Generalized string equations for Hurwitz numbers

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1. Hurwitz numbers of Riemann sphere

The Hurwitz numbers enumerate topologically nonequivalent finite ramified coverings \( \pi : \Gamma \to \Gamma_0 \) of a Riemann surface \( \Gamma_0 \). In the following, we consider the case where \( \Gamma_0 = \mathbb{CP}^1 \).

Partition as ramification data

In a neighborhood of the fiber \( \pi^{-1}(P) \) of a point \( P \), \( \Gamma \) looks like a union of cyclic coverings of degree \( \mu_1, \mu_2, \cdots \). They give a partition

\[
\mu = (\mu_1, \mu_2, \cdots) = (1^{m_1} 2^{m_2} \cdots)
\]

of the degree \( d \) of the covering.
Hurwitz numbers

Given a positive integer $d$, a partition $\mu^{(1)}, \ldots, \mu^{(r)}$ of $d$ and $r$ points $P_1, \ldots, P_r$ of $\mathbb{CP}^1$, we consider coverings $\pi : \Gamma \to \mathbb{CP}^1$ of degree $d$ that are ramified over these points of ramification type $\mu^{(1)}, \ldots, \mu^{(r)}$.

There are only a finite number of topologically nonequivalent coverings of this type. The Hurwitz number counts the equivalence classes $[\pi]$ with weight $\text{Aut}(\pi)$:

$$H_d(\mu^{(1)}, \ldots, \mu^{(r)}) = \sum_{[\pi]} \frac{1}{|\text{Aut}(\pi)|}$$
1. Hurwitz numbers of Riemann sphere

Formula (Burnside’s theorem)

\[
H_d(\mu^{(1)}, \ldots, \mu^{(r)}) = \sum_{|\lambda|=d} \left( \frac{\dim \lambda}{d!} \right)^2 \prod_{k=1}^{r} f_{\lambda}(\mu^{(k)}),
\]

\[
\dim \lambda = \chi_\lambda(C(1^d)), \quad f_{\lambda}(\mu) = \frac{\chi_\lambda(C(\mu))}{\dim \lambda} |C(\mu)|,
\]

where \( \chi_\lambda(C) \) denotes the irreducible character (class function) of the symmetric group \( S_d \) for the partition \( \lambda \), \( C(\mu) \) the conjugacy class of cycle type \( \mu = (1^{m_1}2^{m_2} \cdots) \), and \( |C(\mu)| \) the cardinality of \( C(\mu) \) as a subset of \( S_d \):

\[
|C(\mu)| = d!/z_\mu, \quad z_\mu = \prod_{i \geq 1} m_i!i^{m_i}.
\]
2. Generating functions of Hurwitz numbers

Generating function of almost simple Hurwitz numbers

\[ H_d(1^{d-2}2, \cdots, 1^{d-2}2, \mu) \]

Introduce variables \( \beta, Q \) and \( x = (x_1, x_2, \cdots) \), the power sums \( p_k = \sum_{i \geq 1} x_i^k \) and their products \( p_\mu = p_{\mu_1}p_{\mu_2} \cdots \). Define the generating function \( Z(x) \) as

\[
Z(x) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=d} H_d(1^{d-2}2, \cdots, 1^{d-2}2, \mu) \frac{\beta^r}{r!} Q^d p_\mu
\]
Generating function of double Hurwitz numbers

\[ H_d(\mu, 1^{d-2}, \ldots, 1^{d-2}, \bar{\mu}) \]

Introduce yet another set of variables \( \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots) \) and their power sums \( \bar{p}_k = \sum_{i \geq 1} \bar{x}_i^k \). Define the generating function \( Z(x, \bar{x}) \) as

\[
Z(x, \bar{x}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=|\bar{\mu}|=d} H_d(\mu, 1^{d-2}, \ldots, 1^{d-2}, \bar{\mu}) \frac{\beta^r}{r!} Q^d p_\mu \bar{p}_{\bar{\mu}}
\]
2. Generating functions of Hurwitz numbers

Change of variables for KP and Toda hierarchies

• \( \mathbf{t} = (t_1, t_2, \cdots) \) for KP hierarchy:

\[
\begin{align*}
t_k &= \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k
\end{align*}
\]

