ANALYTIC EXPRESSION OF VOROS COEFFICIENTS AND ITS APPLICATION TO WKB CONNECTION PROBLEM

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1. Introduction

Usually, the WKB method starts from formal solutions (WKB or Liouville-Green solutions) expanded in powers of the Planck constant, and connects these solutions by asymptotic matching at turning points. Voros [V] proposed a resummation prescription of these formal calculations, and argued that his results should be deeply related with Ecalle's theory of "resurgent functions." Further progress along that line has been made by F. Pham and his coworkers [DDP]. We report another approach based upon an idea of Olver [O].

Olver's method for the WKB connection problem, unlike the asymptotic matching at turning points, is based upon analysis at points at infinity. Naturally, one needs semi-global information on a set of solutions for which to consider the connection problem. Olver's idea is very intriguing, because it directly gives an exact connection formula without using any approximation. His connection formulas, however, contain strange quantities whose analytic properties were fairly obscure at that moment; Olver gave only qualitative results on these quantities. From our present standpoint, it it not hard to notice that these quantities (which should be called the "Olver coefficients" or something like that) are nothing else than the "Voros coefficients" in the terminology of Ecalle and Pham. This inspires us with the hope to find some new analytical expressions to the Voros coefficients.

Our basic strategy is just to combine Olver's idea with basic techniques in scattering theory [AKNS], [ZS]. We first convert the problem, as Olver does, to another linear problem by the so called Liouville transformation. This is a quite standard technique [H], and also used in the derivation of Liouville-Green formal solutions. From an analytical point of view, the transformed equation resembles a linear problem in scattering theory; the potential now decays at the rate of inverse squared of the distance from the origin. This fact also lies in the heart of Olver's method, because the definition of his coefficients relies upon that property. Olver's book stops at this stage, but we attempt to go forward further. From the point of view of scattering theory, Voros coefficients may be identified with the "a"-coefficient (inverse transmission coefficients) that is defined along with the "b"-coefficients. (The ratio b/a is called the reflection coefficients.) Further, it is well known in scattering theory that "a" and "b" are connected with Jost solutions by a simple integral relation. From these observations, we are naturally led to an iterated integral series expansion of Voros coefficients.

Under some additional condition, one can further rewrite this iterated integral series into a Laplace integral; the integrand has again an iterated integral series expansion. This part is largely inspired by work of Grigis and Gérard [GG]. Since such a Laplace integral representation is a basic object in the theory of resurgent functions as well as in the work of Voros, we expect to deduce from Olver's connection formulas some new insight into the Ecalle-Voros theory. We shall show only a few examples anticipating further progress in that direction.

The last section is devoted to issues beyond the scope of the second order Sturm-Liouville problem. These are still speculative, but appear to offer various interesting material.

I wish to express my sincere gratitude to Professor Toshihiko Nishimoto for drawing my attention to Olver's work, and to Professor Nobuyuki Tose for encouragement as well as his help for collecting material unavailable in Japan. I am also indebted to Professors Alain Grigis, Kazuhiko Aomoto, Frederic Pham, Koichi Uchiyama and Masafumi Yoshino for discussions and useful information. A more detailed report on the results announced below will be published elsewhere.

2. Second order linear problem and Liouville transformation

We consider the linear equation

$$\psi_{qq} = \lambda^2 f(q)\psi, \tag{1}$$

in the complex plane, where $f(q)$ is a complex analytic function with some nice analytical properties (see below), and λ is a nonzero complex parameter. The subscript "qq" stands for the second derivative, $\psi_{qq} = d^2\psi/dq^2$. We mostly assume that λ is a real positive number; however, we shall see that extending it to complex values becomes crucial later on.

The above linear problem can be converted into a new equation of the form

$$\phi_{ss} = (\lambda^2 + h)\phi, \tag{2}$$

$$h \underset{\text{def}}{=} -f^{-3/4}(f^{-1/4})_{qq}, \tag{3}$$

by the transformation of variables

$$\phi = f^{1/4}\psi, \quad s = \int_{q_0}^{q} dq' f(q')^{1/2}. \tag{4}$$

The determination of $f^{1/4}$ and the point q_0 are suitably chosen subject to the situation in consideration. This transformation is known for years (since, probably, the days of Liouville). We call it the "Liouville transformation," and the s-plane the "Liouville plane." Actually, this can be slightly generalized to the linear equation

$$\psi_{qq} = \left(\lambda^2 f(q) + g(q)\right)\psi, \tag{1'}$$

and leads to the same equation as (2) except that h is then given by

$$h \underset{\text{def}}{=} g/f - f^{-3/4}(f^{-1/4})_{qq}. \tag{3'}$$

Equation (2) may be viewed as a "perturbation" of the equation with $h = 0$, the latter being readily solved by the exponential functions $e^{\pm\lambda s}$. This point of view lies in the heart of the so called "Liouville-Green approximation." For further analysis, we assume that

$$h = O(|s|^{-2}) \quad \text{as } |s| \to \infty. \tag{5}$$

A typical case is the following.

Proposition. *Condition (5) is satisfied if $f(q)$ is a polynomial and $g(q) = 0$.*

Roughly, (5) means that the linear problem on the Liouville plane is of "scattering type" as opposed to the original problem on the q-plane. For our actual analysis of connection problems, condition (5) may be further relaxed; it is frequently sufficient to require the decay property only in a subdomain of the Liouville plane or simply along several curves with both ends at infinity.

Of course, even if (5) is satisfied, the Liouville transformation is a somewhat subtle thing, because $h = h(s)$ is in general a multi-valued function on the s-plane with branch point singularities at the image of zero's of $f(q)$, i.e., at "turning points." It should be noted that the multi-valuedness comes only from that of the inverse map $q = q(s)$ of the Liouville transformation $s = s(q)$. On the q-plane, h in (3) is given by

$$h = \frac{4ff_{qq} - 5{f_q}^2}{16f^3}, \tag{6}$$

hence it has poles at the zeros of $f(q)$, but no branch point singularities. It is the multi-valuedness of $q = q(s)$ that makes $h(s)$ a multi-valued function on the Liouville plane.

The sheet structure of $h = h(s)$ is, in general, very complicated. The following general notions are convenient to understand the geometric situation.

Definition.
- Zeros of $f(q)$ are called "turning points."
- A "Stokes curve" is a curve that starts from a turning point and whose image on the Liouville plane is a half-line or a segment parallel to the real axis.
- A "Principal (or anti-Stokes) curve" is a curve that starts from a turning point and whose image on the Liouville plane is a half-line or a segment parallel to the imaginary axis.

More precisely, these are the definitions for the case of $\lambda > 0$; if $\arg\lambda \neq 0$, the part "parallel to the real (imaginary) axis" in the above definition should be modified as "rotated from the real (imaginary) axis by an angle of $-\arg\lambda$."

