Non-degenerate solutions of universal Whitham hierarchy

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Reference

K. Takasaki, T. Takebe and L.-P. Teo, Non-degenerate solutions of universal Whitham hierarchy, J. Phys. A: Math. Theor. **43** (2010), 325205.

1. Introduction

1.1 Dispersionless Toda hierarchy (in Lax form)

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \tilde{t}_n} = \{\tilde{\mathcal{B}}_n, \mathcal{L}\},$$
$$\frac{\partial \tilde{\mathcal{L}}}{\partial t_n} = \{\mathcal{B}_n, \tilde{\mathcal{L}}\}, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{t}_n} = \{\tilde{\mathcal{B}}_n, \tilde{\mathcal{L}}\},$$

where

$$\mathcal{L} = P + u_1 + u_2 P^{-1} + \cdots , \quad \tilde{\mathcal{L}}^{-1} = \tilde{u}_0 P^{-1} + \tilde{u}_1 + \tilde{u}_2 P + \cdots ,$$

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \quad \tilde{\mathcal{B}}_n = (\tilde{\mathcal{L}}^{-n})_{<0},$$

$$\left(\sum_k a_k P^k \right)_{\geq 0} = \sum_{k \geq 0} a_k P^k, \quad \left(\sum_k a_k P^k \right)_{<0} = \sum_{k \geq 0} a_k P^k,$$

$$\{F, G\} = P \left(\frac{\partial F}{\partial P} \frac{\partial G}{\partial s} - \frac{\partial F}{\partial s} \frac{\partial G}{\partial P} \right) \qquad (P \leftrightarrow e^{\partial_s},\ s\text{: lattice coord.})$$

1.2 Problem

Find a class of general solutions of the dispersionless Toda hierarchy in a geometric perspective.

Cf. General solutions of the KP and Toda hierarchies are described by a geometric structure:

KP hierarchy — Sato Grassmannian $\mathrm{Gr} \simeq \mathrm{GL}(\infty)/P$

Toda hierarchy — $GL(\infty)$ itself

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \langle s | e^{J(\boldsymbol{t})} g e^{-\tilde{J}(\tilde{\boldsymbol{t}})} | s \rangle, \quad g \in \mathrm{GL}(\infty) \quad \text{(ferminonic formula)}$$

This geometric description stems from a linear structure behind the nonlinear systems. It seems hopeless to seek such a structure in dispersionless integrable hierarchies. An alternative approach is a nonlinear Riemann-Hilbert problem.

1.3 Nonlinear Riemann-Hilbert problem

$$\tilde{\mathcal{L}} = f(\mathcal{L}, \mathcal{M}), \quad \tilde{\mathcal{M}} = g(\mathcal{L}, \mathcal{M}),$$

where $f = f(z, w)$ and $g = g(z, w)$ are assumed to satisfy the equation

$$z\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}\right) = f,$$

and $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are assumed to have such an expansion as

$$\mathcal{M} = \sum_{n=1}^{\infty} nt_n \mathcal{L}^n + t_0 + \sum_{n=1}^{\infty} v_n \mathcal{L}^n,$$

$$\tilde{\mathcal{M}} = -\sum_{n=1}^{\infty} nt_{-n} \tilde{\mathcal{L}}^{-n} + t_0 - \sum_{n=1}^{\infty} v_{-n} \tilde{\mathcal{L}}^n \qquad (t_0 = s)$$

1.3 Nonlinear Riemann-Hilbert problem (cont'd)

(i) f and g give a two-dimensional canonical transformation (symplectic map) $(z, w) \mapsto (\tilde{z}, \tilde{w}) = (f(z, w), g(z, w))$ with respect to the symplectic form $\dfrac{dz \wedge dw}{z}$.

(ii) This Riemann-Hilbert problem (also referred to as generalized string equations) is a kind of factorization problem in a group of such symplectic maps (hence a genuinely nonlinear problem).

(iii) $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are the so called Orlov-Schulman functions.

Ref: T & Takebe, Lett. Math. Phys. 23 (1991), 205–214; Reviews in Mathematical Physics 7 (1995), 743–808.

