Integrable structure of various melting crystal models

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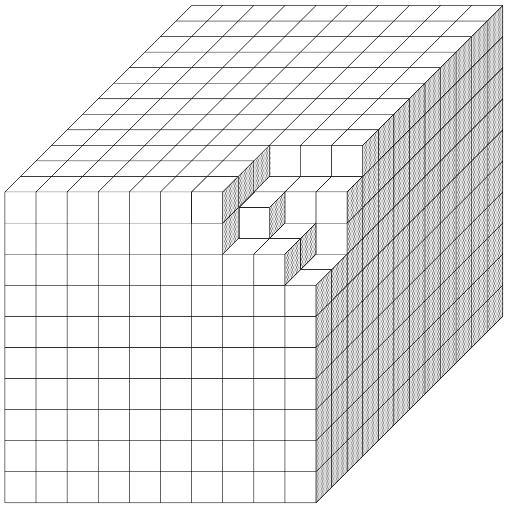
References

1. **K.T. and T. Nakatsu, arXiv:0710.5339 (published)**
2. **K.T., arXiv:1208.4497, arXiv:1302.6129 (published)**
3. **K.T., arXiv:1410.5060**

1. Melting crystal model

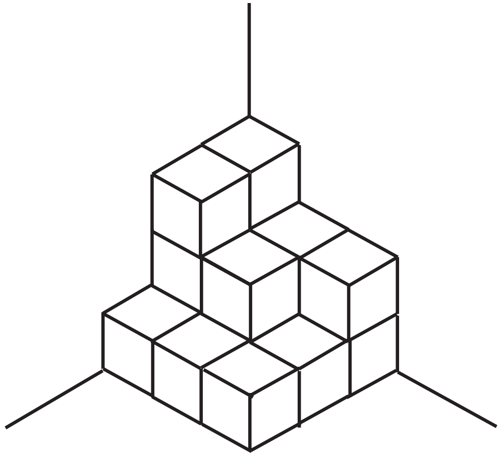
The melting crystal model is a statistical model of a crystal corner in the first octant of the xyz space. The crystal consists of unit cubes, the boundary is a step surface, and the complement in the octant is a 3D Young diagram.



crystal corner

complement

$\longleftrightarrow$



3D Young diagram

Plane partitions and 3D Young diagrams

3D Young diagrams are identified with plane partitions, i.e., non-increasing 2D arrays of non-negative integers:

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{array}{ccc} \pi_{ij} & \geq & \pi_{i,j+1} \\ \rotatebox{90}{$\leq$} & & \\ \pi_{i+1,j} & & \end{array}$$

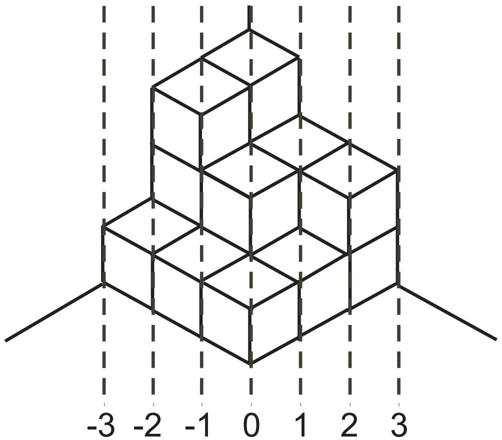
π_{ij} is the height of the stack of cubes on the square $[i-1, i] \times [j-1, j]$ of the xy plane.

Partition function

The Partition function of this model is the sum

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|}, \quad |\pi| = \sum_{i,j=1}^{\infty} \pi_{ij},$$

of the Boltzmann weights $q^{|\pi|}$ $(0 < q < 1)$ over the set $\mathcal{PP}$ of all plane partitions.



This sum can be calculated by the method of diagonal slicing (A. Okounkov and N. Reshetikhin).

$$\pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0, \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0 \end{cases}$$

Reduction to sum over partitions

- There is a one-to-one correspondence

$$\pi \longleftrightarrow (\lambda, T, T'), \quad \lambda = \pi(0),\ T, T' \in \mathrm{SSTab}(\lambda)$$

- Partial sum over T, T' yields the special value $s_\lambda(q^{-\rho})$ of the Schur function $s_\lambda(x_1, x_2, \ldots)$ at

$$q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{i-1/2}, \ldots).$$

- The partition function can be reduced to the sum

$$Z = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2$$

over the set $\mathcal{P}$ of all ordinary partitions.