For illustration, we now consider the case of a harmonic oscillator (see Fig. 1):

$$f(q) = q^2 - E, \quad E > 0 \tag{7}$$

with $\lambda > 0$. In this case, there are two turning points at $q = \pm E^{1/2}$, six Stokes curves and five principal curves. The four points

$$\begin{aligned} \infty_1 &: \ q = -\infty \\ \infty_2 &: \ q = +i\infty \\ \infty_3 &: \ q = -i\infty \\ \infty_4 &: \ q = +\infty \end{aligned}$$

at infinity will play an important role in the formulation of our connection problem. We choose paths for reaching these points as indicated in Fig. 1.

3. Solutions of Liouville-Green form

If one compares the linear problem on the Liouville plane with the situation of potential scattering theory, it would be natural to seek for solutions of the form

$$\phi_\pm = w_\pm e^{\pm s\lambda}. \tag{8}$$

The amplitude part $w_\pm$ should be, in some sense, close to the unity. This can achieved in two different ways.

3.1. Formal solutions

One way is to convert the linear equation by the well known transformation

$$v \underset{\text{def}}{=} (\log \phi)_s \tag{9}$$

to the Riccati equation

$$v_s + v^2 = \lambda^2 + h. \tag{10}$$

Substitution of a formal expansion

$$v_\pm(s, \lambda) = \pm\lambda + \sum_{n=1}^{\infty} v_{\pm,n}(s)\lambda^{1-n} \tag{11}$$

give rise to a set of relations that determine the coefficients recursively as:

$$v_{\pm,1} = 0, \quad v_{\pm,2} = \pm h/2, \quad v_{\pm,3} = h_s/4, \ldots.$$

Solving (9) as

$$\phi_\pm = \exp \int^s ds' v_\pm(s', \lambda), \tag{12}$$

one obtains a pair of solutions as desired, and this is essentially the same as the so called Liouville-Green (or WKB) solutions. (The usual construction is done on the q-plane.)

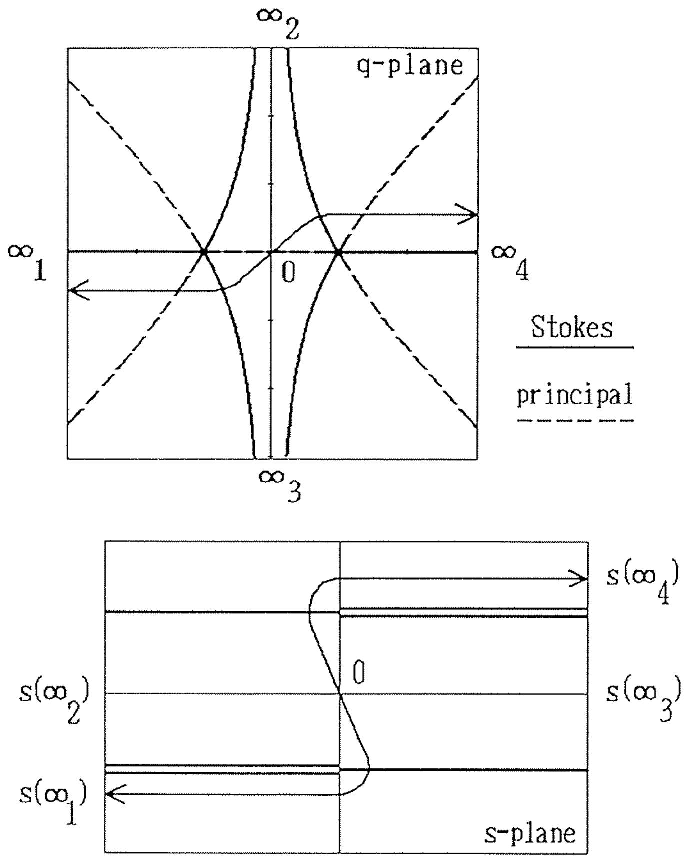

Fig. 1. q-plane and s-plane for $f(q) = q^2 - E,\ E > 0$

These solutions are, however, only *formal* power series of λ, and do not converge in general.

3.2. Analytic solutions

Another way is similar to the usual method in scattering theory. In that approach, one converts the differential equations

$$\left(\frac{d^2}{ds^2} \pm 2\lambda \frac{d}{ds}\right) w_\pm = h w_\pm \tag{13}$$

for $w_\pm$ into the integral equations

$$w_+(s,\lambda) = 1 + \int_{s_0^+}^{s} dt \frac{1 - e^{2(t-s)\lambda}}{2\lambda} h(t) w_+,$$

$$w_-(s,\lambda) = 1 + \int_{s}^{s_0^-} dt \frac{1 - e^{2(s-t)\lambda}}{2\lambda} h(t) w_+, \tag{14}$$

where $s_0^{\pm}$ are some fixed points. These integral equations can be solved by successive substitution (i.e., Neumann series):

$$w_+(s,\lambda) = 1 + \sum_{n=1}^{\infty} w_+^{(n)}(s,\lambda),$$

$$w_+^{(n)}(s,\lambda) \underset{\text{def}}{=} \int_{s_0^+}^{s=t_{n+1}} dt_n \int_{s_0^+}^{t_n} dt_{n-1} \cdots \int_{s_0^+}^{t_2} dt_1 \prod_{i=1}^{n} \frac{h(t_i)(1 - e^{2(t_i - t_{i+1})\lambda})}{2\lambda},$$

$$w_-(s,\lambda) = 1 + \sum_{n=1}^{\infty} w_-^{(n)}(s,\lambda),$$

$$w_-^{(n)}(s,\lambda) \underset{\text{def}}{=} \int_{s=t_0}^{s_0^-} dt_1 \int_{t_1}^{s_0^-} dt_2 \cdots \int_{t_{n-1}}^{s_0^-} dt_n \prod_{i=1}^{n} \frac{(1 - e^{2(t_{i-1} - t_i)\lambda}) h(t_i)}{2\lambda}. \tag{15}$$

Note that the integral equations implement, as well, some initial condition at the point $s_0^{\pm}$. In our treatment of global connection problems, we choose $s_0^{\pm}$ to be points at infinity (just as in the construction of Jost solutions in scattering theory). It is exactly at this stage that decay property (5) play a crucial role. Further, we have to select a suitable path of integration so that the exponential functions in (15) have a uniform bound; the above Neumann series will then converge. In view of these requirements, we now assume that

- the points $s_0^{\pm}$ are put at infinity with Re $s_0^{\pm} = \mp\infty$;
- the integrals in (14) and (15) are along such paths $\Gamma_{\pm}(s)$ that starts from $s_0^{\pm}$ and ends at q;
- along $\Gamma_{\pm}(s)$, $\pm$Re s is monotonously increasing. (Such a path is said to be "progressive.")

Under that situation, one has indeed the uniform bound

$$|1 - e^{2(t_i - t_{i+1})\lambda}| \le 2,$$

and one can prove the following basic result.

