2.5.10), etc. Therefore if one wishes to show that a matrix ξ satisfies, say, (2.5.9), one has only to check (2.5.9) without identifying the form of A_F, which nevertheless automatically agrees with the one given by (2.5.10).

2.6. *Solution of evolution equations.*

PROPOSITION. *Given Cauchy data* $\xi(t = 0)$, *one can solve the evolution equations in a closed form as:*

(2.6.1)
$$\xi(t) = \exp(tF(\Lambda))\xi(t = 0) \cdot h(t)^{-1},$$
$$h(t) = (\exp(tF(\Lambda))\xi(t = 0))_{--},$$

where $\exp(tF(\Lambda)) := \sum_{n=0}^{\infty} t^n F(\Lambda)^n/n!$.

PROOF. The matrix $\tilde{\xi}(t) := \exp(tF(\Lambda))\xi(t = 0)$ evidently satisfies the following:

$$\partial\tilde{\xi}/\partial t = F(\Lambda)\tilde{\xi}, \qquad \tilde{\xi}(t = 0) = \xi(t = 0).$$

From this, one can readily derive (2.5.8) for $\xi(t) := \tilde{\xi}(t) \cdot h(t)^{-1}$ with A_F unidentified, which however turns out to be given by (2.5.10) due to the last remark in §2.5. Q.E.D.

REMARKS. (i) If the differential ring $(\mathscr{R}, \partial)$ is given a more explicit structure, the structure equation written in matrix form (2.5.10), too, can be solved along a parallel way. Consider for example the case where

$$\mathscr{R} = \mathscr{C}[[x]], \qquad \partial = d/dx.$$

The structure equation then takes the same form as the equation of the time evolution generated by $F = \partial$; thereby the result above provides the solution formula

$$\xi = \exp(x\Lambda)\xi(x=0) \cdot h(x)^{-1}, \qquad h(x) = (\exp(x\Lambda)\xi(x=0))_{--}, \tag{2.6.2}$$

which gives, in terms of the *constant* matrix data $\xi(x=0)$, a parametrization of the family $\{\mathcal{J}\}$ of $\mathcal{D}$-submodules of $\mathcal{D}$ with splitting property (2.2.2).

(ii) The argument used to derive (2.6.1) is of universal nature, and also applicable to the evolution equation of η. This leads to the following expression of solutions:

$$\begin{aligned} \eta(t) &= k(t)^{-1} \cdot \eta(t=0)\exp(-tF(\Lambda)), \\ k(t) &= (\eta(t=0)\exp(-tF(\Lambda)))_{++}. \end{aligned} \tag{2.6.4}$$

Note, however, that a new circumstance occurs here; $k(t)$ is an *infinite* matrix. One has to make sense of its inverse and multiplication with other matrices. This is in principle the same issue as we shall encounter in the case of the KP hierarchy in §3. A prescription to be presented therein will give a definite meaning to the above formula.

2.7. *Matrix Riccati equations.* Let us reconsider the meaning of the contents of §2.5 and §2.6 from a more general point of view.

A "matrix Riccati equation" is a system of differential equations of the following form:

$$dU/dt = L_{+-} + L_{--}U - UL_{--} - UL_{-+}U, \tag{2.7.1}$$

where U is an $n \times m$ matrix of unknowns of the independent variable t,

$$U = (u_{ij})_{0 \le i < n, 0 \le j < m}, \tag{2.7.2}$$

n and m positive integers, and $L_{\pm\pm}$ the four blocks

$$L = \begin{pmatrix} L_{--} & L_{-+} \\ L_{+-} & L_{++} \end{pmatrix}, \qquad L_{--} = (l_{ij})_{0 \le i < n, 0 \le j < m}, \quad \text{etc.} \tag{2.7.3}$$

of an $(n+m) \times (n+m)$ square matrix $L = (l_{ij})_{0 \le i,j < m+n}$ of functions of t. The familiar Riccati equation of the second order

$$du/dt = a + bu - u^2 \tag{2.7.4}$$

corresponds to the case where

$$m = n = 1, \qquad L = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}. \tag{2.7.5}$$

Matrix Riccati equations can be "linearized." In the case of the original Riccati equation, one can reduce it to the linear equation

$$d^2\varphi/dt^2 - b\,d\varphi/dt - a = 0 \tag{2.7.6}$$

via the so called Cole-Hopf transformation

$$u = \frac{d\varphi}{dt}/\varphi. \tag{2.7.7}$$

In a matrix case one starts from the linear system

$$d\Phi/dt = L\Phi, \tag{2.7.8}$$

where Φ is a rectangular matrix with m columns

$$\Phi = \begin{pmatrix} \Phi_{--} \\ \Phi_{+-} \end{pmatrix}, \qquad \Phi_{--} = (\varphi_{ij})_{0\le i<m, 0\le j<n}, \quad \text{etc.} \tag{2.7.9}$$

Under the assumption that Φ_{--} is invertible, the "matrix Cole-Hopf transformation"

$$U = \Phi_{+-} \cdot (\Phi_{--})^{-1} \tag{2.7.10}$$

sends any solution of (2.7.8) to a solution of (2.7.1), and vice versa.

"Linearization" can also be achieved in a dual form based on the linear system

$$d\Psi/dt = -\Psi L, \tag{2.7.11}$$

where Ψ is a rectangular matrix with n rows

$$\Psi = (\Psi_{+-} \quad \Psi_{++}), \qquad \Psi_{++} = (\psi_{ij})_{m\le i<m+n, n\le j<m+n}, \quad \text{etc.} \tag{2.7.12}$$

The transformation

$$U = -(\Psi_{++})^{-1} \cdot \Psi_{+-} \tag{2.7.13}$$

combines the linear system with the matrix Riccati equations.

To make contact with the previous setting, we introduce the rectangular matrices

$$\xi := \begin{pmatrix} 1 \\ U \end{pmatrix}, \qquad \eta := (-U \quad 1), \tag{2.7.14}$$

where 1 stands for unit matrices of suitable size. Note that they are perpendicular to each other:

$$\eta\xi = 0. \tag{2.7.15}$$

Matrix Riccati equation (2.7.1) then allows the following two equivalent expressions:

$$d\xi/dt = L\xi - \xi A, \qquad A := L_{--} + L_{-+}U, \tag{2.7.16}$$

$$d\eta/dt = B\eta - \eta L, \qquad B := L_{++} - UL_{-+}. \tag{2.7.17}$$

"Matrix Cole-Hopf transformations" (2.7.10) and (2.7.13) can be rewritten as

$$\xi = \Phi \cdot (\Phi_{--})^{-1}, \qquad \eta = (\Psi_{++})^{-1} \cdot \Psi. \tag{2.7.18}$$

What we have seen in §2.5 and §2.6 is evidently a special case of the setting above; the infinite matrix $W = (w_{ij})_{m\le i<\infty, 0\le j<m}$ corresponds to U.

2.8. *Grassmann manifolds.* Geometrically, a matrix Riccati equation defines a dynamical motion in a Grassmann manifold, the matrices ξ and η playing the role of "frame matrices" representing a point on it.

Following the notation of [Sat-Sat], let $\mathrm{GM}(m, \mathrm{V})$ denote the Grassmann manifold formed by all m-dimensional vector subspaces of a fixed vector space V, which we now define to be the vector space of column vectors of size $m+n$, $\dim \mathrm{V} = m+n$. For such a vector subspace one can take a basis $\{\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(m-1)}\}$ and construct from them a rectangular matrix $\xi := (\xi^{(0)}, \ldots, \xi^{(m-1)})$ of size $(m+n) \times m$. We call ξ a *frame matrix*. A different choice of bases that span the same vector subspace corresponds to a change of the frame matrix of the form

$$(2.8.1) \qquad \xi \to \xi h, \qquad h \in \mathrm{GL}(m).$$

Allowing for this nonuniqueness, one obtains the following expression of $\mathrm{GM}(m, \mathrm{V})$:

$$(2.8.2) \qquad \mathrm{GM}(m, \mathrm{V}) \simeq \mathrm{Fr}(m+n, m)/\mathrm{GL}(m),$$

where $\mathrm{Fr}(m+n, m)$ denotes the set of all frame matrices

$$(2.8.3) \qquad \mathrm{Fr}(m+n, m) := \{(m+n) \times n \text{ matrices of rank } m\},$$

on which $\mathrm{GL}(m)$ acts as shown in (2.8.1).