Final expression of partition function

By the Cauchy identity

$$\sum_{\lambda\in\mathcal{P}} s_\lambda(x_1,x_2,\ldots)s_\lambda(y_1,y_2,\ldots) = \prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1},$$

the reduced sum over $\lambda \in \mathcal{P}$ turns into the so called MacMahon function:

$$Z = \prod_{i,j=1}^{\infty}(1-q^{i+j-1})^{-1} = \prod_{n=1}^{\infty}(1-q^n)^{-n}$$

Slightest generalization

$$
\begin{aligned}
Z &= \sum_{\pi \in \mathcal{PP}} q^{|\pi|} Q^{|\pi(0)|} \quad \text{(definition)} \\
&= \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} \\
&= \prod_{n=1}^{\infty} (1 - Q q^n)^{-n}.
\end{aligned}
$$

This is a kind of deformations of the model by the external potential $|\pi(0)|$ (= area of the 0-th slice) with the coupling constant $\log Q$. An integrable system emerges in deformations by more complicated external potentials.

External potentials $\Phi_k(\lambda, k)$, $k = 1, 2, \ldots$

Heuristic definition (divergent for $0 < q < 1$):

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} q^{k(\lambda_i+s-i+1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}$$

True definition (by recombination of terms):

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty}(q^{k(\lambda_i+s-i+1)} - q^{k(s-i+1)}) + \frac{1-q^{ks}}{1-q^k}q^k$$

They are q-analogues of the eigenvalues of Casimir operators of $U(\infty)$. The parameter $s \in \mathbb{Z}$ plays the role of lattice coordinate in the underlying Toda hierarchy.

Deformed partition function

$$Z(s,t) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda| + s(s+1)/2} e^{\Phi(\lambda,s,t)}$$

where $\Phi(\lambda, s, t) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s)$, and $t = (t_1, t_2, \ldots)$ plays the role of time variables.

(K.T. and T. Nakatsu, 2007) $Z(s,t)$ is related to a tau function $\tau(s,t)$ of the 1D Toda hierarchy as

$$Z(s,t) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k}\right) q^{-s(s+1)(2s+1)/6} \tau(s, \iota(t)),$$

$$\iota(t) = (-t_1, t_2, -t_3, \ldots, (-1)^k t_k, \ldots)$$

2. Modified melting crystal model

Undeformed partition function

$$Z' = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}) Q^{|\lambda|} = \prod_{n=1}^{\infty} (1 + Qq^n)^n$$

where ${}^t\lambda$ denotes the transpose (or conjugate partition) of λ. Formally, this model is obtained from the previous model by replacing

$$s_\lambda(q^{-\rho})^2 \longrightarrow s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}).$$

This model is related to topological string theory on a toric Calabi-Yau threefold called the resolved conifold.

Deformed partition function

$$Z'(s,t,\bar{t}) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}) Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda,s,t,\bar{t})},$$

$$\Phi(\lambda,s,t,\bar{t}) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda,s) + \sum_{k=1}^{\infty} \bar{t}_k \Phi_{-k}(\lambda,s).$$

Results obtained in 2012–13 (K.T.)

(i) $Z'(s,t,\bar{t})$ is related to a tau function $\tau'(s,t,\bar{t})$ of the 2D Toda hierarchy.

(ii) This solution of the 2D Toda hierarchy is actually a solution of the Ablowitz-Ladik (or relativistic Toda) hierarchy embedded in the 2D Toda hierarchy.

2.1 Outline of part (i)

Idea of proof of part (i)

Mostly parallel to the case of $Z(s,t)$:

• Find a fermionic expression of $Z'(s,t,\bar{t})$ in terms of charged free fermions.

• Use a quantum torus algebra (realized by fermion bilinears) and shift symmetries therein to rewrite $Z'(s,t,\bar{t})$.