Proposition. *$w_{\pm}^{(n)}$ satisfy the inequality*

$$|w_{\pm}^{(n)}(s,\lambda)| \le \frac{V_{\pm}(s)^n}{n!|\lambda|}, \quad V_{\pm}(s) \underset{\text{def}}{=} \int_{\Gamma_{\pm}(s)} dt |h(t)|. \tag{16}$$

Therefore the Neumann series converge and obey the inequality

$$|w_{\pm}(s,\lambda) - 1| \le \exp(V_{\pm}(s)/|\lambda|) - 1. \tag{17}$$

Solutions of the original linear equation on the q-plane are now given by

$$\psi_\pm(q,\lambda) = w_\pm(s(q),\lambda)(ds(q)/dq)^{-1/2}e^{\pm s(q)\lambda}, \tag{18}$$

which we call exact solutions of the Liouville-Green form. If $s_0^\pm$ are the images of points $\infty_\pm$ at infinity of the q-plane, these solution are exponentially small ("recessive" in the terminology of the WKB method) in a neighborhood of $\infty_\pm$. Such solutions play a basic role in our treatment of the connection problem; a more precise situation is presented in the beginning of the next section. If $\arg\lambda \neq 0$, all conditions on Re $s_0^\pm$ and Re s should be replaced to Re $s_0^\pm\lambda$ and Re $s\lambda$, and everything goes in a quite parallel way.

3.3. Asymptotic expansion linking formal and analytic solutions

If the above analytic solutions have asymptotic expansion as $\lambda \to \infty$ (in a sector, for example), the asymptotic series should agree, up to a factor independent of q, with a formal solution described in Subsection 3.1 (because asymptotic expansion, if exists, is unique). The Laplace integral representation discussed later on indeed yields such asymptotic expansion.

4. Formulation of connection problem

4.1. General setting

In the following, we consider the case of $\lambda > 0$ alone. This is just for simplicity of presentation; everything carries over to the case of $\arg\lambda \neq 0$.

To consider the WKB connection problem, we first fix a cut-sheet of $f(q)^{1/2}$ and a determination of $s(q)$ and $(ds(q)/dq)^{-1/2}(= f(q)^{-1/4})$, though this is more or less for convenience of calculations. We then select a set of points ∞_I $(I = 1, 2, \ldots)$ and attach to them an exact solution of the Liouville-Green form,

$$\psi_I(q,\lambda) = w_I(s(q),\lambda)(ds/dq)^{-1/2}\exp\epsilon_I s(q)\lambda, \tag{19}$$

where ϵ_I is a sign factor, taking values in ± 1, and subject to the condition that

$$\mathrm{Re}\ \epsilon_I s(q) = -\infty \quad \text{at} \quad q = \infty_I. \tag{20}$$

For each solution, we take a domain D_I comprised of points q with a path $C_I(q)$ that links ∞_I with q (avoiding turning points, of course) and deforms continuously as q moves. Further, its image $\Gamma_I(s(q)) =_{\mathrm{def}} s(C_I(q))$ on the Liouville plane is assumed to be progressive with respect to Re $\epsilon_I s$. The amplitude part $w_I(s,\lambda)$ is given, for each point s of $\Delta_I =_{\mathrm{def}} s(D_I)$, by the Neumann series in the previous section. Such a domain D_I can be chosen farly large, as illustrated below; this fact is crucial in Olver's method.

4.2. Example: harmonic oscillator

For illustration, we now consider again the case of the harmonic oscillator with $E > 0$ and $\lambda > 0$. To fix a determination of s and $(ds/dq)^{-1/2}$, we cut the q-plane along two curves that start, respectively, from the two turning points and tend to infinity in the second and fourth quadrant of the q-plane (see Fig. 2).

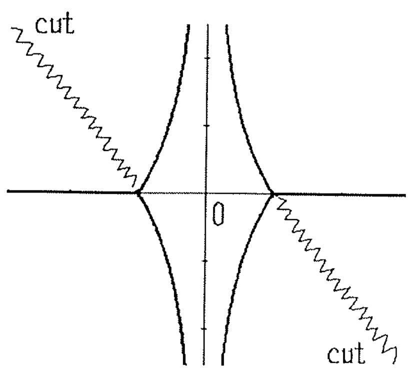

Fig. 2. Cut sheet on the q-plane for the harmonic oscillator

As a set of reference points at infinity, we take the four points $\infty_1, \ldots, \infty_4$ as mentioned in Section 2, and construct four solutions of the Liouville-Green form:

$$\psi_I = w_I(ds/dq)^{-1/2}e^{s\lambda} \quad (I = 1, 2),$$
$$\psi_I = w_I(ds/dq)^{-1/2}e^{-s\lambda} \quad (I = 3, 4). \tag{21}$$

For each ψ_I, one can determine a maximal domain D_I of points q that can be reached by a progressive path $C_I(q)$ from ∞_I. Let us call P_{IJ} the principal curve between ∞_I and ∞_J.

- D_1 is the complement of the union of the interval $[-E^{1/2}, +E^{1/2}]$ and the domain on the right side of $P_{24} \cup P_{34}$
- D_2 is the whole plane cut along P_{13} and P_{34}.
- D_3 is the whole plane cut along P_{12} and P_{24}.
- D_4 is the complement of the union of the interval $[-E^{1/2}, +E^{1/2}]$ and the domain on the left side of $P_{12} \cup P_{13}$.

4.3. Olver's basic idea

In Olver's method to the WKB connection problem, connection formulas are given as a collection of linear relations

$$\psi_I = c_{IJ}\psi_J + c_{IK}\psi_K. \tag{22}$$

among suitable triples $\{\psi_I, \psi_J, \psi_K\}$ that we call "fundamental triplets." We say a triplet is fundamental if each of the three reference points $\infty_I, \infty_J, \infty_K$ at infinity can be reached from the other two through the corresponding two domains in D_I, D_J, D_K, in other words, if there are three progressive paths

$$C_{IJ} \subset D_I \cap D_J, \quad C_{JK} \subset D_J \cap D_K, \quad C_{KI} \subset D_K \cap D_I$$

and C_{IJ} links ∞_I and ∞_J, etc. In the case of the harmonic oscillator above, fundamental triplets are $\{\psi_1, \psi_2, \psi_3\}$ and $\{\psi_2, \psi_3, \psi_4\}$. Olver gives explicit formulas of the coefficients of (22) for such an fundamental triplet. In fact, Olver's consideration in his book is limited to a *generic case*, i.e., the case where the triplet is associated with three Stokes curves that start from a turning point and tend to the three infinite points without meeting any other turning points. (Note that this also implicitly assume that the turning point at the center is of order one, i.e., a zero of $f(q)$ of order one.) We shall show later that Olver's method can be extended to more general cases.

A key to Olver's observation is the following general result.