• \( \mathbf{t} = (t_1, t_2, \cdots) \) and \( \mathbf{\bar{t}} = (\bar{t}_1, \bar{t}_2, \cdots) \) for Toda hierarchy:

\[
\begin{align*}
t_k &= \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k, \quad \bar{t}_k = -\frac{\bar{p}_k}{k} = -\frac{1}{k} \sum_{i \geq 1} \bar{x}_i^k
\end{align*}
\]

Schur functions are redefined as functions of these variables:

\[
\begin{align*}
s_\lambda(x) &= s_\lambda[\mathbf{t}], \quad s_\lambda(\bar{x}) = s_\lambda[-\mathbf{\bar{t}}] \quad (\text{Zinn-Justin’s notation})
\end{align*}
\]
Generating functions in terms of Schur functions

Use Frobenius’ formula
\[ \sum_{|\mu|=d} \frac{\chi_{\lambda}(C'(\mu))}{z_\mu} p_\mu = s_\lambda(\mathbf{x}) \]

to rewrite the generating functions to sums over partitions \( \lambda \) of arbitrary length:

\[
Z(\mathbf{x}) = \sum_{\lambda} \frac{\dim \lambda}{|\lambda|!} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda(\mathbf{x}) = \sum_{\lambda} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda[t] s_\lambda[1, 0, \cdots],
\]

\[
Z(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{\lambda} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda(\mathbf{x}) s_\lambda(\bar{\mathbf{x}}) = \sum_{\lambda} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda[t] s_\lambda[-\bar{t}].
\]

Remark: Relevant formulae

\[
\frac{\dim \lambda}{|\lambda|!} = s_\lambda[1, 0, \cdots], \quad f_\lambda(1^{d-2}2) = \kappa_\lambda = \sum_{i \geq 1} \lambda_i(\lambda_i - 2i + 1).
\]
2. Generating functions of Hurwitz numbers

Cut-and-join operator

\[ M_0 = \frac{1}{2} \sum_{k,l=1}^{\infty} \left( klt_k t_l \frac{\partial}{\partial t_{k+l}} + (k + l) t_{k+l} \frac{\partial^2}{\partial t_k \partial t_l} \right) \]

- Schur functions are eigenfunctions of \( M_0 \):
  \[ M_0 s_\lambda(t) = \frac{\kappa_\lambda}{2} s_\lambda(t). \]

- \( Z(t) = Z(x) \) and \( Z(t, \bar{t}) = Z(x, \bar{x}) \) can be expressed as
  \[ Z(t) = e^{\beta M_0} e^{Q t_1}, \]
  \[ Z(t, \bar{t}) = e^{\beta M_0} \exp \left( - \sum_{k=1}^{\infty} Q^k t_k \bar{t}_k \right). \]
Generating functions as tau functions

• $Z[t]$ is a tau function of the KP hierarchy (Kazarian & Lando, Goulden & Jackson, · · ·).

• $Z[t, \bar{t}]$ is a tau function of the Toda hierarchy at a point, say $s = 0$, of the lattice (Okounkov). In other words, $Z[t, \bar{t}]$ is a tau function of the 2-component KP hierarchy.

Remark: Any tau function $\tau(s, t, \bar{t})$ of the Toda hierarchy is a sequence of tau functions (indexed by $s \in \mathbb{Z}$) of the 2-component KP hierarchy.
3. Fermionic representation of tau functions

Fermionic operators and Fock space

• creation-annihilation operators $\psi_i, \psi_i^* \ (i \in \mathbb{Z})$

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i+j,0}, \quad \psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

• ground states in charge $s$ sector of the Fock space

$$\langle s \rangle = \langle -\infty | \cdots \psi_{s-1}^* \psi_s^* \rangle, \quad |s\rangle = \psi_{-s} \psi_{-s+1} \cdots | -\infty \rangle$$

• Fermion bilinears

$$J_k = \sum_{n \in \mathbb{Z}} :\psi_{-n+k} \psi_n^*: \quad L_0 = \sum_{n \in \mathbb{Z}} n :\psi_{-n} \psi_n^*: \quad W_0 = \sum_{n \in \mathbb{Z}} n^2 :\psi_{-n} \psi_n^*:$$