Charged fermions

- **2D fermion fields**

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n}$$

- **Creation-annihilation operators**

$$\psi_m\psi_n^* + \psi_n^*\psi_n = \delta_{m+n,0}, \quad \psi_m\psi_n + \psi_n\psi_m = \psi_m^*\psi_n^* + \psi_n^*\psi_m^* = 0$$

- **Ground states of the charge s sector in the Fock space**

$$\langle s| = \langle -\infty| \cdots \psi_{s-1}^*\psi_s^*, \quad |s\rangle = \psi_{-s}\psi_{-s+1}\cdots|-\infty\rangle$$

- **Excited states $\langle\lambda, s|$ and $|\lambda, s\rangle$ labelled by partitions $\lambda \in \mathcal{P}$**

Building blocks of fermionic expression

- **Fermion bilinears**

$$L_0 = \sum_{n\in\mathbb{Z}} n{:}\psi_{-n}\psi^*_n{:}, \quad W_0 = \sum_{n\in\mathbb{Z}} n^2{:}\psi_{-n}\psi^*_n{:},$$

$$H_k = \sum_{n\in\mathbb{Z}} q^{kn}{:}\psi_{-n}\psi^*_n{:}, \quad J_k = \sum_{n\in\mathbb{Z}} {:}\psi_{k-n}\psi^*_n{:}$$

- **Vertex operators**

$$\Gamma_\pm(z) = \exp\left(\sum_{k=1}^\infty \frac{z^k}{k} J_{\pm k}\right), \quad \Gamma'_\pm(z) = \exp\left(-\sum_{k=1}^\infty \frac{(-z)^k}{k} J_{\pm k}\right),$$

$$\Gamma_\pm(x_1, x_2, \ldots) = \prod_{i\geq 1} \Gamma_\pm(x_i), \quad \Gamma'_\pm(x_1, x_2, \ldots) = \prod_{i\geq 1} \Gamma'_\pm(x_i)$$

- **Matrix elements**

$$s_{\lambda}(q^{-\rho}) = \langle s|\Gamma_{+}(q^{-\rho})|\lambda, s\rangle,$$

$$s_{{}^{\mathrm{t}}\lambda}(q^{-\rho}) = \langle \lambda, s|\Gamma'_{-}(q^{-\rho})|s\rangle,$$

$$Q^{|\lambda|+s(s+1)/2} = \langle \lambda, s|Q^{L_0}|\lambda, s\rangle,$$

$$\Phi_k(\lambda, s) = \langle \lambda, s|H_k|\lambda, s\rangle$$

Fermionic expression of partition function

$$Z'(s, t, \bar{t}) = \langle s|\Gamma_{+}(q^{-\rho})Q^{L_0}e^{H(t,\bar{t})}\Gamma'_{-}(q^{-\rho})|s\rangle,$$

$$H(t, \bar{t}) = \sum_{k=1}^{\infty} t_k H_k + \sum_{k=1}^{\infty} \bar{t}_k H_{-k}$$

Quantum torus algebra and shift symmetries

- The quantum torus algebra is realized by

$$V_m^{(k)} = q^{-km/2} \sum_{n \in \mathbb{Z}} q^{kn} {:}\psi_{m-n}\psi_n^*{:}, \quad k, m \in \mathbb{Z}.$$

H_k and J_k are contained therein as $H_k = V_0^{(k)}$, $J_k = V_k^{(0)}$.

- Shift symmetries imply the following algebraic relations:

$$\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})H_k$$
$$= \left((-1)^k q^{-W_0/2} J_k q^{W_0/2} + \frac{q^k}{1-q^k}\right)\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}),$$

$$\boldsymbol{H_{-k}}\boldsymbol{\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})}$$

$$\boldsymbol{= \Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})\left(q^{-W_0/2}J_{-k}q^{W_0/2} - \frac{1}{1-q^k}\right)}$$

Partition function as tau function

(K.T., 2012) **The partition function is related to a tau function $\tau'(s,t,\bar{t})$ of the 2D Toda hierarchy as**

$$Z'(s,t,\bar{t}) = \exp\left(\sum_{k=1}^{\infty} \frac{q^k t_k - \bar{t}_k}{1-q^k}\right) \tau'(s,\iota(t),-\bar{t}).$$