Proposition. *Let ψ_I and ψ_J be two solutions of the Liouville-Green form with $\epsilon_I = +1$ and $\epsilon_J = -1$, and suppose that the reference points ∞_I and ∞_J are linked by a curve C_{IJ} whose image Γ_{IJ} on the Liouville plane is progressive (i.e.,* Re $s(q)$ *is increasing as q tends from ∞_I to ∞_J). Then w_I and w_J both have finite boundary values* $\lim_{q\to\infty_J} w_I$ *and* $\lim_{q\to\infty_I} w_J$*, and these boundary values actually coincide.*

The boundary values

$$a_{IJ}(\lambda) \underset{\text{def}}{=} \lim_{q\to\infty_J} w_I = \lim_{q\to\infty_I} w_J \tag{23}$$

are exactly the Voros (or Olver) coefficient in the sense of Section 1. For a generic fundamental triplet $\{\psi_I, \psi_J, \psi_K\}$ (Fig. 3), one has three such quantities

$$a_{IJ} = a_{JI}, \quad a_{JK} = a_{KJ}, \quad a_{KI} = a_{IK}. \tag{24}$$

Olver's basic idea is to determine the coefficients c_{IJ} and c_{IK} by comparing the behavior of the tree terms in (22) as q tends to ∞_J and ∞_K. Note that each term in (22) carries a factor (Liouville-Green factor, so to speak) of the form $(ds/dq)^{-1/2}e^{\pm s\lambda}$, the other part being a quantity with a definite boundary value at each reference point at infinity. The two Liouville-Green factors have opposite behavior, one being exponentially large ("dominant") and the other exponentially small ("recessive"). As q tends to ∞_J or to ∞_K, one can select a dominant one, divide both hand sides of (22) by that factor, and consider the limit. This yields two linear equations that determine c_{IJ} and c_{IK} in terms of the boundary values of w's (i.e., a's). In this calculation, however, one should also take into account the multi-valuedness of $(ds/dq)^{-1/2}$ and $e^{\pm s\lambda}$; to this end, we have

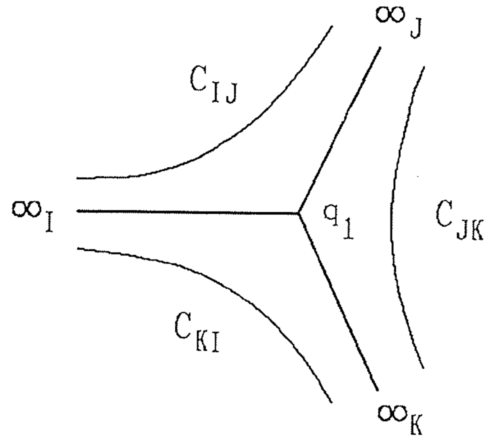

Fig. 3. A generic fundamental triplet

fixed a cut-sheet on the q-plane, and selected a determination of these functions. Across a cut line, they have discontinuity:

$$(ds/dq)^{-/2} \longrightarrow \sigma i (ds/dq)^{-1/2},$$
$$e^{\pm s\lambda} \longrightarrow e^{\pm(-s+2\omega)\lambda}, \tag{25}$$

where σ takes values in $\{+1, -1\}$ depending on the way to go across the cut line; ω is determined by the turning point to which the cut line is hooked. Doing these calculations, one can explicitly write the coefficients c_{IJ} and $_{IK}$ in terms of the following three quantities.

- powers of $i = \sqrt{-1}$,
- exponential functions of the form $e^{-2\omega\lambda}$, $\omega \in \mathbb{C}$,
- the $a_{IJ}(\lambda)$'s introduce above.

4.4 Connection formulas for harmonic oscillator

In the case of the harmonic oscillator with $E > 0$ and $\lambda > 0$, one can thus obtain the following connection formulas for the fundamental triplets $\{\psi_1, \psi_2, \psi_3\}$ and $\{\psi_2, \psi_3, \psi_4\}$.

$$\psi_1 = \frac{1}{a_{23}(\lambda)}\psi_2 + i\frac{e^{-\pi i E\lambda/2}}{a_{23}(\lambda)}\psi_3,$$
$$\psi_4 = i\frac{e^{-\pi i E\lambda/2}}{a_{23}(\lambda)}\psi_2 + \frac{1}{a_{23}(\lambda)}\psi_3. \tag{26}$$

At first sight, a_{IJ}'s other than a_{23} appear to have no contribution, but this is not the case; they are simply reduced to the unity:

$$a_{12}(\lambda) = a_{13}(\lambda) = a_{42}(\lambda) = a_{43}(\lambda) = 1. \tag{27}$$

As a progressive path C_{23} for a_{23}, one may take the imaginary axis of the q-plane, which is mapped to the real axis of the Liouville plane (see Fig. 1).

The case of quartic oscillators as Voros considered can be analyzed in much the same way. A class of equations with nonpolynomial potentials, such as the Mathieu equation, can also be dealt with along the same line.

5. Iterated integrals and Laplace integrals

Olver's method thus gives exact connection formulas without any approximation. An unsatisfactory feature is that analytic structure of the basic quantities a_{IJ} is obscure from the construction. What Olver did is to show some qualitative estimates of a_{IJ}'s as well as conditions under which a_{IJ}'s become trivial as in (25). As Olver discussed, a_{IJ} always behaves as

$$a_{IJ} = 1 + O(\lambda^{-1}) \quad (\lambda \to \infty). \tag{28}$$

The lowest order WKB approximation is simply to replace a_{IJ} by the unity. What, for example, about higher order corrections? This is the place where we now attempt at a more detailed analysis.

We shall also deal with non-generic cases. If Stokes curves and turning points are in some "degenerate" configuration, Olver's method outlined above should be applied very carefully or, in extremely degenerate case, suitably modified.

5.1. General results

Let us recall the fact that the potential h on the Liouville plane decays at infinity as shown in (5). This means that the new linear problem is of "scattering type."

In scattering theory, two basic quantities, usually written "a" and "b," are introduced as connection coefficients of Jost solutions. A remarkable fact is that these coefficients also have an integral representation in terms of Jost solutions. From the definition of a_{IJ} (as well as Olver's proof of the existence of boundary values), one can see that a_{IJ} is essentially the same as the a-coefficient, whereas w_I and w_J play the role of Jost solutions in scattering theory. With this analogy, one can derive an integral representation of a_{IJ}.

Proposition. *$a_{IJ}(\lambda)$ have two integral expressions as:*

$$\begin{aligned} a_{IJ}(\lambda) &= 1 + (2\lambda)^{-1} \int_{\Gamma_{IJ}} ds h(s) w_I(s,\lambda) \\ &= 1 + (2\lambda)^{-1} \int_{\Gamma_{IJ}} ds h(s) w_J(s,\lambda), \end{aligned} \tag{29}$$

where the setting is the same as in the previous section.

Remark. In fact, the analogy with scattering theory is somewhat subtle. In the usual situation of scattering theory, the term λ^2 is rather $-\lambda^2$, and Jost solutions are thereby oscillatory at infinity. In our setting, we are viewing the region where ϕ_I's behave exponentially small or large; the situation of scattering theory takes place on a line parallel to the imaginary axis of the Liouville plane. Nevertheless, the analogy with the a-coefficient turns out to be valid in the present situation.