This Grassmann manifold is covered by a finite number of affine coordinate patches. Each coordinate patch is numbered by a sequence $S = (s_0, s_1, \ldots, s_{m-1})$ of increasing integers, $0 \le s_0 < s_1 < \cdots < s_{m-1} < m+n$, and given by

$$(2.8.4) \qquad \mathrm{GM}(m, n)_S := \{\xi \in \mathrm{Fr}(m; n, m); \det(\xi_{s_i j}) \neq 0\}/\mathrm{GL}(m).$$

In this coordinate patch one can take a distinguished frame matrix that satisfies the conditions

$$(2.8.5) \qquad \xi_{s_i j} = \delta_{ij} \quad \text{for } 0 \le i, j < m,$$

the other mn entries of ξ then giving a system of affine coordinates on $\mathrm{GM}(m, n)_S$.

The unknown functions u_{ij} of matrix Riccati equation (2.7.5) are thus nothing other than the affine coordinates on the coordinate patch $\mathrm{GM}(m, n)_{(0,1,\ldots,m-1)}$. (2.7.16) defines, through (2.8.2), a dynamical system on $\mathrm{GM}(m, n)$. The matrix Riccati equation can be understood as representing the dynamical motion in a particular coordinate system.

In this respect the $\mathrm{GL}(m)$ freedom of frame matrices play the role of "gauge transformations." Up to here we have understood (2.7.16) as equations to the special frame matrices of the form as shown in (2.7.14) (i.e. under normalization (2.8.5) with $S = (0, 1, \ldots, m-1)$), but if this normalization is removed, (2.7.16) allows the transformation

$$(2.8.6) \qquad \xi \to \xi h, \qquad A \to h^{-1} A h + h^{-1}\, dh/dt,$$

where h is an arbitrary $\mathrm{GL}(m)$-valued function of t. The "matrix Cole-Hopf transformation" is a special case of these transformations, connecting a "gauge-fixing" to another one.

As we have already seen in several stages, this dynamical motion admits another representation in terms of the matrix η. This reflects a general duality principle among Grassmann manifolds. Let V^* denote the dual vector space of V, $\langle\ ,\ \rangle$ the canonical pairing of V^* and V. Then a canonical identification

$$\mathrm{GM}(n, \mathrm{V}^*) \simeq \mathrm{GM}(m, \mathrm{V}) \tag{2.8.7}$$

is obtained by assigning to each $V \subset \mathrm{V}$ ($V' \subset \mathrm{V}^*$) its *polar subspace* $V^{\perp} := \{v' \in \mathrm{V}^*; \forall v \in V, \langle v', v\rangle = 0\}$ ($V'^{\perp} := \{v \in \mathrm{V}; \forall v' \in V', \langle v', v\rangle = 0\}$). If one regards V^* as the set of *row* vectors of size $m+n$, the counterpart of (2.8.2) for $\mathrm{GL}(n, \mathrm{V}^*)$ becomes:

$$\mathrm{GM}(n, \mathrm{V}^*) \simeq \mathrm{GL}(n)\backslash\mathrm{Fr}(n, m+n). \tag{2.8.8}$$

The matrix η is thus nothing other than a representative ("dual frame matrix") of the "dual Grassmann manifold" $\mathrm{GL}(n, \mathrm{V}^*)$.

2.9. *Simultaneous time evolutions.* We have discussed, up to here, the case with a single time variable, but this is just for simplifying the presentation. One can actually consider simultaneous time evolutions $\mathscr{I} \to \mathscr{I}(t_1, t_2, \dots)$ caused by the right multiplication of the boost operator $\exp(-t_1F_1 - t_2F_2 - \cdots)$ with a set of differential operators $F_1, F_2, \dots$ with constant coefficients. The time evolutions are described by a system of evolution equations with respect to the multidimensional time variables, each of which takes the same form as presented in §2.4. The contents of §2.5 and §2.6 can be readily extended as well without substantial change.

If $F_i = \partial^i$, $i = 1, 2, \dots$, the evolution equations of the simultaneous time evolutions with time parameters $t = (t_1, t_2, \dots)$ agree, in essence, with the KP hierarchy restricted to a special sector of its solution space. The microdifferential operator $W(t) = 1 + w_1\partial^{-1} + \cdots$ of the KP hierarchy (cf. §3) is connected with $W_m(t)$ as:

$$W(t) = W_m(t)\partial^{-m}.$$

"Rational" and "soliton" solutions of the KP hierarchy fall into this class.

3. KP hierarchy as deformations of $\mathscr{D}$-modules.

3.1. *Setup—microdifferential operators.* Throughout this section $(\mathscr{R}, \partial)$, $\mathscr{C}$, $\mathscr{D}$ are the same as in §2. Also the following notations are used:

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\} \quad \text{(the set of all integers)},$$
$$\mathbb{N} := \{0, 1, 2, \dots\}, \qquad \mathbb{N}^c := \{-1, -2, \dots\}.$$

Since the notion of microdifferential operators is one of the most basic ingredients of the theory of the KP hierarchy, we give a brief account of it and related notions.

Let $\mathscr{E}$ denote the following set of formal Laurent series in ∂, which we call *microdifferential operators* (or, in the traditional language of analysis, *pseudodifferential operators*):

$$(3.1.1)\qquad \mathscr{E} = \mathscr{E}_{\mathscr{R}} := \left\{ \sum_{n\in\mathbf{Z}} a_n\partial^n \;;\; \text{(i) } \forall n, a_n \in \mathscr{R}, \quad \text{(ii) } \exists m, \forall n > m,\ a_n = 0 \right\}.$$

The notions of summation and multiplication are defined by the following rules:

$$(3.1.2a)\qquad \sum a_n\partial^n + \sum b_n\partial^n := \sum (a_n + b_n)\partial^n,$$

(3.1.2b)

$$\sum a_n\partial^n \cdot \sum b_n\partial^n := \sum c_n\partial^n, \qquad c_n = \sum \binom{i}{k} a_i\partial^k(b_{n+k-i}),$$

where the summation $\sum$ in the definition of c_n ranges over all $i \in \mathbf{Z}$ and $k \in \mathbf{N}$. Actually only a finite number of terms can survive in this sum because of the assumption on a_n and b_n and of a property of the binomial coefficients: $\binom{i}{k} = 0$ for $i < 0$. These rules give a natural generalization of the calculus of differential operators, and $\mathscr{E}$ thereby acquires a ring structure, $\mathscr{D}$ being a subring. Another basic operation is the formal adjoint

$$(3.1.3)\qquad \left(\sum a_n\partial^n\right)^* := \sum (-\partial)^n a_n,$$

which causes an anti-automorphism of $\mathscr{E}$.

The left $\mathscr{R}$-submodules

$$(3.1.4)\qquad \mathscr{E}^{(i)} := \left\{\sum a_n\partial^n \in \mathscr{E};\ a_n = 0 \text{ for } n > i\right\}, \qquad i \in \mathbf{Z},$$

give a filtration of $\mathscr{E}$. Just as in the case of differential operators, a microdifferential operator $\sum_{n=-\infty}^{i} a_n\partial^n$ is said to be *of order* i if $a_i \neq 0$, a_i then being called the *leading coefficient*. A microdifferential operator is said to be *monic* if the leading coefficient is equal to the unity. It is not hard to see that a microdifferential operator is invertible (i.e., has an inverse in $\mathscr{E}$) if and only if the leading coefficient is invertible in $\mathscr{R}$. In particular, any monic microdifferential operator is invertible.