The tau function $\tau'(s,t,\bar{t})$ is defined as

$$\tau'(s,t,\bar{t}) = \langle s|\exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g' \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}\right)|s\rangle,$$

$$g' = q^{W_0/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-W_0/2}.$$

2.2 Outline of part (ii)

Idea of proof of part (ii)

- **Translate building blocks of the fermionic expression to the language of $\mathbb{Z} \times \mathbb{Z}$ matrices.**

- **Use a matrix factorization problem to determine the initial values, at $t = \bar{t} = 0$, of the dressing operators $W, \bar{W}$.**

- **Show that the Lax operators $L, \bar{L}$ take a special form that characterizes the Ablowitz-Ladik hierarchy.**

Matrix representation

- Fermion bilinears and $\mathbb{Z} \times \mathbb{Z}$ matrices are related as

$$X = (x_{ij}) = \sum_{i,j \in \mathbb{Z}} x_{ij} E_{ij} \longleftrightarrow \hat{X} = \sum_{i,j \in \mathbb{Z}} x_{ij} {:}\psi_{-i}\psi_j^*{:}$$

This correspondence can be extended to exponentials of fermion bilinears (Clifford operators).

- Matrix representation of building blocks of $Z'(s, t, \bar{t})$:

$$L_0 = \Delta, \quad W_0 = \Delta^2, \quad H_k = q^{k\Delta}, \quad J_k = \Lambda^k,$$
$$\Gamma_\pm(z) = (1 - z\Lambda^{\pm 1})^{-1}, \quad \Gamma'_\pm(z) = 1 + z\Lambda^{\pm 1}$$

where $\Delta = \sum_{i \in \mathbb{Z}} iE_{ii}$, $\Lambda = \sum_{i \in \mathbb{Z}} E_{i,i+1}$.

Matrix factorization problem

In principle, all solutions of the 2D Toda hierarchy can be captured by the factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1}\bar{W}.$$

U is a $\mathbb{Z} \times \mathbb{Z}$ matrix that represents the generating operator g of a tau function. Given such a matrix U, the problem is to find $\mathbb{Z} \times \mathbb{Z}$ matrices W and $\bar{W}$ that are lower triangular and upper triangular, respectively. $L = W\Lambda W^{-1}$ and $\bar{L} = \bar{W}\Lambda\bar{W}^{-1}$ are the associated Lax operators.

Initial values of $W, \bar{W}$

The generating operator g' of $\tau'(s,t,\bar{t})$ has the matrix representation

$$U' = q^{\Delta^2/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{\Delta}\Gamma'_-(q^{-\rho})\Gamma'_+(q^{-\rho})q^{-\Delta^2/2}.$$

The matrix factorization problem in this case can be solved at the initial time $t = \bar{t} = 0$ explicitly as

$$W(0,0) = q^{\Delta^2/2}\Gamma'_-(Qq^{-\rho})^{-1}\Gamma_-(q^{-\rho})^{-1}q^{-\Delta^2/2},$$

$$\bar{W}(0,0) = q^{\Delta^2/2}Q^{\Delta}\Gamma_+(Qq^{-\rho})\Gamma'_+(q^{-\rho})q^{-\Delta^2/2}.$$

Initial values of Lax operators

The initial values of $L = W\Lambda W^{-1}$ and $\bar{L}^{-1} = \bar{W}\Lambda^{-1}\bar{W}^{-1}$ take a factorized form:

$$L(0,0) = (\Lambda - q^{\Delta})(1 + Qq^{\Delta-1}\Lambda^{-1})^{-1},$$
$$\bar{L}(0,0)^{-1} = -(1 + Qq^{\Delta-1}\Lambda^{-1})(\Lambda - q^{\Delta})^{-1}.$$

Remark **Associativity breaks down partly in the set of $\mathbb{Z} \times \mathbb{Z}$ matrices. In particular,**

$$\bar{L}(0,0) = (\bar{L}(0,0)^{-1})^{-1} \neq -L(0,0).$$

Structure of Lax operators

The factorized form of the initial values of L and $\bar{L}$ is preserved by time evolutions of the 2D Toda hierarchy:

> **(K.T., 2013)** **The Lax operators have the factorized form**
>
> $$L = BC^{-1}, \quad \bar{L}^{-1} = -CB^{-1},$$
>
> $$B = \Lambda - b, \quad C = 1 + c\Lambda^{-1},$$
>
> **b and c are diagonal matrices.**

According to Brini et al., arXiv:1002.0582, this factorized form of L and $\bar{L}$ characterizes the Ablowitz-Ladik hierarchy embedded in the 2D Toda hierarchy.