From this integral representation, we can derive remarkable conclusions:

• First, substituting the iterated integral series for w_I and w_J, one can obtain the following expression of a_{IJ}.

Proposition. *$a_{IJ}(\lambda)$ has the iterated integral expansion*

$$\begin{aligned} a_{IJ}(\lambda) =& 1 + \frac{1}{2\lambda}\int_{\Gamma_{IJ}} ds\; h(s) \\ &+ \sum_{n=1}^{\infty} \frac{1}{2\lambda}\int_{(*)} dt_0 \cdots dt_n h(t_0) \prod_{i=1}^{n} \frac{(1 - e^{2(t_{i-1}-t_i)\lambda})h(t_i)}{2\lambda}. \\ (*):&\quad t_0, \ldots, t_n \in \Gamma_{IJ}, \quad t_0 \preceq \ldots \preceq t_n, \end{aligned} \tag{30}$$

where $\preceq$ means that the points are ordered in that way along Γ_{IJ} from $s(\infty_I)$ to $s(\infty_J)$.

• Further, if Γ_{IJ} can be chosen to be a *straight line* on the Liouville plane, one can rewrite the above iterated integral into a Laplace integral:

$$a_{IJ}(\lambda) = 1 + (2\lambda)^{-1}\int_{\Gamma_{IJ}} dsh(s) + (2\lambda)^{-1}\int_0^{e^{i\theta}\infty} dte^{-2t\lambda} A_{IJ}(t), \tag{31}$$

where θ is the angle between Γ_{IJ} and the real axis of the Liouville plane. This is due to the obvious identity

$$\frac{1 - e^{2(t_{i-1}-t_i)\lambda}}{2\lambda} = \int_{t_{i-1}}^{t_i} ds_i\; e^{2(s_i - t_i)\lambda}. \tag{32}$$

Changing the integration variables from (s_i, t_i) to (x_i, y_i) as

$$x_i = s_i - t_{i-1}, \quad y_i = t_i - s_{i-1},$$

one can indeed derive from (28) an expression like (31).

Proposition. *$A_{IJ}(t)$ has the iterated integral expansion*

$$\begin{aligned} A_{IJ}(t) =& \sum_{n=1}^{\infty}\int_{(**)} dx_1 \cdots dx_n\; dy_1 \cdots dy_n \prod_{i=1}^{n} h\Big(s + \sum_{j=1}^{i}(x_j + y_j)\Big) \\ (**):&\quad x_1, \ldots, x_n, y_1, \ldots, y_n \in [0, e^{i\theta}\infty), \quad \sum_{j=1}^{n} y_j = t \end{aligned} \tag{33}$$

A somewhat careful consideration shows that $A_{IJ}(t)$ gives a holomorphic function in a neighborhood of the path of integration above.

• One can show, in the same way, that w_I's themselves have a similar Laplace integral representation. The above Laplace integral representation is, in fact, derived from such a Laplace integral of w_I's and basic relation (29).

These (iterated) integral representations provide detailed information on the analytic structure of a_{IJ} as well as ψ_I's themselves. An immediate consequence of the Laplace integral representation is that $a_{IJ}(\lambda)$ has an asymptotic expansion

$$a_{IJ}(\lambda) \sim 1 + \sum_{n=1}^{\infty} a_{IJ,n}/(2\lambda)^n \quad (\lambda \to \infty, |\arg\lambda + \theta| < \pi/2). \tag{34}$$

The coefficients can be read out from the Taylor coefficients of $A_{IJ}(t)$ at $t = 0$. As Voros observed (in a different formulation), the coefficients $a_{IJ,n}$ can also be evaluated, independently, by the WKB formal solutions mentioned in Section 3. The above result implies that such a formal series expansion is Borel summable in the sector arising in (34).

For the harmonic oscillator discussed above, only a_{23} is non-trivial. For a_{23}, one can choose Γ_{23} to be the real axis of the Liouville plane. In that case, $\theta = 0$. In fact, one can further move Γ_{23} as far as it does not meet the images of turning points. This, in particular, shows that the path of the Laplace integral can be rotated within the range $|\theta| < \pi/2$. A similar analysis can be done in more general cases.

5.2. A degenerate configuration of Stokes curves

Let us now consider the case that allows Stokes curves linking two turning points. This is a kind of "denereracy," which can be resolved by slightly changing the potential or the phase $\arg\lambda$. Voros indeed relies upon the latter trick in actual calculations. Olver, in his book, also avoids to deal with such degeneracy directly; however, Olver's method can be extended to degenerate cases, and this reveals an interesting phenomena. Let us consider a simple case where two turning points, say q_1 and q_2 are connected by a finite Stokes curve (see Fig. 4). This type of configuration frequently occurs in applications. For example, the previous harmonic oscillator with $\arg\lambda = \pm\pi/2$ has such has such Stokes curves.

For calculations, we put two cut lines, one between the Stokes curves S_1 and S_3 and the other between S_2 and S_4. We also assume that

$$\begin{aligned} \operatorname{Re} s(q) &= -\infty \quad \text{at } q = \infty_1, \infty_3, \\ \operatorname{Re} s(q) &= +\infty \quad \text{at } q = \infty_2, \infty_4, \end{aligned} \tag{35}$$

so that the associated exact solutions of the Liouville-Green form are written

$$\begin{aligned} \psi_I &= w_I (ds/dq)^{-1/2} e^{s\lambda} \quad \text{for } I = 1, 3, \\ \psi_I &= w_I (ds/dq)^{-1/2} e^{-s\lambda} \quad \text{for } I = 2, 4. \end{aligned} \tag{36}$$

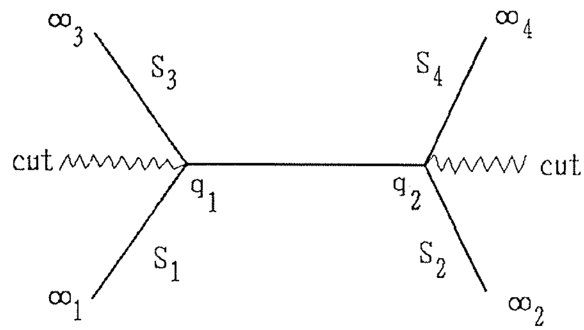

Fig. 4. A degenerate configuration of Stokes curves

This configuration may be part of a larger set of Stokes curves; other part does not affect the treatment of fundamental triplets in $\psi_1, \ldots, \psi_4$. Any triplet of the four ψ_I's is "fundamental," and one can derive a set of connection formulas just as in the non-degenerate case discussed above (with slightest modification due to the difference of the configuration of Stokes curves). Changing triplets and their combinations in (22), one obtains seemingly different formulas for the same triplet. Since such two formulas must have the same contents, the coefficients should satisfy some algebraic relations (consistency conditions). In the present situation, one can thus obtain the relation

$$a_{14}a_{23} = a_{12}a_{34} + e^{-2\omega\lambda}a_{13}a_{24}, \tag{37}$$

where

$$\omega \underset{\text{def}}{=} s(q_2) - s(q_1). \tag{38}$$

In fact, this kind of relations are not specific to "degenerate" configurations. They do exist even in "nondegenerate" configurations, but are simply hidden.