$\mathscr{E}$ can be decomposed as:

$$(3.1.5)\qquad \mathscr{E} = \mathscr{D} \oplus \mathscr{E}^{(-1)} \quad \text{(direct sum of left } \mathscr{R}\text{-modules).}$$

Let $(\ )_\pm$ denote the projections onto the first (+) and second (−) components:

$$(3.1.6)\qquad \left(\sum a_n\partial^n\right)_+ := \sum_{n\geq 0} a_n\partial^n, \qquad \left(\sum a_n\partial^n\right)_- := \sum_{n<0} a_n\partial^n.$$

3.2. *KP Hierarchy—Review.* The KP hierarchy is made up of a set of evolution equations that describe an infinite number of simultaneous time evolutions of a monic microdifferential operator of order zero:

$$(3.2.1)\qquad W := 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots.$$

Three equivalent representations are now available to the KP hierarchy, two of which are referred to as the "Lax" and "Zakharov-Shabat" representations, respectively. The third one has no widely accepted name at present; let us call it, for the moment, the "W-representation."

The Lax representation reads as:

$$\partial L/\partial t_n = [B_n, L], \qquad n = 1, 2, \ldots, \tag{3.2.2}$$

where L (the Lax operator) and B_n denote the following:

$$L := W \cdot \partial \cdot W^{-1}, \tag{3.2.3}$$

$$B_n := (L^n)_+. \tag{3.2.4}$$

Since W is monic, as remarked in §3.1, it is invertible and the definitions above are meaningful. L becomes a monic microdifferential operator of order one, and B_n a monic differential operator of order n.

The Zakharov-Shabat representation is the following system of equations:

$$\partial B_m/\partial t_n - \partial B_n/\partial t_m + [B_m, B_n] = 0, \qquad m, n = 1, 2, \ldots. \tag{3.2.5}$$

If B_n are parametrized by a single Lax operator L as in (3.2.4), this indeed becomes equivalent to the Lax representation above.

The W-representation takes the form of evolution equations to W:

$$\partial W/\partial t_n = B_n W - W\partial^n, \qquad n = 1, 2, \ldots, \tag{3.2.6}$$

where the B_n are understood to be defined through (3.2.3) and (3.2.4). Note however that (3.2.6) itself, under the requirement that B_n be differential operators, uniquely determines the relation of B_n and W; from (3.2.6) the B_n are written

$$B_n = W \cdot \partial^n \cdot W^{-1} + (\partial W/\partial t_n)W^{-1},$$

and the $(\ )_+$ part of both sides gives the relation

$$B_n = (W \cdot \partial^n \cdot W^{-1})_+, \tag{3.2.7}$$

which is a restatement of the definition of B_n in (3.2.3) and (3.2.4). This circumstance is of the same nature as we met in §2 in various representations of evolution equations.

3.3. *$\mathscr{D}$-modules to be deformed.* As $\mathscr{D}$-modules, we consider left $\mathscr{D}$-submodules $\{\mathscr{I}\}$ of $\mathscr{E}$ with the splitting property

$$\mathscr{E} = \mathscr{I} \oplus \mathscr{E}^{(-1)} \quad \text{(direct sum of left } \mathscr{R}\text{-modules)}. \tag{3.3.1}$$

One can specify the structure of such $\mathscr{D}$-modules with basically the same reasoning as presented in §2.2, as follows.

First, under splitting condition (3.3.1) there is a unique $\mathscr{R}$-generator system $\{W_i;\ i \geq 0\}$ of $\mathscr{I}$ of the form

$$W_i = \partial^i - \sum_{j<0} w_{ij}\partial^j \tag{3.3.2}$$

with the relations ("structure equations")

$$W_{i+1} - \partial \cdot W_i - w_{i,-1} W_0 = 0. \tag{3.3.3}$$

The existence of such a $\mathscr{R}$-generator system, conversely, characterizes left $\mathscr{D}$-submodules of $\mathscr{E}$ with the splitting property.

Second, as a corollary of this result, $\mathscr{I}$ turns out to be generated, over $\mathscr{D}$, by a single element:

$$\mathscr{I} = \mathscr{D} W_0. \tag{3.3.4}$$

This conversely characterizes left $\mathscr{D}$-submodules of $\mathscr{E}$, the generator W_0 being an arbitrary monic element of $\mathscr{E}$ of order zero.

3.4. *Deformations and evolution equations.* The contents of §§2.3, 2.4, and 2.9 carry over to the case above without any change. One can thereby introduce an infinite set of time evolutions caused by the monomials ∂^n, $n = 1, 2, \ldots$, with time variables t_n, $n = 1, 2, \ldots$. Let $\mathscr{I}(t) = \mathscr{I}(t_1, t_2, \ldots)$ be the result of these time evolutions, which is now a $\mathscr{D}_{\mathscr{R}[[t]]}$-submodule of $\mathscr{E}_{\mathscr{R}[[t]]}$ that satisfies splitting condition (3.3.1) with $\mathscr{E} = \mathscr{E}_{\mathscr{R}}$ replaced by $\mathscr{E}_{\mathscr{R}[[t]]}$. According to §2.3 an "informal" definition of $\mathscr{I}(t)$ reads (cf. (2.3.1)):

$$\text{“}\mathscr{I}(t) = \mathscr{I} \exp(-t_1\partial - t_2\partial^2 - \cdots).\text{”} \tag{3.4.1}$$

A more rigorous formulation is due to the relation (cf. (2.4.1))

$$\{\partial P/\partial t_n + P\partial^n;\ P \in \mathscr{I}(t)\} \subset \mathscr{I}(t), \qquad n \geq 1. \tag{3.4.2}$$

With the $\mathscr{R}[[t]]$-generators $W_i(t)$, $i \geq 0$, of $\mathscr{I}(t)$ one can write (3.4.2) in the form of evolution equations, which reads:

$$\partial W_i(t)/\partial t_n + W_i(t)\partial^n = \sum_{j<0} b_{ijn}(t) W_j(t), \qquad n \geq 1, \tag{3.4.3}$$

where $b_{ijk}(t)$ are elements of $\mathscr{R}[[t]]$ uniquely determined from the equations themselves (cf. (2.4.4)). Another representation of the time evolutions is due to the $\mathscr{D}_{\mathscr{R}[[t]]}$-generator $W_0(t)$:

$$\partial W_0(t)/\partial t_n + W_0(t)\partial^n = B_n(t) W_0(t), \qquad n \geq 1, \tag{3.4.4}$$

$B_n(t)$ being elements of $\mathscr{D}_{\mathscr{R}[[t]]}$. This is nothing other than the W-representation of the KP hierarchy under the identification $W = W_0$. One can thus reproduce the KP hierarchy as deformations (time evolutions) of $\mathscr{D}$-modules.