Remark: $M_0 \leftrightarrow \frac{1}{2} \sum_{n \in \mathbb{Z}} (n - 1/2)^2 :\psi_{-n} \psi_n^*: = \frac{1}{2} W_0 - \frac{1}{2} L_0 + \frac{1}{8} J_0$
The special GL(∞) element

\[ g = e^{\beta W_0 / 2} Q^{L_0} \]

determines a tau function

\[ \tau(s, t, \bar{t}) = \langle s | \exp \left( \sum_{k=1}^{\infty} t_k J_k \right) e^{\beta W_0 / 2} Q^{L_0} \exp \left( - \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) | s \rangle \]

of the Toda hierarchy. This tau function has the Schur function expansion

\[ \tau(s, t, \bar{t}) = e^{\beta s(s+1)(2s+1)/12} Q^{s(s+1)/2} \times \sum_{\lambda} e^{\beta \kappa_\lambda / 2} (e^{\beta (2s+1)/2} Q)^{|\lambda|} s_\lambda[t] s_\lambda[-\bar{t}] \]

and reduces to \( Z[t, \bar{t}] \) upon renormalizing \( Q \) and setting \( s = 0 \).
4. Generalized string equations

**Intertwining relations of fermion bilinears**

\[ g = e^{-\beta W_0/2} Q^{L_0} \] intertwines special fermion bilinears as

\[ J_k g = g Q^k e^{-\beta k^2/2} \sum_{n \in \mathbb{Z}} e^{\beta kn} :\psi_{-n+k} \psi_n^* :g, \]

\[ g J_{-k} = Q^k e^{\beta k^2/2} \sum_{n \in \mathbb{Z}} e^{\beta kn} :\psi_{-n-k} \psi_n^* :g \]

Remark: These relations play a role in an integrable structure of the melting crystal model as well.
4. Generalized string equations

Lax and Orlov-Schulman operators

\[ L = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(-n+1)\partial_s}, \quad \bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \sum_{n=1}^{\infty} \bar{u}_n e^{(n-1)\partial_s}, \]

\[ M = \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \quad \bar{M} = -\sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n \]

Lax equations

\[ \frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad k = 1, 2, \ldots \]

same equations replacing \( L \to M, \bar{L}, \bar{M} \)

Twisted canonical commutation relations

\[ [L, M] = L, \quad [\bar{L}, \bar{M}] = \bar{L} \]
4. Generalized string equations

**Theorem**

- The generalized string equations

\[ L^k = Q^k e^{-\beta k^2/2} \bar{L}^k e^{\beta k \bar{M}}, \quad \bar{L}^{-k} = Q^k e^{\beta k^2/2} \bar{L}^{-k} e^{\beta k M} \]

hold for \( k = 1, 2, \ldots \).

- These equations reduce to the lowest ones

\[ L = Q e^{-\beta/2} \bar{L} e^{\beta \bar{M}}, \quad \bar{L}^{-1} = Q e^{\beta/2} L^{-1} e^{\beta M} \]

Remark: Generalized string equations in \( c = 1 \) string theory

\[ L = \bar{L} \bar{M} + \text{const.}\bar{L}, \quad \bar{L}^{-1} = L^{-1} M + \text{const.} L^{-1} \]
5. Classical limit of generalized string equations

Classical (= dispersionless) limit of Toda hierarchy

• Introduce a new parameter $\hbar$, allow the tau function itself to depend on $\hbar$, and assume that the rescaled tau function $\tau_{\hbar}(s, t, \bar{t}) = \tau(\hbar, \hbar^{-1} s, \hbar^{-1} t, \hbar^{-1} \bar{t})$ behaves as

$$\log \tau_{\hbar}(s, t, \bar{t}) \sim \hbar^{-2} F(s, t, \bar{t}) + O(\hbar^{-1}) \quad (\hbar \to 0).$$

$F(s, t, \bar{t})$ is called “free energy”, etc.