Unfortunately, there is no general method for solving this nonlinear Riemann-Hilbert problem efficiently.

1.4 Teo's idea

L.-P. Teo proposed to consider the symplectic map $(z,w) \mapsto (f(z,w), g(z,w))$ defined (implicitly) by a generating function $H(z,\tilde{z})$ as

$$w = zH_z(z,\tilde{z}), \quad \tilde{w} = -\tilde{z}H_{\tilde{z}}(z,\tilde{z}).$$

The nonlinear Riemann-Hilbert problem thereby turns into a more tractable form

$$\mathcal{M} = \mathcal{L}H_z(\mathcal{L},\tilde{\mathcal{L}}), \quad \tilde{\mathcal{M}} = -\tilde{\mathcal{L}}H_{\tilde{z}}(\mathcal{L},\tilde{\mathcal{L}}),$$

where $H_z(z,\tilde{z}) = \partial H(z,\tilde{z})/\partial z$, $H_{\tilde{z}}(z,\tilde{z}) = \partial H(z,\tilde{z})/\partial \tilde{z}$.

Remark: If $H(z,\tilde{z}) = z\tilde{z}^{-1}$, then the problem reduces to the string equations $\mathcal{M} = \mathcal{L}\tilde{\mathcal{L}}^{-1} = \tilde{\mathcal{M}}$ of a growth model extensively studied in the last decade by Krichever, Marshakov, Mineev-Weinstein, Wiegmann, Zabrodin, ...

2. Non-degenerate solutions of dispersionless Toda hierarchy

Ref: L.-P. Teo, Commun. Math. Phys. **297** (2010), 447–474.

2.1 Twisted Riemann-Hilbert problem

Given a holomorphic function $H(z, \tilde{z})$ with the non-degeneracy condition $H_{z\tilde{z}}(z, \tilde{z}) \neq 0$, find four functions $\mathcal{L} = \mathcal{L}(P)$, $\mathcal{M} = \mathcal{M}(P)$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(P)$, $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(P)$ of a complex variable P (and extra variables t_n, $n \in \mathbf{Z}$) with the following properties:

(i) $\mathcal{L}$ and $\mathcal{M}$ are holomorphic functions in the punctured disk $1 < |P| < \infty$, $\mathcal{L}$ being univalent therein, and have a Laurent expansion of the form

$$\mathcal{L} = P + \sum_{n=1}^{\infty} u_n P^{-n+1}, \quad \mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + t_0 + \sum_{n=1}^{\infty} v_n \mathcal{L}^n.$$

2.1 Twisted Riemann-Hilbert problem (cont'd)

(ii) $\tilde{\mathcal{L}}^{-1}$ and $\tilde{\mathcal{M}}$ are holomorphic functions in the punctured disk $0 < |P| < 1$, $\tilde{\mathcal{L}}$ being univalent therein, and have a Laurent expansion of the form

$$\tilde{\mathcal{L}}^{-1} = \sum_{n=0}^{\infty} \tilde{u}_n P^{n-1}, \quad \tilde{\mathcal{M}} = -\sum_{n=1}^{\infty} n t_{-n} \tilde{\mathcal{L}}^{-n} + t_0 - \sum_{n=1}^{\infty} v_{-n} \tilde{\mathcal{L}}^{n}.$$

(iii) These functions can be analytically continued to a neighborhood of the unit circle $|P| = 1$ and satisfy the functional equations

$$\mathcal{M} = \mathcal{L} H_z(\mathcal{L}, \tilde{\mathcal{L}}), \quad \tilde{\mathcal{M}} = -\tilde{\mathcal{L}} H_{\tilde{z}}(\mathcal{L}, \tilde{\mathcal{L}}).$$