3.5. *Matrix form—Bridge to Grassmann manifold.* From the coefficients $w_{ij}(t)$ of the $\mathscr{R}[[t]]$-generators we construct the following matrices (cf. (2.5.1), (2.5.2)):

$$\xi(t) := (w_{ij}(t))_{i\in\mathbf{Z}, j\in\mathbf{N}^c}, \tag{3.5.1}$$

$$\eta(t) := (-w_{ij}(t))_{i\in\mathbf{N}, j\in\mathbf{Z}}. \tag{3.5.2}$$

As in §2.5, we supplement the w_{ij}'s outside the original range as:

$$w_{ij}(t) := \delta_{ij} \quad (i, j \in \mathbf{N}^c); \qquad -\delta_{ij} \quad (i, j \in \mathbf{N}). \tag{3.5.3}$$

Also we use the signatures ++, +−, −+, −− as subscripts to indicate the four blocks of an infinite matrix whose rows and columns are numbered by integers. E.g., if $A = (a_{ij})_{i,j\in\mathbf{Z}}$,

$$A = \begin{pmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{pmatrix}, \qquad A_{--} = (a_{ij})_{i,j\in\mathbf{N}^c}, \quad \text{etc.} \tag{3.5.4}$$

Another important set of matrices are

$$\Lambda^n := (\delta_{i,j-n})_{i,j\in\mathbf{Z}}, \qquad n \in \mathbf{Z}. \tag{3.5.5}$$

Following the argument of §2.5 one can rewrite (3.4.3) in terms of $\xi(t)$ and $\eta(t)$. This results in the following two matrix systems, which are equivalent to each other.

$$\partial\xi(t)/\partial t_n = \Lambda^n\xi(t) - \xi(t)A_n(t), \qquad n \geq 1, \tag{3.5.6}$$

$$\partial\eta(t)/\partial t_n = B_n(t)\eta(t) - \eta(t)\Lambda^n, \qquad n \geq 1, \tag{3.5.7}$$

where

$$A_n(t) := (\Lambda^n\xi(t))_{--}, \qquad B_n(t) := (\eta(t)\Lambda^n)_{++}. \tag{3.5.8}$$

One can also rewrite (3.3.3), which are structure equations for W_i's to form a $\mathscr{D}$-module, into a similar matrix form.

The matrices $\xi(t)$ and $\eta(t)$ may be thought of as "frame matrices" (cf. §2.8) representing a point (which moves as t varies) of the "universal Grassmann manifold" of Sato and Sato [Sat-Sat]. Segal and Wilson [Seg-Wil] argued basically the same fact from a somewhat different point of view. To be more precise, these two approaches are "dual" to each other; Sato and Sato adopted the ξ representation as the fundamental picture, whereas the argument of Segal and Wilson, based on the notion of "Baker functions," may be thought of as providing a realization of the η representation. In the case of $\mathscr{R} = \mathscr{C}[[x]]$ and $\partial = \partial/\partial x$, for example, a complete system of Baker functions is given the following formal Laurent series of a new formal variable λ ("spectral parameter")

$$\psi_i(x,t,\lambda) := \sum_{j\in\mathbf{Z}} w_{ij}\lambda^j \exp\left(x\lambda + \sum_{n=1}^{\infty} t_n\lambda^n\right), \qquad i \geq 0. \tag{3.5.9}$$

Their linear combinations with coefficients in $\mathscr{R} = \mathscr{C}[[x]]$ form a $\mathscr{D}$-module isomorphic to $\mathscr{I}$. In terms of matrices the relation to $\mathscr{I}$ takes the following very compact form:

$$\eta \cdot \left(\lambda^j \exp\left(x\lambda + \sum_{n=1}^{\infty} t_n\lambda^n\right)\right)_{j\in\mathbf{Z}} = (\psi_i)_{i\in\mathbf{Z}}, \tag{3.5.10}$$

which shows how to connect the setting of Segal and Wilson with ours.

Finally, let us discuss the meaning of a modification of the η representation. To this end we use the matrix

$$\xi^*(t) := J^t\eta(t) = (-w_{-i-1,-j-1}(t))_{i\in\mathbf{Z},j\in\mathbf{N}^c}, \tag{3.5.11}$$

where the superscript "t" denotes the transpose, and J the $\mathbf{Z} \times \mathbf{Z}$ matrix

$$J := (\delta_{i+j+1,0})_{i,j\in\mathbf{Z}}. \tag{3.5.12}$$

(3.5.7) can be transferred into a matrix system for $\xi^*(t)$:

$$\partial\xi^*(t)/\partial t_n = -\Lambda^n\xi^*(t) - \xi^*(t)A_n^*(t), \tag{3.5.13}$$

where

$$A_n^*(t) := -(\Lambda^n\xi^*(t))_{--}. \tag{3.5.14}$$

Thus $\xi^*(t)$ satisfies a matrix system of basically the same form as that of $\xi(t)$, except that the first term on the right side has an *opposite* sign. This is equivalent to saying that $\xi^*(t)$ obeys the same law of time evolution as $\xi(-t)$, i.e., the "time reversal" of the KP hierarchy.

The circumstance above is, in fact, specific to the case of the KP hierarchy. The relationship to the time reversal above is rather a *coincidence*, which is due to the "self-duality" of the fundamental index set, $\mathbf{Z}$, under the involution caused by the action of J. This is not the case in multidimensional theories (cf. §4).

3.6. *Solution—Linear algebra of infinite matrices.* An advantage of writing evolution equations in matrix forms (3.5.6) and (3.5.7) is, as we have argued in §2.6, that one can solve them in a "closed form." Thus one obtains the following solution formulas (cf. (2.6.1), (2.6.4)):

$$\xi(t) = \exp\left(\sum_{n=1}^{\infty} t_n\Lambda^n\right)\xi(t=0)\cdot h(t)^{-1}, \tag{3.6.1}$$

$$\eta(t) = k(t)^{-1}\cdot\eta(t=0)\exp\left(-\sum_{n=1}^{\infty} t_n\Lambda^n\right), \tag{3.6.2}$$

where

$$h(t) := \left(\exp\left(\sum_{n=1}^{\infty} t_n\Lambda^n\right)\xi(t=0)\right)_{--}, \tag{3.6.3}$$

$$k(t) := \left(\eta(t=0)\exp\left(-\sum_{n=1}^{\infty} t_n\Lambda^n\right)\right)_{++}. \tag{3.6.4}$$

Since these formulas include the inversion and multiplication of infinite matrices, one has to show how to make sense of these operations. (We have encountered a similar issue even in the case of the model discussed in §2; see the last remark in §2.6.)

Assigning to each monomial in t an integer-valued weight (or what physicists call "dimensions") as

$$\mathrm{weight}(t_1^{\nu_1}\cdot t_2^{\nu_2}\cdots) := \sum_{n\geq 1} n\nu_n, \tag{3.6.5}$$

one can introduce into the ring $\mathscr{R}[[t]]$ a filtration $\{\mathscr{R}[[t]]_\nu;\ \nu \in \mathbb{N}\}$ of $\mathscr{R}$-submodules:

$$\mathscr{R}[[t]]_\nu := \{\text{linear combinations of monomials in } t \text{ with weight} \geq \nu\}. \tag{3.6.6}$$

This filtration plays the role of a "norm" that measures the convergence of infinite series in $\mathscr{R}[[t]]$. A basic property is summarized as:

LEMMA. *Given a sequence $a_n \in \mathscr{R}[[t]]_{\nu_n}$, $n = 1, 2, \ldots$, with $\nu_n \to \infty$ as $n \to \infty$, the infinite sum $\sum_{n=1}^\infty a_n$ converges and defines an element of $\mathscr{R}[[t]]_{\min\{\nu_n; n \geq 0\}}$.*

PROOF. Let us consider the coefficient of a monomial in t arising in the infinite series $\sum_{n=1}^\infty a_n$. It takes the form of an infinite sum of elements of $\mathscr{R}$. Under the assumption, however, only a finite number of terms can survive in that sum, which has thereby a definite meaning within $\mathscr{R}$. The last part of the lemma is evident because the coefficients of monomials less than $\min\{\nu_n; n \geq 1\}$ all vanish in that situation. Q.E.D.