• In the Lax formalism, this amounts to replacing the difference operators $L, M, \bar{L}, \bar{M}$ (quantum observables) by functions $\mathcal{L}, \mathcal{M}, \tilde{L}, \tilde{M}$ (classical observables):

$$\sum_n a_n(\hbar, s) e^{n\delta_s} \to \sum_n a_n^{(0)}(s) p^n, \quad a_n(\hbar, s) = a_n^{(0)}(s) + O(\hbar).$$
Lax and Orlov-Schulman functions

\[ \mathcal{L} = p + \sum_{n=1}^{\infty} u_n^{(0)} p^{1-n}, \quad \bar{\mathcal{L}}^{-1} = \bar{u}_0^{(0)} p^{-1} + \sum_{n=1}^{\infty} \bar{u}_n^{(0)} p^{n-1}, \]

\[ \mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}, \quad \bar{\mathcal{M}} = -\sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n^{(0)} \bar{\mathcal{L}}^n \]

Lax equations and twisted canonical Poisson relations

\[ \frac{\partial \mathcal{L}}{\partial t_k} = \{ B_k, \mathcal{L} \}, \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_k} = \{ \bar{B}_k, \mathcal{L} \}, \quad k = 1, 2, \ldots \]

same equations replacing \( \mathcal{L} \to \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}, \)

\[ \{ \mathcal{L}, \mathcal{M} \} = \mathcal{L}, \quad \{ \bar{\mathcal{L}}, \bar{\mathcal{M}} \} = \bar{\mathcal{L}} \]

with respect to Poisson bracket \( \{ F, G \} = p \frac{\partial F}{\partial p} \frac{\partial G}{\partial s} - \frac{\partial F}{\partial s} p \frac{\partial G}{\partial p} \).
Classical limit of generalized string equation

- Rescaling $s, t, \bar{t}$ as $s, t, \bar{t} \to \hbar^{-1} s, \hbar^{-1} t, \hbar^{-1} \bar{t}$ changes the generalized string equation as

\[
L = Q e^{-\beta/2} \bar{L} e^{\beta \hbar^{-1} \bar{M}}, \quad \bar{L}^{-1} = Q e^{\beta/2} L^{-1} e^{\beta \hbar^{-1} M}
\]

These equations themselves do not have a limit as $\hbar \to 0$.

- By rescaling $\beta$ as $\beta \to \hbar^{-1} \beta$, one can take the classical limit as $\hbar \to 0$.

- Thus the generalized string equations

\[
\mathcal{L} = Q \bar{\mathcal{L}} e^{\beta \bar{M}}, \quad \bar{\mathcal{L}}^{-1} = Q \mathcal{L}^{-1} e^{\beta M}
\]

in the classical limit is obtained.
There is a unique solution of the classical limit of the generalized string equations that has power series expansion with respect to $t, \bar{t}$:

$$
\log \bar{u}_0 = \log Q + \beta s + \text{ (terms of positive orders in } t, \bar{t}),
$$

$$
u_n = -\beta n \bar{t}_n \bar{u}_0^n + \text{ (terms of higher orders in } t, \bar{t}),
$$

$$\bar{u}_n = \beta n t_n \bar{u}_0 + \text{ (terms of higher orders in } t, \bar{t}).$$

(“(0)” is omitted.) This solution is quasi-homogeneous with respect to rescaling

$$
t_n \to c^{-n} t_n, \quad \bar{t}_n \to c^n \bar{t}_n, \quad s \to s, \quad p \to c^{-1} p,
$$

$$
u_n \to c^n \nu_n, \quad \bar{u}_n \to c^{-n} \bar{u}_n, \quad v_n \to c^n v_n, \quad \bar{v}_n \to c^{-n} \bar{v}_n.$$
Solution at special values of $t, \bar{t}$

When $\bar{t}$ is specialized to $\bar{t}_k = \bar{t}_1 \delta_{k1}$ (which amounts to specializing to almost-simple Hurwitz numbers), $\mathcal{L}$ simplifies as

$$\mathcal{L} = pe^{-\beta \bar{t}_1 \bar{u}_0 p^{-1}},$$

and $\bar{u}_0 (= \partial^2 \mathcal{F}/\partial s^2)$ is determined by the single equation

$$\log \bar{u}_0 = \log Q + \beta s + \beta \sum_{k=1}^{\infty} k t_k \frac{(-k\beta \bar{t}_1 \bar{u}_0)^k}{k!}.$$

Remarkably, the functional structure of $\mathcal{L}$ essentially coincides with the defining equation $z = we^w$ of Lambert’s W-function $w = W(z)$ ($\mathcal{L}^{-1} \leftrightarrow z$, $p^{-1} \leftrightarrow w$). Presumably, this will be related to the spectral curve in the sense of Eynard and Orantin.