2.2 Solution of Riemann-Hilbert problem

An equivalent expression of the generalized string equations:

$$nt_n = \frac{1}{2\pi i}\oint_{|P|=1} H_z(\mathcal{L}(P),\tilde{\mathcal{L}}(P))\mathcal{L}(P)^{-n}d\mathcal{L}(P),$$

$$nt_{-n} = \frac{1}{2\pi i}\oint_{|P|=1} H_{\tilde{z}}(\mathcal{L}(P),\tilde{\mathcal{L}}(P))\tilde{\mathcal{L}}(P)^{n}d\tilde{\mathcal{L}}(P),$$

$$t_0 = \frac{1}{2\pi i}\oint_{|P|=1} H_z(\mathcal{L}(P),\tilde{\mathcal{L}}(P))d\mathcal{L}(P) = -\frac{1}{2\pi i}\oint_{|P|=1} H_{\tilde{z}}(\mathcal{L}(P),\tilde{\mathcal{L}}(P))d\tilde{\mathcal{L}}(P),$$

$$v_n = \frac{1}{2\pi i}\oint_{|P|=1} H_z(\mathcal{L}(P),\tilde{\mathcal{L}}(P))\mathcal{L}(P)^{n}d\mathcal{L}(P),$$

$$v_{-n} = \frac{1}{2\pi i}\oint_{|P|=1} H_{\tilde{z}}(\mathcal{L}(P),\tilde{\mathcal{L}}(P))\tilde{\mathcal{L}}(P)^{-n}d\tilde{\mathcal{L}}(P), \quad n = 1, 2, \cdots$$

Remark: If $H(z,\tilde{z}) = z/\tilde{z}$, the contour integrals reduce to harmonic moments of a conformal map.

2.2 Solution of Riemann-Hilbert problem (cont'd)

Theorem (L.-P. Teo)
(i) t_n's give a system of local coordinates on the space $\mathcal{Z}$ of the pairs $(\mathcal{L}, \tilde{\mathcal{L}})$ of conformal maps. In other words, the period map $\Phi : (\mathcal{L}, \tilde{\mathcal{L}}) \mapsto (t_n)_{n\in\mathbf{Z}}$ is locally invertible.
(ii) The composition $\Psi \circ \Phi^{-1}$ of another period map $\Psi : (\mathcal{L}, \tilde{\mathcal{L}}) \mapsto (v_n)_{n\neq 0}$ and the inverse period map Φ^{-1} gives a solution of the Riemann-Hilbert problem (hence, of the dispersionless Toda hierarchy).
(iii) The associated free energy (dispersionless tau function) $\mathcal{F}$ is obtained explicitly in terms of contour integrals.

These solutions are called non-degenerate solutions. They form a class of general solutions of the dispersionless Toda hierarchy.

3. Universal Whitham hierarchy

Ref: I.M. Krichever, Comm. Pure. Appl. Math. **47** (1994), 437–475.

3.1 Lax functions

The Lax functions $z_\alpha(p)$, $\alpha = 0, 1, \ldots, M$, are functions with Laurent expansions of the form

$$z_0(p) = p + \sum_{j=2}^{\infty} u_{0j} p^{-j+1},$$

$$z_a(p) = \frac{r_a}{p - q_a} + \sum_{j=1}^{\infty} u_{aj}(p - q_a)^{j-1} \quad (a = 1, \ldots, M),$$

in a neighborhood of $p = \infty$ and $p = q_a$, respectively. The coefficients $u_{\alpha j}$ ($r_a = u_{a0}$) and the centers q_a are dynamical variables.

3.2 Lax equations

The hierarchy has $1 + M$ series of time evolutions with time variables t_{0n}, $n = 1, 2, \ldots$ and t_{an}, $a = 1, \ldots, M$, $n = 0, 1, 2, \ldots$. The time evolutions of the Lax functions are defined by the Lax equations

$$\partial_{\alpha n} z_\beta(p) = \{\Omega_{\alpha n}(p),\, z_\beta(p)\}, \quad \partial_{\alpha n} = \partial/\partial t_{\alpha n},$$

with respect to the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial t_{01}} - \frac{\partial f}{\partial t_{01}}\frac{\partial g}{\partial p}.$$

Remark: This Poisson bracket is an analogue of the Poisson bracket used in the formulation of the dispersionless KP hierarchy.