Bearing this in mind, let us turn to the analysis of $h(t)$. To this end we split $\xi(t = 0)$ into two pieces as

$$\xi(t = 0) = \begin{pmatrix} 1 \\ \mathbb{O} \end{pmatrix} + \begin{pmatrix} \mathbb{O} \\ W \end{pmatrix}, \qquad W := (\xi(t = 0))_{+-},$$

and apply $\exp(\sum t_n \Lambda^n)$ to each term. Then $h(t)$ can be written

$$\begin{aligned} h(t) &= h_0(t) + h_1(t), \\ h_0(t) &:= \left(\exp\left(\sum t_n \Lambda^n\right)\right)_{--}, \\ h_1(t) &:= \left(\exp\left(\sum t_n \Lambda^n\right) \begin{pmatrix} \mathbb{O} \\ W \end{pmatrix}\right)_{--}. \end{aligned} \tag{3.6.7}$$

The first term $h_0(t)$ evidently has a unique inverse because it is an upper triangular matrix whose diagonal part is the unit matrix. Hence we seek to construct $h(t)^{-1}$ as:

$$h(t)^{-1} := \sum_{n=0}^{\infty} (-)^n h_0(t)^{-1} (h_1(t) h_0(t)^{-1})^n. \tag{3.6.8}$$

We have to deal with the following two issues:

(i) Does the multiplication of matrices in each term on the right side make sense? Since these are infinite matrices, each entry of the matrix product is formed by the sum of an infinite number of terms.

(ii) Does the infinite series on the right side converge? This is also related to infinite series in $\mathscr{R}[[t]]$.

To settle these issues, let us first note the following properties of the entries of $h_0(t) = (h_{0ij}(t))$ and $h_1(t) = (h_{1ij}(t))$, which one can readily verify from the construction:

(3.6.9a) $h_{0ij}(t) \in \mathscr{R}[[t]]_{j-i}$ for all i, j;

(3.6.9b) $h_{0ij}(t) = 0$ for $i > j$;

(3.6.9c) $h_{0ii}(t)$ are invertible in $\mathscr{R}[[t]]$ for all i;

(3.6.9d) $h_{1ij}(t) \in \mathscr{R}[[t]]_{-i}$ for all i, j.

PROPOSITION. *The set of all* $\mathbf{N}^c \times \mathbf{N}^c$ *matrices* $h(t) = h_0(t) + h_1(t)$ *with each factor satisfying* (3.6.9) *forms a group under matrix multiplication, the inverse* $h(t)^{-1}$ *being given by* (3.6.8).

PROOF. Let us focus on the invertibility of each $h(t)$; the other part is much easier to check. From the first three conditions of (3.6.9) $h_0(t)$ is invertible with an inverse that satisfies just the same properties. From (3.6.9) one can show, using the lemma above, the convergence of the infinite series that occur in the evaluation of each entry of the nth power of the matrix $h_1(t)h_0(t)^{-1}$, which thereby acquires a definite meaning. As a by-product of this estimate, the (i, j)th entry of the nth power turns out to lie in $\mathscr{R}[[t]]_{-i+n}$. Hence, again by virtue of the lemma, the infinite sum over $n = 1, 2, \ldots$ on the right side of (3.6.8) is entry-wise convergent and fulfills the last condition of (3.6.9). Q.E.D.

One can thus make sense of $h(t)^{-1}$ in (3.6.1) with the following estimate:

(3.6.11) $$(h(t)^{-1})_{ij} \in \mathscr{R}[[t]]_{-i-1} \quad \text{for all } i, j.$$

From this, once again appealing to the lemma, one deduces that the matrix product on the right side of (3.6.1) makes sense as a matrix with entries lying in $\mathscr{R}[[t]]$.

One can justify (3.6.2) with a parallel reasoning.

4. Attempt at multidimensional generalizations.

4.1. *Microdifferential operators in multidimensions.* The notion of microdifferential operators becomes more involved in multidimensions. This is mostly due to the fact that one has to distinguish various "co-directions" (i.e., rays of cotangent vectors), to each of which corresponds a different way of "microlocalization." In the following we give an abstract version of microdifferential operators with the co-direction fixed to a special one throughout.

Let $\mathscr{R}$ be a differential ring with s derivations $\partial_0, \partial_1, \ldots, \partial_{s-1}$ which commute with each other, $\mathscr{C}$ its ring of constants, which we require to include all rational numbers $\mathbf{Q}$, and $\mathscr{D} = \mathscr{D}_{\mathscr{R}} := \mathscr{R}[\partial_0, \ldots, \partial_{s-1}]$ the ring of differential operators with coefficients in $\mathscr{R}$. The "order" of a differential operator is, as in the one-dimensional case, the maximal degree of powers of derivations included therein. This gives rise to a filtration of $\mathscr{D}$ with the $\mathscr{R}$-submodules $\mathscr{D}^{(i)} := \{P \in \mathscr{D};\ P \text{ is of order} \le i\}$, $i \ge 0$. To simplify the presentation, we freely use the multi-index notation as mentioned in §1:

$$\partial^\nu := (\partial_0)^{\nu_0}(\partial_1)^{\nu_1}\cdots(\partial_{s-1})^{\nu_{s-1}}, \qquad |\nu| := \nu_0 + \cdots + \nu_{s-1},$$
$$\binom{\nu}{\kappa} := \binom{\nu_0}{\kappa_0}\binom{\nu_1}{\kappa_1}\cdots\binom{\nu_{s-1}}{\kappa_{s-1}},$$

for $\nu = (\nu_0, \nu_1, \ldots, \nu_{s-1}) \in \mathbf{Z}^s$ and $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_{s-1}) \in \mathbf{Z} \times \mathbf{N}^{s-1}$.

We now define a ring $\mathscr{E} = \mathscr{E}_{\mathscr{R}}$ of (s-dimensional) microdifferential operators as the following set of formal Laurent series in $\partial_0, \ldots, \partial_{s-1}$ that *allow negative powers of* ∂_0:

$$\text{(4.1.1)} \qquad \mathscr{E} := \Big\{\sum a_\nu \partial^\nu \ ; \ \text{(i) } \forall \nu,\ a_\nu \in \mathscr{R}, \quad \text{(ii) } \exists m,\ a_\nu = 0 \text{ if } |\nu| > m\Big\},$$

where the summation $\sum$ ranges over all $\nu \in \mathbf{Z} \times \mathbf{N}^{s-1}$ (i.e., $\nu_0 \in \mathbf{Z}$, $\nu_1 \in \mathbf{N}, \ldots, \nu_{s-1} \in \mathbf{N}$}. The notions of addition and multiplication are introduced under the following rules:

$$\text{(4.1.2a)} \qquad \sum a_\nu \partial^\nu + \sum b_\nu \partial^\nu := \sum (a_\nu + b_\nu)\partial^\nu,$$

(4.1.2b)

$$\sum a_\nu \partial^\nu \cdot \sum b_\nu \partial^\nu := \sum c_\nu \partial^\nu, \qquad c_\nu := \sum \binom{\alpha}{\kappa} a_\alpha \partial^\kappa b_{\nu+\kappa-\alpha},$$

where the summation $\sum$ in the definition of c_ν ranges over all $\kappa \in \mathbf{N}^s$ and $\alpha \in \mathbf{Z} \times \mathbf{N}^{s-1}$, which effectively becomes, as in the one-dimensional case, a finite sum. The formal adjoint

$$\text{(4.1.3)} \qquad \Big(\sum a_\nu \partial^\nu\Big)^* := \sum (-\partial)^\nu a_\nu$$

causes an anti-isomorphism of $\mathscr{E}$. The notion of order can be readily extended to $\mathscr{E}$, and gives rise to a filtration with the $\mathscr{R}$-submodules

$$\text{(4.1.4)} \qquad \mathscr{E}^{(i)} := \Big\{\sum a_\nu \partial^\nu \in \mathscr{E};\ a_\nu = 0 \text{ for } |\nu| > i\Big\}, \qquad i \in \mathbf{Z}.$$

A crucial difference is that there is no natural analogue of (3.1.5) for the multidimensional case; e.g., $\mathscr{D} \cap \mathscr{E}^{(-1)} = 0$, but $\mathscr{D} + \mathscr{E}^{(-1)}$ does not span the whole $\mathscr{E}$. This is a major difficulty that occurs inevitably when one attempts to extend the theory of the KP hierarchy beyond the barrier of one dimension.

