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Isomonodromic Deformations and Whitham Equations

K. Takasaki (Kyoto University)

- Int. J. Mod. Phys. A11 (381) (1996), 5505.
  (With T. Nakatsu) hep-th/......
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$\boxed{\text{Outline}}$

- isospectral v.s. isomonodromic problems
  — motivation
- isomonodromic problem as MODULATION of isospectral problem — general idea
- examples

Isospectral Problem

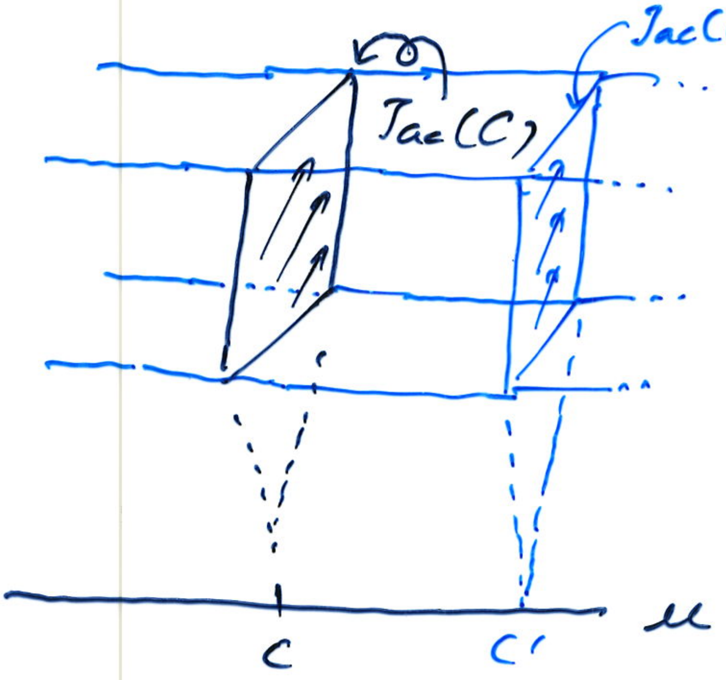
$$\frac{dM(\lambda)}{dt} = [A(\lambda), M(\lambda)] \qquad \text{(Lax representation)}$$

$$\Rightarrow \boxed{\frac{d}{dt} f(\lambda,\mu) = 0}, \quad \text{where}$$

$$f(\lambda,\mu) = \det(\mu I - M(\lambda)).$$
$$= \mu^n + c_1(\lambda)\mu^{n-1} + \dots + c_n(\lambda)$$

$C : f(\lambda,\mu) = 0$ Spectral curve

Lax equation $\longrightarrow$ linear flow on $Jac(C)$



$$\simeq \underbrace{C \times \dots \times C}_{g} / \mathfrak{S}_g$$

$\mathcal{M}$: moduli space of spectral curves

($\dim \mathcal{M} = g$)

Isomonodromic Problem

$$\frac{\partial M(\lambda)}{\partial t} - \frac{\partial A(\lambda)}{\partial \lambda} = [A(\lambda), M(\lambda)]$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} f(\lambda,\mu) \neq 0}$$

$$\left(\frac{\partial}{\partial t} \log f(\lambda,\mu) = - \mathrm{Tr} \frac{\partial A(\lambda)}{\partial \lambda} (\mu I - M(\lambda))^{-1} \right)$$

Spectral curve depends on t.

($\Rightarrow$ usual methods for solving isospectral problems do not work.)

Remark $\mathrm{Jac}(C) \simeq \underbrace{C \times \cdots \times C}_{g} / \mathfrak{S}_g$ is still useful for construction of "Darboux coordinates"
("spectral Darboux coordinates" Harnad et al)

$\downarrow$

"Hamiltonian structure" of isomonodromic problems
(K. Okamoto, et al)

cf. Manin's work (P_{VI} etc) ?

$$\boxed{\text{isomonodromic problem} \neq \text{isospectral problem}}$$

However, there is much evidence that

$$\boxed{\text{isomonodromic problem} \approx \text{modulation of isospectral problem}}$$

asymptotically

$x \to \infty$

$\varepsilon \to 0$

Examples:

- Boutroux (1913) $P_I, P_{II}, \cdots$
- Garnier (1919) Schlesinger equation

  ⋮

- Dubrovin, Krichever, Novikov (around '90)

  string equations of 2d q. gravity

  $(\approx P_I, \cdots)$

- Vereschagin ('96) $P_I \sim P_{VI}$

$\boxed{\text{Problem}}$

Give asymptotic description of isomonodromic problems as **modulation** of isospectral problem!