5.3. Algebraic relation of a_{IJ}'s and resurgence

From the geometric situation on the Liouville plane, one can see that linear paths Γ_{IJ} for the definition of $A_{IJ}(t)$ can be selected as indicated in Fig. 5. Accordingly, each of $a_{IJ}(\lambda)$ has the following Laplace integral representation.

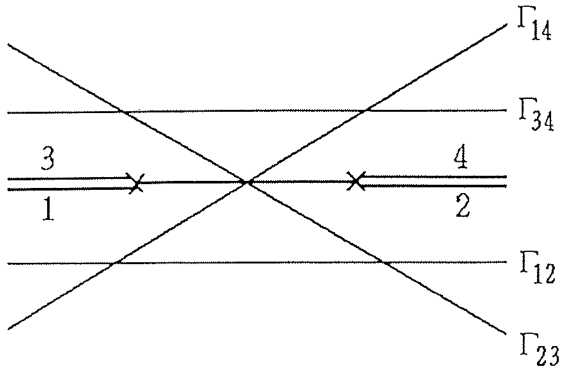

Fig. 5. Linear paths on the Liouville plane

$$a_{14}(\lambda) = 1 + \frac{1}{2\lambda}\int_{\Gamma_{14}} ds\ h(s) + \frac{1}{2\lambda}\int_0^{e^{i\theta}\infty} dt\ e^{2t\lambda} A_{14}(t) \quad (-\theta_0 < \theta < \theta_0), \tag{39}$$

$$a_{23}(\lambda) = 1 + \frac{1}{2\lambda}\int_{\Gamma_{23}} ds\ h(s) + \frac{1}{2\lambda}\int_0^{e^{i\theta}\infty} dt\ e^{2t\lambda} A_{23}(t) \quad (-\theta_0 < \theta < \theta_0). \tag{40}$$

$$a_{14}(\lambda) = 1 + \frac{1}{2\lambda}\int_{\Gamma_{14}} ds\ h(s) + \frac{1}{2\lambda}\int_0^{e^{i\theta}\infty} dt\ e^{2t\lambda} A_{14}(t) \quad (0 < \theta < \theta_0), \tag{41}$$

$$a_{23}(\lambda) = 1 + \frac{1}{2\lambda}\int_{\Gamma_{23}} ds\ h(s) + \frac{1}{2\lambda}\int_0^{e^{i\theta}\infty} dt\ e^{2t\lambda} A_{23}(t) \quad (-\theta_0 < \theta < 0). \tag{42}$$

(θ_0 is a positive constant to be determined by the configuration of Stokes curves and turning points.) In (41) and (42), one will not be able to put $\theta = 0$; A_{14} and A_{23} are expected to have singularities on the positive real axis.

Remarkably, (37) carries precise information on these singularities. To see this, we first rewrite (37) as:

$$a_{14} = \frac{a_{12}a_{34}}{a_{23}} + e^{-2\omega\lambda}\frac{a_{13}a_{24}}{a_{23}}. \tag{37'}$$

All the ingredients of the right hand side, except the exponential function, have a Laplace integral representation along the half line $[0, e^{i\theta}\infty)$ with $-\theta_0 < \theta < 0$. On the other hand, a_{14} has a Laplace integral representation as in (41) along the half line $[0, e^{i\theta}\infty)$ for $0 < \theta < \theta_0$. What will occur if one rotates the path in (41) downward across the positive real axis? That should result in a Laplace integral along $[0, e^{i\theta}\infty)$ with $\theta_0 < \theta < 0$ (the main part) plus some contributions from paths running around singularities of $A_{14}(t)$ (see Fig. 6). The right hand side of (37′) takes exactly such a form; the first term is the main part to be obtained as an analytic continuation of $A_{14}(t)$ through a neighborhood of $t = 0$, whereas the second term is a contribution from a singularity at $t = \omega$. A similar interpretation can be found for a_{23} if one writes (37) as:

$$a_{23} = \frac{a_{12}a_{34}}{a_{14}} + e^{-2\omega\lambda}\frac{a_{13}a_{24}}{a_{14}}. \tag{37''}$$

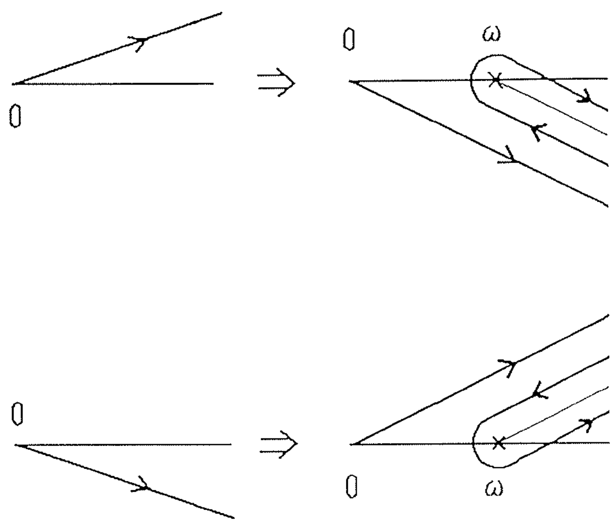

Fig. 6. Interpretation of (37) by deformations of the path

Of particular importance is the fact that the discontinuity along a path on the Liouville plane (now issuing from ω) is again written in a closed form in terms of a_{IJ}'s. This is obviously the same phenomena as originally called "resurgence" by Ecalle, and rediscovered by Voros in the context of the WKB connection problem. The Liouville plane now plays the role of the "Borel plane" in the Ecalle theory. Note, further, that ω is a "half-period" of $f(q)^{1/2}$; this is a general phenomena, as already observed by Voros, Ecalle and Pham.

One can interpret our connection formulas in the same way; now w_I's as well as a_{IJ}'s join the game. The connection formulas, too, thus turn out to be a concise representation of the "resurgence structure" of w_I's on the Borel plane. This also agrees with observations of Voros, Ecalle and Pham. It should be noted that in our treatment based upon Olver's idea, connection formulas are derived without any analysis on singularities on the Borel(=Liouville) plane. Singularity structures can be read out from the connection formulas as a consequence. In this respect, our approach is opposite, or complementary, to the method of Voros; Voros rather starts from a hypothesis on such singularities, and derives his connection formulas from that postulate.

5.4. More degenerate case

A further degenerate case takes place if the two turning points in the above example coalesce to a turning point of second order. In that case, the calculation of connection coefficients cannot be reduced to that of triplets. One should therefore consider the four solutions altogether and seek for a 2×2-connection relation of the form

$$\begin{aligned} \psi_1 &= c_{13}\psi_3 + c_{14}\psi_4, \\ \psi_2 &= c_{23}\psi_3 + c_{24}\psi_4. \end{aligned}$$

From the boundary behavior of each term as $q \to \infty_1, \ldots, \infty_4$, one can derive four linear relations. These relations, however, are not independent and cannot determine the four coefficients. Olver's method thus breaks down in this case.