4.2. *Splitting condition in multidimensions.* Since there is no a priori way of splitting $\mathscr{E}$ in multidimensional cases, our strategy is to introduce a splitting *by hand*; a different choice of such a splitting may lead to a distinct theory of deformations of $\mathscr{D}$-modules.

In the one-dimensional case, the splitting of $\mathscr{E}$ as $\mathscr{E} = \mathscr{D} \oplus \mathscr{E}^{(-1)}$ corresponds to the splitting of $\mathbf{Z}$, the index set to the exponents of powers of ∂, into the two pieces $\mathbf{N}$ and $\mathbf{N}^c$. We then replace $\mathscr{D}$ in $\mathscr{E} = \mathscr{D} \oplus \mathscr{E}^{(-1)}$ by a left $\mathscr{D}$-submodule $\mathscr{J}$ retaining the splitting as $\mathscr{E} = \mathscr{J} \oplus \mathscr{E}^{(-1)}$, and build up a deformation theory within the family of such $\mathscr{D}$-modules. This is a rough sketch of the program we have pursued.

To deal with multidimensional cases along the same program we first choose a splitting of the whole index set. The whole index set in the s-dimensional case is

$$I := \mathbf{Z} \times \mathbf{N}^{s-1}, \tag{4.2.1}$$

which represents all the exponents of powers of $\partial_0, \ldots, \partial_{s-1}$ occurring in $\mathscr{E}$. We divide it into two pieces $I_\pm$ as:

$$I = I_+ \amalg I_-, \tag{4.2.2}$$

where $\amalg$ means a *disjoint sum*, i.e., $I = I_+ \cup I_-$, $I_+ \cap I_- = \emptyset$. For some reasons we further require the following condition:

$$\forall \nu \in I_+, \quad \forall \kappa \in \mathbf{N}^s, \quad \nu + \kappa \in I_+. \tag{4.2.3}$$

This condition could be relaxed, but we shall not discuss that possibility in the following.

Such a splitting of the index set induces a splitting of $\mathscr{E}$:

$$\mathscr{E} = \mathscr{E}(I_+) \oplus \mathscr{E}(I_-) \quad \text{(direct sum of left } \mathscr{R}\text{-modules)}, \tag{4.2.4}$$

where

$$\mathscr{E}(I_\pm) := \left\{ \sum a_\nu \partial^\nu \in \mathscr{E};\ a_\nu = 0 \text{ unless } \nu \in I_\pm \right\}. \tag{4.2.5}$$

An immediate consequence of (4.2.3) is that $\mathscr{E}(I_+)$ becomes a left $\mathscr{D}$-module. In multidimensional cases there are of course an infinite number of choices for $I_\pm$.

We then proceed to consider left $\mathscr{D}$-submodules $\{\mathscr{J}\}$ of $\mathscr{E}$ under the splitting condition

$$\mathscr{E} = \mathscr{J} \oplus \mathscr{E}(I_-) \quad \text{(direct sum of left } \mathscr{R}\text{-modules)}. \tag{4.2.6}$$

What one has to do first of all is to make clear the structure of such $\mathscr{D}$-modules.

4.3. *Structure of $\mathscr{D}$-module—Symptom of difficulty.* It is not hard to see, with the same argument as employed in the one-dimensional case, that such a $\mathscr{D}$-module $\mathscr{J}$ has an $\mathscr{R}$-generator system $\{W_\alpha;\ \alpha \in I_+\}$ of the form

$$W_\alpha = \partial^\alpha - \sum_{\beta \in I_-} w_{\alpha\beta} \partial^\beta, \qquad w_{\alpha\beta} \in \mathscr{R}, \tag{4.3.1}$$

with the structure equations

$$W_{\alpha+1_\sigma} - \partial_\sigma \cdot W_\alpha - \sum_{\beta \in I_+ \cap (I_- + 1_\sigma)} w_{\alpha\beta} W_\beta = 0, \qquad 0 \le \sigma < s, \tag{4.3.2}$$

where 1_σ denotes the unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the σth site, and $I_- + 1_\sigma$ the translation of I_- into the direction of 1_σ. These structure equations conversely ensure that all $\mathscr{R}$-linear combinations of W_α form a left $\mathscr{D}$-submodule of $\mathscr{E}$ with splitting property (4.2.6).

This result, in fact, indicates a seed of the difficulties that we shall encounter in the description of time evolutions. To see this, let us draw attention to the fact that the intersection $I_+ \cap (I_- + 1_\sigma)$ becomes an *infinite set* for some direction. This occurs for any choice of $I_\pm$ as far as s (= space dimensions) is greater than one. If one rewrites (4.3.2) into the form of differential equations to the coefficients $w_{\alpha\beta}$, this means that each resultant equation inevitably includes an infinite number of terms. In other words, the structure equations are by no means a collection of *algebraic differential equations*. This never occurs in the one-dimensional case.

One might think of introducing some topological linear space with a differential ring structure, instead of an abstract differential ring $\mathscr{R}$, so as to put the infinite sums above under some analytical control. We do not adopt such an approach because, first, we wish to pursue the present issue on the basis of the theory of algebraic differential equations, and second, there are several evidences showing that at least for some choices of $I_\pm$, one can never make such a theory allowing infinite sums without extending $\mathscr{E}$ so as to include operators of *infinite* order.

We shall take another approach. Before discussing it, let us examine what kind of difficulties arise in the formulation of time evolutions.

4.4. *Structure of evolution equations.* The way of introducing time evolutions is entirely parallel to the one-dimensional case. A time evolution is caused by a generator $F = \sum f_\alpha \partial^\alpha$, which is a differential operator with constant coefficients, $f_\alpha \in \mathscr{C}$. The whole theory can be built up within the ring $\mathscr{E}_{\mathscr{R}[[t]]}$ of microdifferential operators with coefficients in $\mathscr{R}[[t]]$, t being a time variable. The law of time evolution reads:

$$\{\partial P/\partial t + PF;\ P \in \mathscr{J}(t)\} \subset \mathscr{J}(t), \tag{4.4.1}$$

where $\mathscr{J}(t)$ is a left $\mathscr{D}_{\mathscr{R}[[t]]}$-submodule of $\mathscr{E}_{\mathscr{R}[[t]]}$ under splitting condition (4.2.6) with $\mathscr{E}$ etc. replaced by $\mathscr{E}_{\mathscr{R}[[t]]}$ etc. This takes a more explicit form in terms of the $\mathscr{R}[[t]]$-generators $W_\alpha(t)$, $\alpha \in I_+$, as the following set of evolution equations.

