Generalized string equations for Hurwitz numbers

Kanehisa Takasaki

December 17, 2010

1. Hurwitz numbers of Riemann sphere
2. Generating functions of Hurwitz numbers
3. Fermionic representation of tau functions
4. Generalized string equations
5. Classical limit of generalized string equations

1. Hurwitz numbers of Riemann sphere

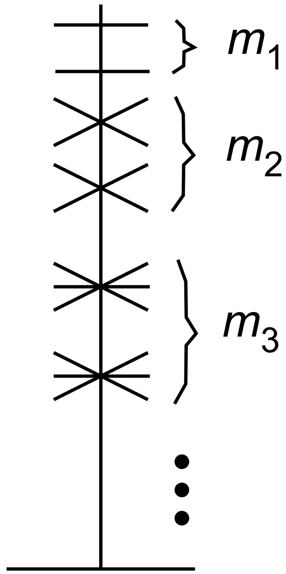
The Hurwitz numbers enumerate topologically nonequivalent finite ramified coverings $\pi : \Gamma \to \Gamma_0$ of a Riemann surface Γ_0. In the following, we consider the case where $\Gamma_0 = \mathbf{CP}^1$.

Partition as ramification data

In a neighborhood of the fiber $\pi^{-1}(P)$ of a point P, Γ looks like a union of cyclic coverings of degree $\mu_1, \mu_2, \cdots$. They give a partition

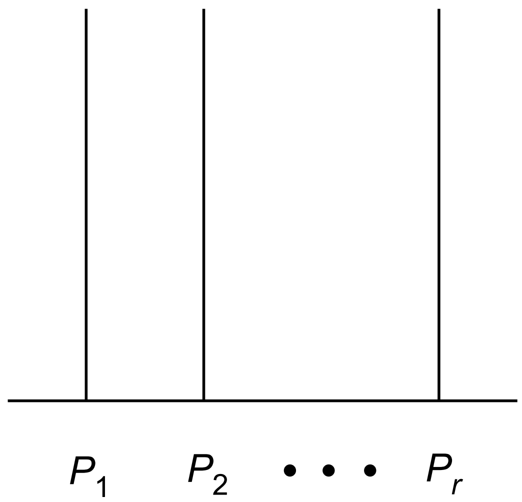
$$\mu = (\mu_1, \mu_2, \cdots) = (1^{m_1} 2^{m_2} \cdots)$$

of the degree d of the covering.



Hurwitz numbers

Given a positive integer d, a partition $\mu^{(1)}, \cdots, \mu^{(r)}$ of d and r points $P_1, \cdots, P_r$ of $\mathbf{CP}^1$, we consider coverings $\pi : \Gamma \to \mathbf{CP}^1$ of degree d that are ramified over these points of ramification type $\mu^{(1)}, \cdots, \mu^{(r)}$.



There are only a finite number of topologically nonequivalent coverings of this type. The Hurwitz number counts the equivalence classes $[\pi]$ with weight $\mathrm{Aut}(\pi)$:

$$H_d(\mu^{(1)}, \cdots, \mu^{(r)}) = \sum_{[\pi]} \frac{1}{|\mathrm{Aut}(\pi)|}$$

Formula (Burnside's theorem)

$$H_d(\mu^{(1)}, \cdots, \mu^{(r)}) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{k=1}^{r} f_\lambda(\mu^{(k)}),$$

$$\dim \lambda = \chi_\lambda(C(1^d)), \quad f_\lambda(\mu) = \frac{\chi_\lambda(C(\mu))}{\dim \lambda} |C(\mu)|,$$

where $\chi_\lambda(C)$ denotes the irreducible character (class function) of the symmetric group S_d for the partition λ, $C(\mu)$ the conjugacy class of cycle type $\mu = (1^{m_1} 2^{m_2} \cdots)$, and $|C(\mu)|$ the cardinality of $C(\mu)$ as a subset of S_d:

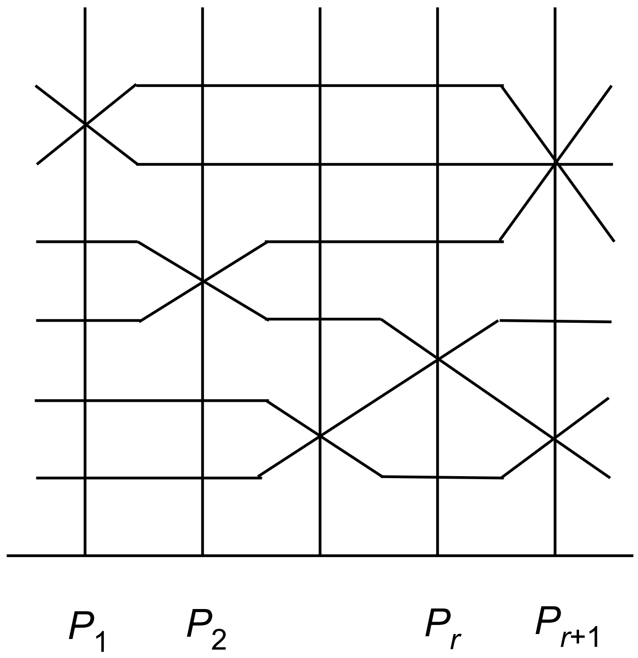
$$|C(\mu)| = d!/z_\mu, \quad z_\mu = \prod_{i \geq 1} m_i! i^{m_i}.$$

2. Generating functions of Hurwitz numbers

Generating function of almost simple Hurwitz numbers

$$H_d(\underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \mu)$$

Introduce variables β, Q and $\boldsymbol{x} = (x_1, x_2, \cdots)$, the power sums $p_k = \sum_{i \geq 1} x_i^k$ and their products $p_\mu = p_{\mu_1} p_{\mu_2} \cdots$. Define the generating function $Z(\boldsymbol{x})$ as

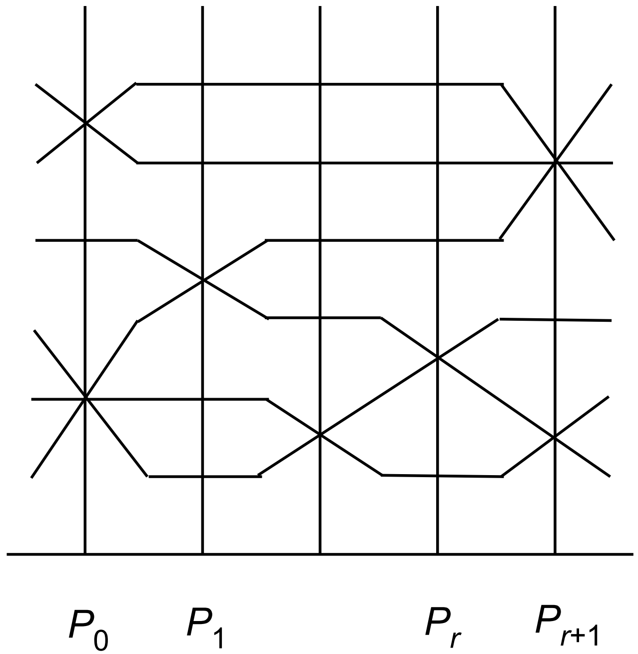


$$Z(\boldsymbol{x}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=d} H_d(\underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \mu) \frac{\beta^r}{r!} Q^d p_\mu$$

Generating function of double Hurwitz numbers

$$H_d(\mu, \underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \bar{\mu})$$

Introduce yet another set of variables $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \cdots)$ and their power sums $\bar{p}_k = \sum_{i\geq 1} \bar{x}_i^k$. Define the generating function $Z(\boldsymbol{x}, \bar{\boldsymbol{x}})$ as



$$Z(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=|\bar{\mu}|=d} H_d(\mu, \underbrace{1^{d-2}2, \cdots, 1^{d-2}2}_{r}, \bar{\mu}) \frac{\beta^r}{r!} Q^d p_\mu \bar{p}_{\bar{\mu}}$$