6. Beyond second order Sturm-Liouville linear problem

6.1. Dirac equations

There are obviously a number of possible directions that deserve further studies. An immediate idea is to develop a similar approach to "Dirac equations" of the form

$$\phi_s = \begin{pmatrix} \lambda & q(s) \\ r(s) & -\lambda \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}, \tag{43}$$

where potentials $q(s)$ and $r(s)$ are assumed to have some analytic properties parallel to the potential $h(s)$ in the "Schrödinger equation" on the Liouville plane. In this case, we directly start from the s-plane rather than seeking for an analogue of the q-plane. Actually, even in the study of the second order Sturm-Liouville problem, Dirac equations have been rather a standard framework; Ecalle and Grigis deal with this problem by converting the original second order equation into a first order Dirac-type system. Further, scattering theory has been also developed for the Dirac equations (for example, by Zakharov and Shabat [ZS], and Ablowitz et al. [AKNS] with applications to soliton theory); our iterated integral series and Laplace integral representation can be naturally extended to that direction.

From our point of view, the analysis on the s-plane is more fundamental than the q-plane, and has enough contents in itself as well as further extensions. One may even forget of the origin of $h(s)$ (the q-plane, the Liouville transformation, etc.); the only thing that plays a crucial role seems to be the "resurgence" of the potential $h(s)$ (here s is identified with the Borel variable). Ecalle indeed reports intriguing results along that line for a more general class of linear systems [E, vol. III].

6.2. String equations

Nonlinear equations should be the next, and even more important subject. Of course, Ecalle has made, in that direction as well, a number of suggestive observations, dealing with quite general situations. Let us rather seek for more concrete examples; a special equation should have its own interesting properties.

In that respect, recent progress in theoretical physics seems to offer a family of very interesting nonlinear differential equations, the "string equations," which are expected to describe the physics of low dimensional string and gravity theory [BK], [DS], [GM]. In the simplest case, the string equation becomes the first Painlevé equation:

$$u_{xx} = u^2 - x, \quad u = u(x). \tag{44}$$

From a physical reason, it is better to put a small parameter G on the left hand side as:

$$Gu_{xx} = u^2 - x, \quad u = u(x), \tag{44'}$$

but for the moment, let us discuss the case of $G = 1$. Physicists seek for solutions with the boundary condition

$$u \sim x^{1/2} \quad \text{as } x \to +\infty, \tag{45}$$

and further ask, for example, if there is any natural condition that determines a remaining arbitrary constant. Several candidates for such a condition have been discussed: – some requirements on the location of possible poles on the real axis (as conjectured by the classical theory of Boutroux [B]), a boundary condition as $x \to -\infty$ (as suggested by F. David [D]), etc. Along with the first Painlevé equation, physicists have introduced an infinite sequence of higher string equations that describe different physical contents:

$$P_{2k+1}[u] = x \quad (k = 1, 2, \ldots), \tag{46}$$

where $P_{2k+1}[u]$ is the generator (the variational derivative of a Hamiltonian $H_{2k+1}[u]$) of the higher (k-th) Korteweg-de Vries (KdV) equation,

$$u_{t_{2k+1}} = P_{2k+1}[u]_x. \tag{47}$$

It is well known in soliton theory [GD] that $P_{2k+1}[u]$'s can be recursively determined as coefficients of a formal solution of the Riccati equation associated with the linear equation

$$\phi_{xx} + u\phi = \lambda^2\phi. \tag{48}$$

(Recall the construction of formal WKB solutions in Section 3!) For example,

$$\begin{aligned} &P_3[u] = (3u^2 - u_{xx})/16, \\ &P_5[u] = -(10u^3 - 10uu_{xx} - 5(u_x)^2 + u_{xxxx})/64, \\ &\text{etc} \ldots \end{aligned} \tag{49}$$

After suitable rescaling of u and x, equation (47) with $k = 1$ reproduce the first Riccati equation. In general, $P_{2k+1}[u]$ is a differential polynomial of u including at most $2k$ order derivatives of u, the highest order term being linear in u. For the higher string equations, physicists require the boundary condition

$$u \sim \text{const}.x^{1/(k+1)} \quad \text{as } x \to +\infty. \tag{50}$$

The constant on the right hand side is to be determined by the equation itself. As in the $k = 1$ case, this boundary condition is still too weak to fix a physical solution. According to physicists, a candidate for additional conditions is to require that u is real-valued, has no singularity on the real axis and, further, satisfies the boundary condition

$$u \sim \text{const}.x^{1/(k+1)} \quad \text{as } x \to -\infty \tag{51}$$

for the same constant as in (50). It is conjectured, by numerical analysis, that such a solution exists if and only if k is even [BMP]; in particular, the first Painlevé equation fails to have such a solution. Mathematically, this issue is obviously related to a global connection problem of the above nonlinear equations.

6.3. Possible link with Ecalle theory

Is such a global connection problem feasible from the point of view of the Ecalle theory or anything along that line? A final answer is far beyond our present scope, but let us show two observations suggesting a possible link with the Ecalle theory. For simplicity, we consider the first Painlevé equation alone.

One observation is due to physicists [DS], [GM]: Suppose that u is given as a linear combination of the form

$$u = x^{1/2} + v. \tag{52}$$

the second term v being, of course, expected to describe subleading contributions on the right hand side of (45). In terms v, the first Painlevé equation can be rewritten into a kind of nonlinear Sturm-Liouville eqation,

$$v_{xx} - 2x^{1/2}v = v^2 + \frac{1}{4}x^{-3/2}, \tag{53}$$

with an inhomogeneous term and a nonlinear term on the right hand side. Physicists infer from this fact that there should be "instanton effects" (i.e., subdominant effects like quantum tunneling) governed by the WKB solutions

$$v_0^{\pm} \sim x^{-1/8} \exp(\pm\frac{\sqrt{32}}{5}x^{5/4}) \tag{54}$$

of the linearized and homogenized equation. Mathematically, this simply means that (53) has an irregular singularity of the Poincaré rank one with respect the new variable

$$s = \frac{\sqrt{32}}{5}x^{5/4}. \tag{55}$$

As far as only a neighborhood of $s = \infty$ is concerned (i.e., within a *local* theory), this already takes a "prepared form" for the Ecalle theory. The boundary problem with two boundary condition at $x = \pm\infty$ is obviously related a the connection problem around $s = \infty$, hence with some *nonlinear Stokes phenomena*. Ecalle seems to suggest a general framework for studying such issues within his "alien calculus."