Boutroux (1913) Painlevé I, II, ...

appoximation as modulated elliptic function

$$(P_I) \quad \frac{d^2u}{dx^2} = 6u^2 - x, \quad x \to \infty$$

$$u \sim \wp(x \mid g_2 = x, g_3 = \cdots)$$

$$(P_I) \quad \varepsilon^2 \frac{d^2u}{dX^2} = 6u^2 - X, \quad \varepsilon \to 0$$

$$u \sim \wp\Big(\underset{\overset{\shortparallel}{x}}{\boxed{\tfrac{X}{\varepsilon}}} \,\Big|\, \underbrace{g_2 = X,\ g_3 = g_3(X)}_{\text{slow dynamics}}\Big)$$

Garnier (1919)

Schlesinger equation

$$\frac{\partial A_i}{\partial t_j} = (1-\delta_{ij})\frac{[A_i, A_j]}{t_i - t_j} - \delta_{ij}\sum_{k(\neq i)}\frac{[A_i, A_k]}{t_i - t_k}$$

e.g. Asymptotics as $t_j \to \infty$ (etc..)

autonomous (isospectral) analogue

$\boxed{t_i \longrightarrow c_i}$ on RHS

$$\frac{\partial A_i}{\partial t_j} = (1-\delta_{ij})\frac{[A_i, A_j]}{c_i - c_j} - \delta_{ij}\sum_{k(\neq i)}\frac{[A_i, A_k]}{c_i - c_k}$$

$\longrightarrow$ spectral curve C

$$\boxed{\text{What is slow dynamics of spectral curve ?}}$$

What is "modulation"?

"finite-gap solution" of isospectral problem (e.g. KdV, Toda chain, Neumann, ...)

$$u = u_0\left(\sum U_j t_j \;\middle|\; \{I_n\}\right)$$

$u_0(z \,|\, \{I_n\})$ ... Abelian function

$z \in \mathbb{C}^g$ $\quad (Jac = \mathbb{C}^g / L)$

$\{I_n\}$ ... constants of motion

$\{U_j\}$ ... constant vectors

period of Abelian differentials on "spectral curve" C

(time independent)