3.2 Lax equations (cont'd)

$\Omega_{0n}(p)$ and $\Omega_{an}(p)$, $n = 1, 2, \cdots$, are polynomials in p and $(p - q_a)^{-1}$ of the form

$$\Omega_{0n}(p) = p^n + nu_{02}p^{n-2} + \cdots + *,$$

$$\Omega_{an}(p) = \frac{r_a^n}{(p - q_a)^n} + \cdots + \frac{*}{p - q_a}$$

that give the singular part of $z_0(p)^n$ and $z_a(p)^n$, i.e.,

$$z_0(p)^n = \Omega_{0n}(p) + O(p^{-1}) \quad (p \to \infty),$$

$$z_a(p)^n = \Omega_{an}(p) + O(1) \quad (p \to q_a)$$

$\Omega_{a0}(p)$'s are exceptional and defined as

$$\Omega_{a0}(p) = -\log(p - q_a).$$

3.3 Relation to dispersionless Toda hierarchy

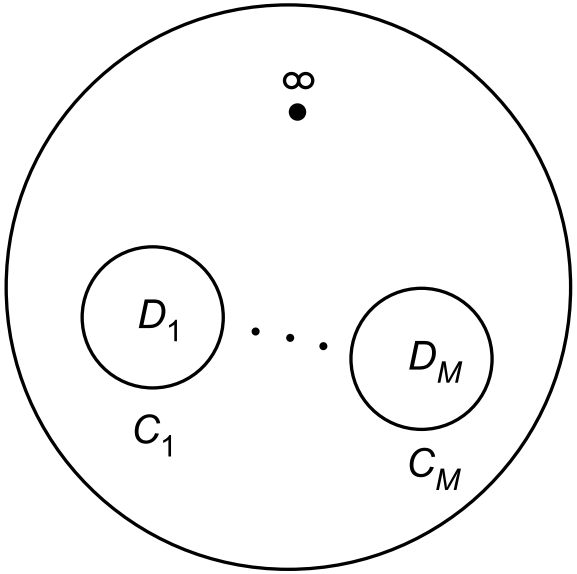
The dispersionless Toda hierarchy amounts to the case where $M = 1$ (two marked points):

$$z_0(p) = \mathcal{L}(P), \quad z_1(p) = \tilde{\mathcal{L}}(P)^{-1}, \quad p = P + u_1,$$

$$t_{0n} = t_n, \quad t_{1n} = t_{-n}, \quad t_{10} = t_0.$$

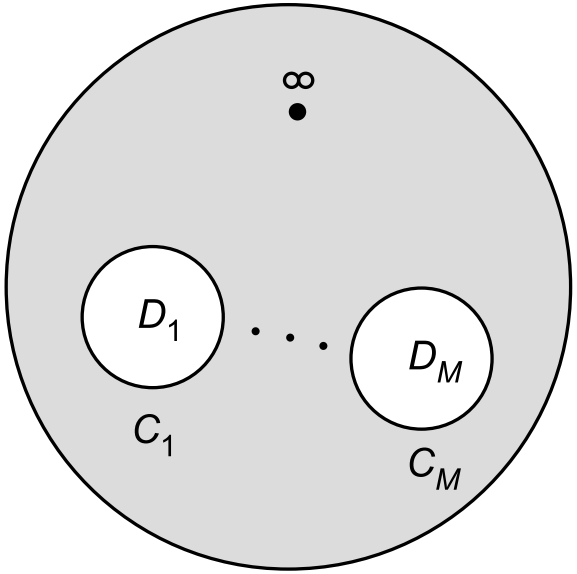
4. Non-degenerate solutions of universal Whitham hierarchy

4.1 Twisted Riemann-Hilbert problem



Choose disjoint simple closed curves $C_1, \cdots, C_M$ in the finite part of the Riemann sphere $\mathbf{CP}^1$. Let $D_1, \cdots, D_M$ denote their inside domains. For given M functions $H_a(z_0, z_a)$, $a = 1, \cdots, M$, with the non-degeneracy conditions $H_{a,z_0 z_a}(z_0, z_a) \neq 0$, find $2 + 2M$ functions $z_\alpha(p), \zeta_\alpha(p)$, $\alpha = 0, 1, \cdots, M$ with the following properties (i), (ii), (iii):