Another observation is due to Pham in the "Epilogue" of his forthcoming book [CNP]. As a typical example, Pham considers therein the Riccati equation

$$u_x = u^2 - x. \tag{56}$$

Obviously, this equation allows basically the same treatment as we have mentioned for the first Painlevé equation. By the substitution

$$u = x^{1/2} + v, \tag{57}$$

one indeed obtains the equation

$$v_x - 2x^{1/2}v = v^2 - \frac{1}{2}x^{1/2}, \tag{58}$$

which has a prepared form of the Ecalle theory.

Of course, the Riccati equation is far simpler than the first Painlevé equation. For example, the Riccati equation can be converted to the Airy equation

$$p_x - xp = 0 \tag{59}$$

by the well known transformation

$$u = -p_x/p. \tag{60}$$

This is never the case for the Painlevé equation; even a stronger property ("irreducibility") is known [N], [U], which implies its highly transcendental nature. Nevertheless, the Riccati equation may be viewed as a nice example to get some insight into the issues on the Painlevé equation mentioned above.

Let us show a result of numerical computation to the Riccati equation (Fig. 7). This is, actually, just a redrawing of a figure quoted in the book of Pham. [The caption therein says that it is reproduced from: Artigue and Gautheron, *Systèmes différentiels, étude graphique* (Cedic, Paris 1983).] The curves other than the axis in the figure are graphs, in the (x, u)-plane, of solutions of the Riccati equation with various initial conditions. On the right half plane, one will be able to see a parabola, which is exactly the curve $u^2 - x = 0$. The upper half of this parabola is "repulsive" in the sense that every point close to this part soon get far away as x increases; the lower half is "attractive" in the opposite sense. (A rigorous analysis of this phenomena can be found in Hille's book [H].) Note that drawing this kind of figure in the Painlevé case requires a fine tuning of two arbitrary constants in a general solutions; one has to find a nice one-parameter family of solutions, which is, even numerically, a hard problem.

6.4. New resurgence from τ function?

It seems thus very plausible that the Ecalle theory will be a useful tool for the study of the string equations at infinity. There is, however, another issue related to possible poles of solutions. At least for the case of the Painlevé equation, it is known that poles are all of second order, and come from zeros of the so called "τ function," and the τ function is proven to be an entire function. Actually, the notion of the "τ function" is

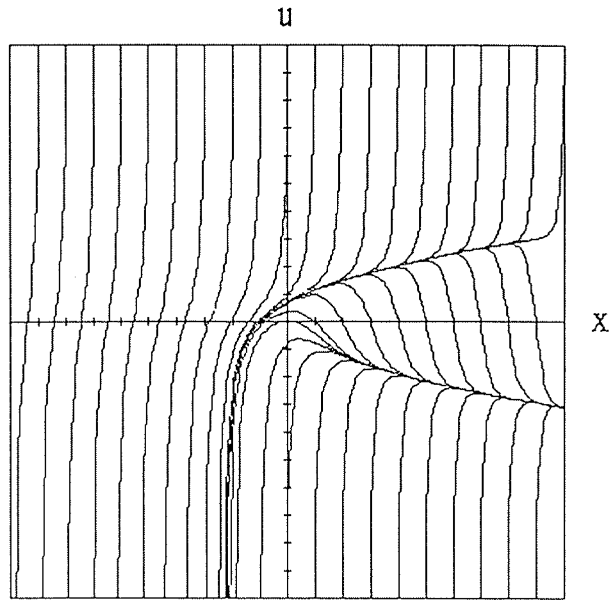

Fig. 7. Graphs of solutions to $u_x = u^2 - x$

also introduced in the KdV and higher KdV equations. For the study of poles of u, the τ function should play a key role. Its relation to the Ecalle theory is now far from our understanding; this might lead to a new aspect of resurgence. At this stage, we already have two variables, s in (55) and λ in (48), which will be responsible for two distinct resurgence phenomena ("equational" and "quantum" in the terminology of Ecalle). The small parameter G, if incorporated as in (44′), might yield a third resurgence that will be specific to the τ function.

References

[AKNS] Ablowitz, M.J., Kaup, D.J., Newell, A.C., and Segur, H., The inverse scattering transform — Fourier analysis for nonlinear problems, Stud. Appl. Math. vol. 53 (1974), 249-315.

[B] Boutroux, P., Recherches sur les transcendantes de M. Painlevé et l'etude asymptotique des équations différentielles du second ordre, Ann. Sci. Ecole Norm. Sup. (3) 30 (1913), 255-375; (3) 31 (1914), 99-159.

[BK] Brézin, E., and Kazakov, V.A., Exactly solvable field theories of closed strings, Phys. Lett. B236 (1990) 144-150.

[BMP] Brézin, E., Marinari, E.,and Parisi, G., A non-perturbative ambiguity free solution of a string model, Lett. Math. Phys. B242 (1990), 35-38.

[CNP] Candelpergher, B., Nosmas, J.C., and Pham, F., Approche de la résurgence (to appear).

[D] David, F., Loop equations and non-perturbative effects in two-dimensional quantum gravity, Mod. Phys. Lett. A 5 (1990), 1019-1029.

[DDP] Delabaere, E., Dillinger, H., and Pham, F., Développements semi-classiques exacts des niveau d'énergie d'un oscillateur à une dimension, C. R. Acad. Sci. Série I, 310 (1990), 141-146.

[DS] Douglas, M.R., and Shenker, H., Strings in less than one-dimension, Nucl. Phys. B335 (1990), 635-654.

[E] Ecalle, J., Les fonctions résurgentes, I-III, Publ. Math. Orsay 1981, 1985.

[GD] Gelfand, I.M., and Dikii, L.A., Asymptotic behavior of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equations, Russian Math. Surveys 30:5 (1975), 77-113.

[GG] Gérard, C., and Grigis, A., Precise estimates of tunneling and eigenvalues near a potential barrier, J. Diff. Eq. 72 (1985), 149-177; Grigis, G., Sur l'équation de Hill analytique, Sem. Bony-Sjöstrand-Meyer 1984/85.

[GM] Gross, D.J., and Migdal, A.A., Nonperturbative two-dimensional quantum gravity, Phys. Rev. Lett. 64 (1990), 127-130.

[H] Hille, E., Ordinary differential equations in the complex domain (John Wiley and Sons, New York, 1976).

[N] Nishioka, K., A note on the transcendency of Painlevé's first transcendent, Nagoya Math. J. 109 (1988), 63-67.

[O] Olver, F.W.J., Asymptotics and special functions, Academic Press 1974.

[U] Umemura, H., On the irreducibility of the first differential equation of Painlevé, in *Algebraic geometry and commutative algebra, in honor of Masayoshi Nagata*, pp. 101-119 (Kinokuniya, Tokyo, 1988).

[V] Voros, A., The return of quartic oscillator, Ann. Inst. H. Poincaré, Section A, 39 (1983), 211-338.

[W] Wasow, W., Linear turning point problem, Springer-Verlag 1985.

[ZS] Zakharov, V.E., and Shabat, A.B., A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem, Funct. Anal. and Appl. 8 (1974), 226-235.