$\downarrow$

isospectral

"modulation" of finite-gap solution

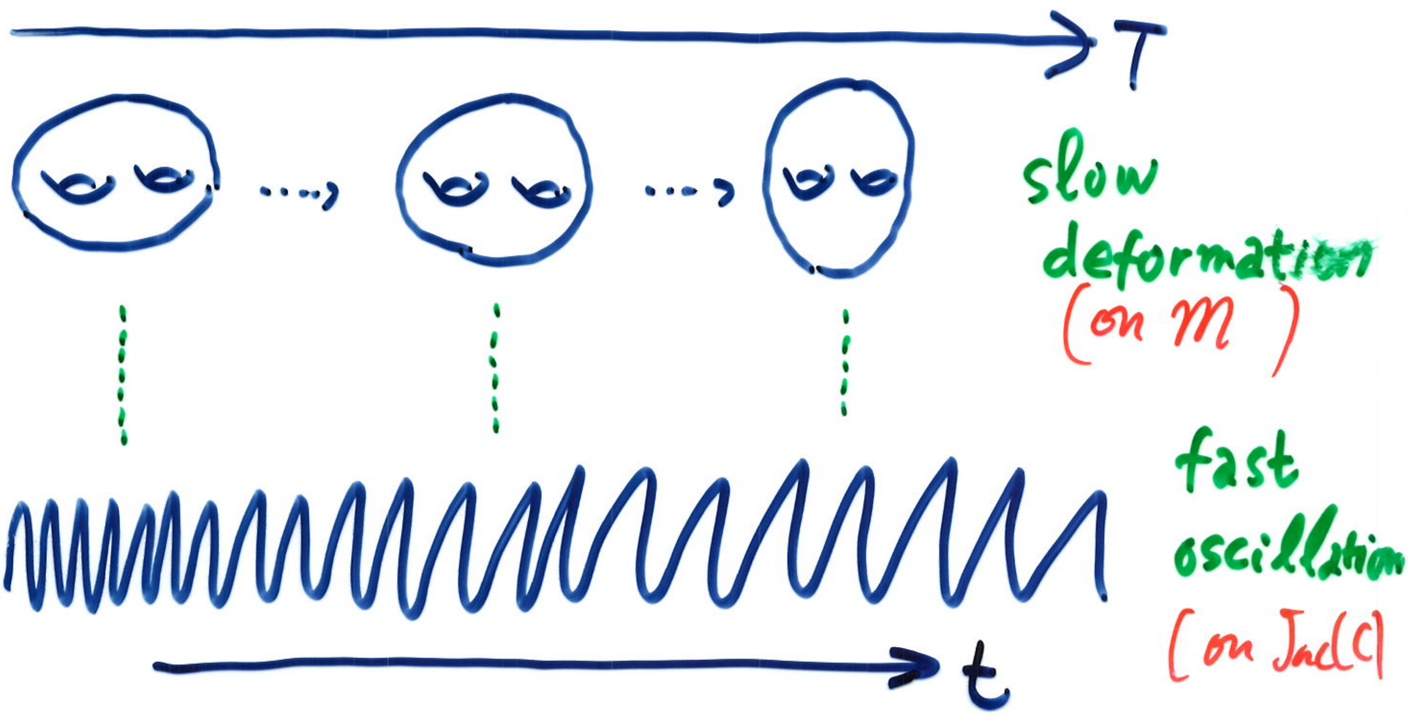
$$\boxed{\varepsilon \to 0}$$

$$u \sim u_0\left(\sum U_j(T)\, t_j \,\Big/\, \{I_n(T)\}\right)$$

t_j : fast variables

$T_j = \varepsilon t_j$: slow variables

$$\left.\begin{array}{l} U_j \to U_j(T) \\ I_n \to I_n(T) \end{array}\right\} \text{functions of slow variables}$$



and now for more general cases

It is known (at least for KdV, SG, etc) that slow dynamics is described by equations of the form

$$\boxed{\frac{\partial}{\partial T_i} d\Omega_j = \frac{\partial}{\partial T_j} d\Omega_i}$$

(Flaschka. Forest. McLaughlin equations)

$d\Omega_i$: meromorphic differentials

$$\psi = \frac{\prod(\text{theta functions})}{\text{theta functions}} \exp\left(\sum t_i \int^{(\lambda,\mu)} d\Omega_i \right)$$

This kind of equations describing modulation are called "Whitham equations" or "modulation equations".

isomonodromic problem with small parameter

– general prescription

$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$ $\quad$ T : slow variable

$\frac{\partial}{\partial \lambda} \to \varepsilon \frac{\partial}{\partial \lambda}$ $\quad$ t : fast —

$$u = u_0(t, T / I(T))\Big|_{T=\varepsilon t} + \varepsilon u_1(t, T/I(T))\Big|_{T=\varepsilon t} + \cdots$$

$$A(\lambda) \to A^{(0)}(\lambda) + \varepsilon A^{(1)}(\lambda) + \cdots$$

$$M(\lambda) \to M^{(0)}(\lambda) + \varepsilon M^{(1)}(\lambda) + \cdots$$

$$\frac{\partial M}{\partial t} + \varepsilon \frac{\partial M}{\partial T} - \varepsilon \frac{\partial A}{\partial \lambda} = [A(\lambda), M(\lambda)]$$