4.1 Twisted Riemann-Hilbert problem (cont'd)



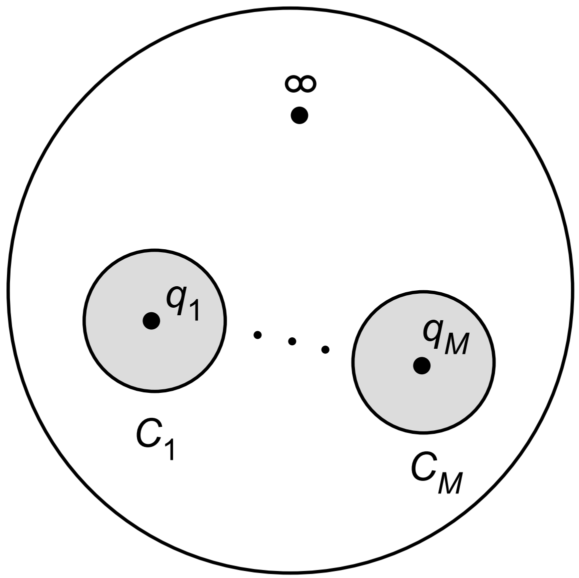
(i) $z_0(p)$ and $\zeta_0(p)$ are holomorphic functions on $\mathbf{C} \setminus (D_1 \cup \ldots \cup D_M)$, $z_0(p)$ is univalent therein and, as $p \to \infty$,

$$z_0(p) = p + O(p^{-1}),$$

$$\zeta_0(p) = \sum_{n=1}^{\infty} n t_{0n} z_0(p)^{n-1} + \frac{t_{00}}{z_0(p)} + O(p^{-2}),$$

where $t_{00} = -t_{10} - \cdots - t_{M0}$.

4.1 Twisted Riemann-Hilbert problem (cont'd)



(ii) $z_a(p)$ and $\zeta_a(p)$, $a = 1, \cdots, M$, are holomorphic functions on D_a punctured at a point $q_a \in D_a$, $z_a^{-1}(p)$ is univalent on D_a and, as $p \to q_a$,

$$z_a(p) = \frac{r_a}{p - q_a} + O(1),$$

$$\zeta_a(p) = \sum_{n=1}^{\infty} n t_{an} z_a(p)^{n-1} + \frac{t_{a0}}{z_a(p)} + O((p - q_a)^2).$$

(iii) For $a = 1, \ldots, M$, the four functions $z_0(p), \zeta_0(p), z_a(p), \zeta_a(p)$ can be analytically continued to a neighborhood of C_a and satisfy the functional equations

$$z_a(p) = H_{a,z_0}(z_0(p), \zeta_0(p)), \quad \zeta_a(p) = -H_{a,z_a}(z_0(p), \zeta_0(p)).$$

4.2 Solution by inversion of period map

Period map $\Phi : (z_\alpha)_{\alpha=0,\cdots,M} \mapsto (t_{0n}, t_{an}, t_{a0})_{n=1,2,\cdots,\ a=1,\cdots,M}$ on the space $\mathcal{Z}$ of $(1+M)$-tuples of conformal maps defined by

$$nt_{0n} = \sum_{a=1}^{M} \frac{1}{2\pi i} \oint_{C_a} H_{a,z_0}(z_0(p), z_a(p)) z_0(p)^{-n} dz_0(p),$$

$$nt_{an} = \frac{1}{2\pi i} \oint_{C_a} H_{a,z_a}(z_0(p), z_a(p)) z_a(p)^{-n} dz_a(p),$$

$$t_{a0} = \frac{1}{2\pi i} \oint_{C_a} H_{a,z_a}(z_0(p), z_a(p)) dz_a(p),$$

4.2 Solution by inversion of period map (cont'd)

Another period map $\Psi : (z_\alpha)_{\alpha=0,\cdots,M} \mapsto (v_{\alpha n})_{n=1,2,\cdots,\ \alpha=0,\cdots,M}$ defined by

$$v_{0n} = \sum_{a=1}^{M} \frac{1}{2\pi i} \oint_{C_a} H_{a,z_0}(z_0(p), z_a(p)) z_0(p)^n dz_0(p),$$

$$v_{an} = \frac{1}{2\pi i} \oint_{C_a} H_{a,z_a}(z_0(p), z_a(p)) z_a(p)^n dz_a(p).$$

Theorem (T, Takabe & Teo) (i) $t_{\alpha n}$'s give a system of local coordinates on $\mathcal{Z}$, and Φ is locally invertible.
(ii) $\Psi \circ \Phi^{-1}$ gives a solution of the Riemann-Hilbert problem (hence, of the universal Whitham hierarchy).
(iii) The free energy $\mathcal{F}$ can be obtained explicitly in terms of contour integrals.