$$\frac{\partial W_\alpha(t)}{\partial t} + W_\alpha(t)F = \sum_{\beta \in I_+ \cap (I_- + \mathrm{Supp}(F))} b_{\alpha\beta}(t) W_\beta(t), \tag{4.4.2}$$

where $b_{\alpha\beta}(t)$ are elements of $\mathscr{R}[[t]]$ that are unique and determined by the equations themselves (cf. §2.4), $\mathrm{Supp}(F)$ denotes the set $\{\alpha \in I;\ f_\alpha \neq 0\}$, and $I_- + \mathrm{Supp}(F)$ the set of all multi-indices of the form $\alpha + \beta$, $\alpha \in I_-$, $\beta \in \mathrm{Supp}(F)$. Further, one can write these evolution equations in a matrix form as:

$$\partial\xi/\partial t = F(\Lambda)\xi - \xi A_F, \qquad \partial\eta/\partial t = B_F\eta - \eta F(\Lambda), \tag{4.4.3}$$

where $\xi := (w_{\alpha\beta})_{\alpha\in I, \beta \in I_-}$, $\eta := (-w_{\alpha\beta})_{\alpha \in I_+, \beta \in I}$, with $w_{\alpha\beta}$ extended outside the original range of multi-indices as:

$$w_{\alpha\beta} = \delta_{\alpha\beta} \quad \text{for } \alpha, \beta \in I_-; \qquad -\delta_{\alpha\beta} \quad \text{for } \alpha, \beta \in I_+, \tag{4.4.4}$$

and $F(\Lambda)$ is the matrix obtained from F by replacing ∂_σ by

$$\Lambda_\sigma := (\delta_{\alpha+1_\sigma,\beta})_{\alpha,\beta\in I} \quad \text{for } 1 \leq \sigma < s.$$

We now encounter the same difficulty that occurs in §4.3. To see this, let us rewrite (4.4.2) in terms of their coefficients. Then we obtain the equations

$$\text{(4.4.5)} \qquad \frac{\partial w_{\alpha\beta}}{\partial t} + \sum_{\gamma\in \mathrm{Supp}(F)} w_{\alpha,\beta-\gamma} f_\gamma = \sum_{\gamma\in I_+\cap(I_- + \mathrm{Supp}(F))} b_{\alpha\gamma} w_{\gamma\beta}$$

for $\alpha \in I_+$, $\beta \in I_-$, together with the explicit form of $b_{\alpha\beta}$

$$\text{(4.4.6)} \qquad b_{\alpha\beta} = \sum_{\gamma\in \mathrm{Supp}(F)} w_{\alpha,\beta-\gamma} f_\gamma, \qquad \alpha, \beta \in I_+.$$

The trouble is that the right side of (4.4.5) can become an infinite sum in general.

4.5. *Program for constructing reasonable theories.* The occurrence of infinite terms in the evolution equations is specific to multidimensional cases. One-dimensional cases, as we have seen in the preceding sections, are free of this difficulty. We wish to develop, on one hand, a theory within the realm of algebraic differential equations, but on the other hand, the basic part of the previous setting should be also retained as far as possible.

A possible remedy, which we illustrate below, is to impose some additional relations ("constraints") on $w_{\alpha\beta}$ besides structure equations (4.3.2) so as to make the infinite sums in both the structure equations and the evolution equations *effectively finite.* As such constraints, we consider only those that are independent of t. Geometrically, this means taking a submanifold of an infinite-dimensional Grassmann manifold and making a theory of dynamical motion "constrained on it." Here are several points that one should take into account:

(i) The constraints should be consistent with the time evolution, i.e., if it is satisfied at the initial time $t = 0$, so is it at any t.

(ii) The requirement above may be reinterpreted, rather, as a condition to the time evolution. In other words, given a set of constraints on $w_{\alpha\beta}$, only those time evolutions are permitted that are consistent with the constraints.

(iii) Besides the time evolutions, deformations caused by a generator F with *variable* coefficients (i.e., a general element of $\mathscr{E}$) give rise to transformations of $\mathscr{D}$-modules $\{\mathscr{J}\}$, which correspond to the notion of "transformations of solutions" in the theory of integrable systems. Under the constraints such transformations are also limited to those that are consistent with the constraints. Let us call such deformations *admissible* ones. Time evolutions form an abelian subgroup of the group of these admissible transformations.

(iv) Of course the constraints should be such that the infinitesimal action of these admissible transformations, which takes just the same form as (4.4.2), be free of any infinite sum.

(v) It is desirable that the constraints have as large a group of admissible transformations (or a Lie algebra of infinitesimal transformations) as possible. This is a criterion that, in a sense, measures the "integrability" of the resultant system (cf. §5).

Up to now only three cases of examples are known that appear to fit into this program. One of them is concerned with theta functions, and arises through moduli of algebraic varieties and line bundles over them. A brief account of it was mentioned in Sato's lectures at the summer institute. Research along this line is now in progress by Sato and his group; we omit the details here. We present the other two cases below, which are related to gauge and gravitational fields.

4.6. *Examples—Integrable gauge fields.* Typical examples of this class are self-dual gauge fields in four dimensions and their extensions to higher dimensions (cf. **[Ward]**, for example). We now formulate such integrable gauge fields within our differential-algebraic language.

Let $\mathscr{R}'$ be a differential ring with $s-1$ derivations ∂_σ, $1 \le \sigma < s$, which are mutually commutative. We also assume, as in the preceding sections, that the ring of constants $\mathscr{C}$ includes all rational numbers. For example, one may set $\mathscr{R}' = \mathscr{C}[[x_1,\dots,x_{s-1}]]$, $\partial_\sigma = \partial/\partial x_\sigma$, where $\mathscr{C}$ is a commutative field of characteristic zero. On the other hand, let r be an integer greater than one and $\mathfrak{gl}(r,\mathscr{R}')$ the set of all square matrices of size r. $\mathfrak{gl}(r,\mathscr{R}')$ naturally becomes a differential ring with the same derivations ∂_σ. This is our basic setting for considering $\mathrm{GL}(r)$ gauge fields.

A fundamental ingredient of our abstract treatment is an element W of $\mathfrak{gl}(r,\mathscr{R}'[[\lambda^{-1}]])$ of the form

(4.6.1) $$W = 1 + w_1\lambda^{-1} + w_2\lambda^{-2} + \cdots, \qquad w_n \in \mathfrak{gl}(r,\mathscr{R}'),$$

where λ is a formal variable ("spectral parameter"). We interpret an integrable system of gauge fields equations as a set of simultaneous time evolutions $W \to W(t)$ with time variables $t = (t_1,t_2,\dots)$, $W(t)$ taking the same form as W in (4.6.1) with $\mathscr{R}'$ replaced by $\mathscr{R}'[[t]]$.

The time evolutions are defined by the evolution equations

(4.6.2) $$\partial W(t)/\partial t_i = (F_i(\lambda,\partial') + A_i(t,\lambda))W(t), \qquad i = 1,2,\dots,$$

where $F_i(\lambda,\partial')$, the generators of the time evolutions, are derivations of the form $F_i(\lambda,\partial') = \sum_{\sigma=1}^{s-1} f_{i\sigma}(\lambda)\partial_\sigma$, $f_{i\sigma}(\lambda) \in \mathscr{C}[\lambda]$, and $A_i(t,\lambda)$ elements of $\mathfrak{gl}(r,\mathscr{R}'[[t]][\lambda])$, which are uniquely determined by the equations themselves. In the traditional interpretation the equations in (4.2.2) are nothing other than the linear system, the coefficients of $A_i(t,\lambda)$ in λ playing the role of gauge potentials.

This is the basic setting adopted in **[Tak1]**, in which the author presented an attempt to extend the work of **[Sat-Sat]** to the case of self-dual gauge fields, using essentially the same Grassmann manifold. From the point of view of $\mathscr{D}$-modules the solution space should be embedded into a larger Grassmann manifold in the sense mentioned in §4.5.