($\to$ ε - expansion) perturbation theory

lowest order :

$$\frac{\partial M^{(0)}(\lambda)}{\partial t} = [A^{(0)}(\lambda), M^{(0)}(\lambda)]$$: isospectral problem!

$$f(\lambda, \mu) = \det(\mu I - M^{(0)}(\lambda))$$

$$\frac{\partial}{\partial t} f(\lambda, \mu) = 0, \quad \boxed{\frac{\partial}{\partial T} f(\lambda, \mu) \neq 0}$$

$\downarrow$

slow dynamics (on $\mathcal{M}$)

"multi-scale analysis"

For all known cases, slow dynamics of the spectral curve in isomonodromic problems obeys an equation of the form

$$\boxed{\frac{\partial}{\partial T} dS = d\Omega}$$

- $dS = \mu\, d\lambda$
- $d\Omega$ = meromorphic differential that arises in the algebro-geometric construction of ψ:

$$\psi = \frac{\prod(\text{theta functions})}{\text{theta function}} \exp\left(t \int^{(\lambda,\mu)} d\Omega\right)$$

- $\frac{\partial}{\partial T} d\lambda = 0$

$$\left.\frac{\partial}{\partial T} dS\right|_{\lambda=\text{const}} = \nabla_{\partial/\partial T} dS \ : \ \text{Connection}$$

(Krichever - Phong)

Remark: A symplectic basis of $H_1(C, \mathbb{Z})$ has to be fixed in order to determine $d\Omega$.

If there are several isomonodromic flows $(t_1, \cdots, t_m)$, then the equations of motion of slow dynamics can be written

$$\boxed{\frac{\partial}{\partial T_i} dS = d\Omega_i}$$

$d\Omega_i$:

$$\psi = \frac{\Pi\,(\text{theta function})}{\text{theta function}} \exp\left(\sum t_i \int^{A(p)} d\Omega_i\right)$$

$$\oint_{A_I} d\Omega_i = 0 \qquad (I = 1 \cdots g)$$

$$\left(\quad \langle A_I, B_J \rangle = \delta_{IJ} \qquad \right)$$

$\boxed{\text{Examples}}$ and $\boxed{\text{OPEN PROBLEMS ...}}$

- $P_{I}, P_{II}, P_{III}, \ldots, P_{VI}$: all related to elliptic curve

- Self-similar reduction of periodic 2D Toda equation

  $\rightarrow$ $\boxed{dS = \text{Seiberg - Witten differential}}$

  $\frac{\partial}{\partial a_I} dS = d\omega_I$: holomorphic differentials

  (Gorsky, Krichever, Marshakov, Mironov, Morozov) ... SU(2)

  (.... many people .... SU(N), ...)

- $\left\{ \begin{array}{l} \text{Schlesinger equation} \\ \text{Jimbo - Miwa - Mori - Sato equation} \end{array} \right\}$ solv-int/...

  NOT a self-similar reduction of soliton equations

  (cf) WDVV equation with self-similarity condition

  → special case of Schlesinger equation

  (Dubrovin)

  quasi-homog.

- isomonodromic problems on torus ?

  - construction of isomonodromic problems itself is a problem

    (K. Okamoto, K. Iwasaki, ... 2nd order scalar isomonodromic problems)

    What about matrix systems ?

    Hint : Gaudin models

    XXX → XYZ
    rational → elliptic

    isospectral problems = Hitchin system (on punctured torus)

    possible relation to KZ(B) equations ?

    ...... in progress !

- Is derivation of slow dynamics rigorous ?