3. Orbifold melting crystal model

Partition functions

The foregoing results have been extended to $\mathbb{Z}_a \times \mathbb{Z}_b$ orbifold models (K.T., 2014). The deformed partition functions read

$$Z_{a,b}(s,t) = \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1 q^{-\rho},\ldots,p_a q^{-\rho}) s_\lambda(r_1 q^{-\rho},\ldots,r_b q^{-\rho}) \times Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda,s,t)},$$

$$Z'_{a,b}(s,t,\bar{t}) = \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1 q^{-\rho},\ldots,p_a q^{-\rho}) s_{{}^t\lambda}(r_1 q^{-\rho},\ldots,r_b q^{-\rho}) \times Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda,s,t,\bar{t})}.$$

$p_1,\ldots,p_a$ and $r_1,\ldots,r_b$ are new parameters.

(K.T., 2014) **The partition functions are related to 2D Toda tau functions $\tau_{a,b}(s,t,\bar{t})$ and $\tau'_{a,b}(s,t,\bar{t})$ as**

$$Z_{a,b}(s,t) = f_{a,b}(s,t)\tau_{a,b}(s,T,0) = f_{a,b}(s,t)\tau_{a,b}(s,0,-\bar{T})$$

$$Z'_{a,b}(s,t,\bar{t}) = f'_{a,b}(s,t,\bar{t})\tau'_{a,b}(T,-\bar{T}),$$

$$T = (\underbrace{0,\ldots,0}_{a-1},T_1,\underbrace{0,\ldots,0}_{a-1},T_2,\ldots\underbrace{0,\ldots,0}_{a-1},T_k,\ldots),$$

$$\bar{T} = (\underbrace{0,\cdots,0}_{b-1},\bar{T}_1,\underbrace{0,\cdots,0}_{b-1},\bar{T}_2,\ldots,\underbrace{0,\cdots,0}_{b-1},\bar{T}_k,\ldots)$$

where $T_k, \bar{T}_k \propto t_k$ in the first model, and $T_k \propto t_k$, $\bar{T}_k \propto \bar{t}_k$ in the second model. $f_{a,b}(s,t,\bar{t})$ and $f'_{a,b}(s,t,\bar{t})$ are simple functions of $s,t,\bar{t}$.

Lax operators

(K.T., 2014) **The Lax operators of the first model $Z_{a,b}(s,t)$ satisfy the algebraic relation**

$$L^a = D^{-1}\bar{L}^{-b}$$

where D is a constant.

This implies that the underlying integrable structure is the bigraded Toda hierarchy (in the terminology of G. Carlet). When $a = b = 1$, this integrable hierarchy reduces to the 1D Toda hierarchy.

(K.T., 2014) **The Lax operators of the second model $Z'_{a,b}(s,t,\bar{t})$ have the factorized form**

$$L^a = BC^{-1}, \quad \bar{L}^{-b} = DCB^{-1}$$

where D is a constant, and

$$B = \Lambda^a + \beta_1 \Lambda^{a-1} + \beta_a,$$

$$C = 1 + \gamma_1 \Lambda^{-1} + \cdots + \gamma_b \Lambda^{-b},$$

β_i's and γ_j's are diagonal matrices.

This implies that the underlying integrable structure is the rational reduction of bi-degree (a,b) studied by A. Brini et al., arXiv:1401.5725.

Conclusion

All melting crystal models considered here fall into particular reductions of the 2D Toda hierarchy:

Melting crystal model	Integrable structure
ordinary model $Z(s,t)$	1D Toda hierarchy
modified model $Z'(s,t)$	Ablowitz-Ladik hierarchy
orbifold model $Z_{a,b}(s,t)$	bi-graded Toda hierarchy
orbifold model $Z'_{a,b}(s,t,\bar{t})$	(a,b) rational reduction

The relation between the ordinary and modified models resembles that of the Hermitian and unitary matrix models. An interesting open problem is to extend these results to models defined by supersymmetric Schur functions.