We now show an interpretation in the context of $\mathscr{D}$-modules. We introduce a *dummy variable* x_0 to enlarge $\mathfrak{gl}(r, \mathscr{R}')$ into

$$\mathscr{R} := \mathfrak{gl}(r, \mathscr{R}'[[x_0]]), \tag{4.6.3}$$

with a new derivation $\partial_0 := \partial/\partial x_0$ besides the preassigned ones $\partial_1, \ldots, \partial_{s-1}$. In fact x_0 itself plays no substantial role; what we really need is ∂_0, which we identify with λ. To be more specific, we define a microdifferential operator (of order zero) $W(\partial_0)$ from $W = W(\lambda)$ as

$$W(\partial_0) := 1 + w_1 \partial_0^{-1} + w_2 \partial_0^{-2} + \cdots \tag{4.6.4}$$

and consider the $\mathscr{D}_{\mathscr{R}}$-module

$$\mathscr{I} := \mathscr{D}_{\mathscr{R}} W(\partial_0). \tag{4.6.5}$$

This $\mathscr{D}$-module satisfies the splitting condition with the choice of index sets:

$$I_+ := \mathbf{N} \times \mathbf{N}^{s-1}, \qquad I_- := \mathbf{N}^c \times \mathbf{N}^{s-1}. \tag{4.6.6}$$

Evidently these $\mathscr{D}$-modules not only satisfy the splitting condition but also have a very special structure. We understand this as a consequence of "constraints" in the sense of §4.5 that are implicit in the construction above.

The generators of time evolutions are $F_i(\partial_0, \partial')$, which are now differential operators in s dimensions with constant coefficients. It is not hard to check that the evolution equations as $\mathscr{D}$-modules do reproduce (4.6.2). Therefore, in particular, one does not encounter the difficulty of infinite sums.

The consideration on transformation groups presented in [Tak1] can be as well translated into the language of $\mathscr{D}$-modules above, as "admissible deformations" of $\mathscr{I}$'s in the sense of §2.5. A probably maximal Lie algebra of the generators (which are microdifferential operators like $F_i(\partial_0, \partial')$) of such admissible deformations would be:

$$\mathfrak{g} := \mathfrak{gl}(r, \mathscr{R}'[[\partial_0^{-1}]][\partial_0]) + \sum_{\sigma=1}^{s-1} \mathscr{R}'[\partial_0]\partial_\sigma. \tag{4.6.10}$$

The first part on the right-hand side represents the so-called Riemann-Hilbert transformations [Uen-Nak], which, roughly speaking, act transitively on the solution space of the gauge field equations [Tak1].

4.7. *Examples—Integrable gravitational fields.* Self-dual (in four dimensions) and hyper-Kähler (in $4n$ dimensions) metrics are typical examples of this case (cf. [H-K-L-R] for their mathematical properties and physical relevance). We take, also here, an abstract formulation on the basis of the differential ring $\mathscr{R}' := \mathscr{C}[[x_1, \ldots, x_{s-1}]]$, $\partial_\sigma := \partial/\partial x_\sigma$ for $1 \le \sigma < s$, $\mathscr{C}$ being a commutative field of characteristic zero. In place of W let us now consider an s-tuple $\varphi = (\varphi_1(x', \lambda), \ldots, \varphi_{s-1}(x', \lambda))$, $x' := (x_1, \ldots, x_{s-1})$, of formal Laurent series of the following form:

$$\varphi_\sigma(x', \lambda) = x_\sigma + \varphi_{\sigma 1}\lambda^{-1} + \varphi_{\sigma 2}\lambda^{-2} + \cdots, \qquad \varphi_{\sigma n} \in \mathscr{R}'. \tag{4.7.1}$$

One may understand this as an abstract setting for considering a group of "local diffeomorphisms" (with a parameter λ): $x' = (x_\sigma) \to \varphi = (\varphi_\sigma(x', \lambda))$ in the x'-space.

We omit the details of how to introduce time evolutions and to derive the field equations of, e.g., self-dual and hyper-Kähler metrics; cf. [Tak2] for the case of self-dual metrics.

$\mathscr{D}$-modules relevant to the present case can be singled out as follows. We again use a dummy variable x_0, extend $\mathscr{R}'$ into $\mathscr{R} := \mathscr{R}'[\partial_0]$, and identify λ with $\partial_0 := \partial/\partial x_0$. From $\varphi(\lambda)$ we construct the formal differential operator (of *infinite* order)

$$(4.7.2) \qquad W_\varphi(\lambda, \partial') := \sum_{\nu_1, \dots, \nu_{s-1} \geq 0} \frac{(\varphi(x', \lambda) - x')^\nu}{\nu!} \partial'^\nu,$$

and the microdifferential operator (of order *zero*)

$$(4.7.3) \qquad W_\varphi(\partial_0, \partial') := W_\varphi(\lambda, \partial')|_{\lambda \to \partial_0},$$

where we use the multi-index notations as: $\nu! := \nu_1! \cdots \nu_{s-1}!$, $x'^\nu := (x_1)^{\nu_1} \cdots (x_{s-1})^{\nu_{s-1}}$, $\partial'^\nu := (\partial_1)^{\nu_1} \cdots (\partial_{s-1})^{\nu_{s-1}}$. A fundamental property of these operators is that the composition of "diffeomorphisms" corresponds to the multiplication of operators in the *reversed order*:

$$(4.7.4) \qquad W_{\psi \circ \varphi} = W_\varphi \cdot W_\psi, \qquad (\psi \circ \varphi)(x, \lambda) := \psi(\varphi(x, \lambda), \lambda).$$

The left $\mathscr{D}_\mathscr{R}$-submodule

$$(4.7.5) \qquad \mathscr{J} := \mathscr{D}_\mathscr{R} W_\varphi(\partial_0, \partial')$$

of $\mathscr{E}_\mathscr{R}$ then satisfies the splitting property for the same index sets $I_\pm$ as in (4.6.6). These are the $\mathscr{D}$-modules with which a deformation theory is to be constructed. "Constraints" in the sense of §4.5 are again implicit in the structure of $\mathscr{D}$-modules as above.

A Lie algebra of admissible deformations (transformations), which appears to be maximal, is given by

$$(4.7.6) \qquad \mathfrak{g} := \sum_{\sigma=1}^{s-1} \mathscr{R}'[[\partial_0^{-1}]][\partial_0]\partial_\sigma.$$

Elements of $\sum_{\sigma=1}^{s-1} \mathscr{C}[\partial_0]\partial_\sigma$ generate time evolutions that agree with the evolution equations to φ_σ mentioned above. In the case of self-dual metrics, this reproduces the group of "nonlinear superposition" found by Boyer and Plebanski [Boy-Ple].

The author wishes to take this occasion to correct some errors in [Tak2]. The contents of §§5.5–5.7 of [Tak2] are erroneous. This is concerned exactly with the issue pointed out in §§4.3–4.5 of this article. To correct these errors, one has only to throw away the picture of dynamical flows on the Grassmann manifold used therein, and to reconstruct the whole theory on a submanifold made up of points that originate in the φ's above. This corresponds to picking out only those $\mathscr{D}$-modules that take a form as in (4.7.5). The contents of §§5.2–5.4 of [Tak2] are unaffected and remain true.

5. Conclusion. The notion of deformations of $\mathscr{D}$-modules appears to provide very suggestive material to the theory of multidimensional integrable systems. As we have seen, a naive extension to multidimensions leads to undesirable results, and some more refined setting is required in general. It seems likely that Grassmann manifolds themselves, in contrast to the one-dimensional case, can no longer play the role of "phase space," and we are forced to develop a theory on a suitably chosen submanifold of the naive Grassmann manifold or, equivalently, under the presence of some "constraints" to the $\mathscr{D}$-modules to be deformed. The field equations of integrable gauge and gravitational theory may be understood as examples advocating such a point of view.

Since the general situation in multidimensions becomes thus somewhat involved, we should rather restart by making clear the meaning of the term "integrability." It would be a fascinating idea to *redefine* the notion of integrability, in an entirely general context of algebraic differential equations, as the "transitivity" of the action of a transformation group that the system in question admits. We are thus naturally led to the problem of classifying all such Lie algebras (groups) and its homogeneous spaces. The deformation theory of $\mathscr{D}$-modules offers a laboratory to develop such an idea.

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