5. Dispersive analogue of non-degenerate solutions

5.1 $M = 1$

A dispersive analogue of the non-degenerate solutions in the case of $M = 1$ can be found in Adler and van Moerbeke's work on a system of bi-orthogonal polynomials and their relation to the Toda hierarchy (on a semi-infinite lattice $\mathbf{Z}_{\geq 0}$).

Ref: M. Adler and P. van Moerbeke, Comm. Pure. Appl. Math. **50** (1997), 241–290.

5.1 $M = 1$ (cont'd)

(i) (Adler & van Moerbeke) The system of bi-orthogonal polynomials $P_s(z) = z^s + \cdots$, $Q_s(\tilde{z}) = c_s \tilde{z}^s + \cdots$, $s = 0, 1, \cdots$,

$$\int_C \int_{\tilde{C}} dz d\tilde{z} P_i(z) Q_j(\tilde{z}) e^{U(z,\tilde{z})} = 0 \quad \text{for } i \neq j,$$

gives a solution of the Toda hierarchy if $U(z, \tilde{z})$ is deformed as

$$U(z, \tilde{z}) \to U(z, \tilde{z}) + \sum_{k=1}^{\infty} t_k z^k - \sum_{k=1}^{\infty} \tilde{t}_k \tilde{z}^k.$$

(ii) (T, Takebe & Teo, unpublished) In the large-s limit, this solution turns into the non-degenerate solution of the dispersionless Toda hierarchy with the generating function

$$H(z, \tilde{z}) = -U(z, \tilde{z}^{-1}).$$

5.2 General M

(i) The system of multiple bi-orthogonal polynomials
$P_{\boldsymbol{\nu}}(z_0) = z_0^{\nu_0} + \cdots$, $Q_{a,\boldsymbol{\nu}}(z_a) = c_{a,\boldsymbol{\nu}} z_a^{\nu_s - 1} + \cdots$ $(a = 1, \cdots, M,$
$\nu = (\nu_1, \cdots, \nu_M) \in \mathbf{Z}_{\geq 0}^M$, $\nu_0 = \nu_1 + \cdots + \nu_M)$,

$$\int_{C_0} \int_{C_a} dz_0 dz_a P_{\boldsymbol{\nu}}(z_0) z_a^j e^{U_a(z_0,z_a)} = 0 \quad \text{for } j < \nu_a,$$

$$\int_{C_0} \int_{C_a} dz_0 dz_a z_0^i Q_{a,\boldsymbol{\nu}}(z_0) e^{U_a(z_0,z_a)} = \delta_{i,\nu_0} \quad \text{for } i \leq \nu_0,$$

Ref. M. Adler, P. Vanhaecke & P. van Moerbeke, Commun. Math. Phys. **286** (2008), 1–38.

gives a solution of the $1 + M$-component charged KP hierarchy (if U_a's are suitably deformed by time variables).

5.2 General M (cont'd)

(ii) Conjecture: In the large-$\boldsymbol{\nu}$ limit, this solution turns into the non-degenerate solution of the universal Whitham hierarchy with the generating functions

$$H_a(z_0, z_a) = -U_a(z_0, z_a), \quad a = 1, \cdots, M.$$

6. Conclusion

- The notion of non-degenerate solutions of the dispersionless Toda hierarchy is based on a non-standard (twisted) Riemann-Hilbert problem.
- The existence of these solutions is formulated in a geometric language.
- This result can be generalized to the universal Whitham hierarchy.
- Bi-orthogonal polynomials give a dispersive analogue of these solutions in the Toda case. Presumably, multiple bi-orthogonal polynomials will give a generaization to the Whitham case.