  - At least my derivation is heuristic.
  - choice of cycles $\{A_I, B_I\}$

Schlesinger equation with small parameter

① $$\frac{\partial A_i}{\partial t_j} = (1-\delta_{ij})\frac{[A_i, A_j]}{t_i - t_j} - \delta_{ij}\sum_{k(\neq i)}\frac{[A_i, A_k]}{t_i - t_k}$$

② $$\frac{\partial Y}{\partial \lambda} = M(\lambda)Y, \quad M(\lambda) := \sum_{i=1}^{N}\frac{A_i}{\lambda - t_i}$$

$$\frac{\partial Y}{\partial t_i} = -\frac{A_i}{\lambda - t_i}Y. \quad (i = 1 \cdots N)$$

$\downarrow$ $t_i \to T_i$, $\frac{\partial}{\partial t_i} \to \varepsilon\frac{\partial}{\partial T_i}$. $\frac{\partial}{\partial \lambda} \to \varepsilon\frac{\partial}{\partial \lambda}$

① $$\varepsilon\frac{\partial A_i}{\partial T_j} = (1-\delta_{ij})\frac{[A_i, A_j]}{T_i - T_j} - \delta_{ij}\sum_{k(\neq i)}\frac{[A_i, A_k]}{T_i - T_k},$$

② $$\varepsilon\frac{\partial Y}{\partial \lambda} = M(\lambda)Y, \quad M(\lambda) := \sum_{i=1}^{N}\frac{A_i}{\lambda - T_i}$$

$$\varepsilon\frac{\partial Y}{\partial T_i} = -\frac{A_i}{\lambda - T_i}Y. \quad (i = 1 \ldots N)$$

$$\varepsilon\frac{\partial}{\partial T_i} \longrightarrow \frac{\partial}{\partial t_i} + \varepsilon\frac{\partial}{\partial T_i}$$

t_i: fast variables

Modulation Equation

(to Garnier's problem)

Spectral curve (discovered by Garnier)

$$\det(M(\lambda)-\mu I)=0.$$

Assumption

eigenvalues $\theta_{i\alpha}$ ($\alpha=1\ldots r$)
of A_i are pairwise distinct

(stronger assumption: $\theta_{i\alpha}-\theta_{i\beta}\notin\mathbb{Z}$
(if $\alpha\neq\beta$)

$\rightarrow$ $\pi^{-1}(T_i)$ compactified by r points
(not ramified over $\lambda=T_i$)

Modulation equation (derived by heuristic argument)

$$\boxed{\begin{array}{l}\dfrac{\partial}{\partial T_i}dS=d\Omega_i, \quad \text{where} \quad \left(\dfrac{\partial}{\partial T_i}\lambda=0\right)\\ dS=\mu\, d\lambda\\ d\Omega_i:\ \text{mero. diff. of 2nd kind}\\ \qquad\ \text{poles at } \pi^{-1}(T_i),\ \text{and}\ldots\end{array}}$$

$$\boxed{\begin{array}{l}\text{normalized as} \\ \oint_{A_I} d\Omega_i = 0 \quad I = 1 \cdots g\end{array}}$$

A_I, B_I $I = 1 \cdots g$: symplectic homology basis

(*) parameters of spectral curve

$$F(\lambda, \mu) = \det(M(\lambda) - \mu I) = 0$$

① $\{\theta_{i\alpha}\}$ ⋯ constant (coad. orbit invariants)

② $\{T_i\}$ ⋯ deformation variables

③ $\{h_I\}$ $(I = 1 \cdots g)$ ⋯ other coeff.

($\simeq$ isospectral Hamiltonians)

$\boxed{\text{moduli of spectral curve}}$

modulation equation $\longrightarrow$ $h_I = h_I(\underline{T})$

solution by inverse period map

period map:

$$C = C(\{h_I\}, \{T_i\})$$

$$\boxed{a_I = \oint_{A_I} dS}$$

$$h = \{h_I\} \longmapsto a = \{a_I\}$$

invertibility:

$h \longmapsto a$ is (locally) invertible

inverse period map:

$$a \longmapsto h, \quad h_I = h_I(\{a_J\}, \underset{\text{parameter}}{\underline{\{T_i\}}})$$

① This solves the modulation eq:

$$\frac{\partial}{\partial T_i} dS = d\Omega_i$$

② Also satisfies the deformation equation of Seiberg-Witten type:

$$\frac{\partial}{\partial a_I} dS = d\omega_I \quad \left(\begin{array}{c}\text{normalized} \\ \text{diff. of } \underline{\underline{\text{1st}}} \text{ kind}\end{array}\right)$$