Change of variables for KP and Toda hierarchies

- $\boldsymbol{t} = (t_1, t_2, \cdots)$ for KP hierarchy:

$$t_k = \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k$$

- $\boldsymbol{t} = (t_1, t_2, \cdots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \cdots)$ for Toda hierarchy:

$$t_k = \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k, \quad \bar{t}_k = -\frac{\bar{p}_k}{k} = -\frac{1}{k} \sum_{i \geq 1} \bar{x}_i^k$$

Schur functions are redefined as functions of these variables:

$$s_\lambda(\boldsymbol{x}) = s_\lambda[\boldsymbol{t}], \quad s_\lambda(\bar{\boldsymbol{x}}) = s_\lambda[-\bar{\boldsymbol{t}}] \quad \text{(Zinn-Justin's notation)}$$

Generating functions in terms of Schur functions

Use Frobenius' formula $\sum_{|\mu|=d} \frac{\chi_\lambda(C(\mu))}{z_\mu} p_\mu = s_\lambda(\boldsymbol{x})$ to rewrite the generating functions to sums over partitions λ of arbitrary length:

$$Z(\boldsymbol{x}) = \sum_\lambda \frac{\dim\lambda}{|\lambda|!} e^{\beta\kappa_\lambda/2} Q^{|\lambda|} s_\lambda(\boldsymbol{x}) = \sum_\lambda e^{\beta\kappa_\lambda/2} Q^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[1,0,\cdots],$$

$$Z(\boldsymbol{x},\bar{\boldsymbol{x}}) = \sum_\lambda e^{\beta\kappa_\lambda/2} Q^{|\lambda|} s_\lambda(\boldsymbol{x}) s_\lambda(\bar{\boldsymbol{x}}) = \sum_\lambda e^{\beta\kappa_\lambda/2} Q^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[-\bar{\boldsymbol{t}}].$$

Remark: Relevant formulae

$$\frac{\dim\lambda}{|\lambda|!} = s_\lambda[1,0,\cdots], \quad f_\lambda(1^{d-2}2) = \kappa_\lambda = \sum_{i\ge 1} \lambda_i(\lambda_i - 2i + 1).$$

Cut-and-join operator

$$M_0 = \frac{1}{2} \sum_{k,l=1}^{\infty} \left(k l t_k t_l \frac{\partial}{\partial t_{k+l}} + (k+l) t_{k+l} \frac{\partial^2}{\partial t_k \partial t_l} \right)$$

- Schur functions are eigenfunctions of M_0:

$$M_0 s_\lambda[\boldsymbol{t}] = \frac{\kappa_\lambda}{2} s_\lambda[\boldsymbol{t}].$$

- $Z[\boldsymbol{t}] = Z(\boldsymbol{x})$ and $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}] = Z(\boldsymbol{x}, \bar{\boldsymbol{x}})$ can be expressed as

$$Z[\boldsymbol{t}] = e^{\beta M_0} e^{Q t_1},$$

$$Z[\boldsymbol{t}, \bar{\boldsymbol{t}}] = e^{\beta M_0} \exp\left(- \sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k \right).$$

Generating functions as tau functions

• $Z[\boldsymbol{t}]$ is a tau function of the KP hierarchy (Kazarian & Lando, Goulden & Jackson, $\cdots$).

• $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ is a tau function of the Toda hierarchy at a point, say $s = 0$, of the lattice (Okounkov). In other words, $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ is a tau function of the 2-component KP hierarchy.

Remark: Any tau function $\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ of the Toda hierarchy is a sequence of tau functions (indexed by $s \in \mathbf{Z}$) of the 2-component KP hierarchy.

3. Fermionic representation of tau functions

Fermionic operators and Fock space

- creation-annihilation operators ψ_i, ψ_i^* $(i \in \mathbf{Z})$

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{i+j,0}, \quad \psi_i\psi_j + \psi_j\psi_i = 0, \quad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0$$

- ground states in charge s sector of the Fock space

$$\langle s| = \langle -\infty| \cdots \psi_{s-1}^*\psi_s^*, \quad |s\rangle = \psi_{-s}\psi_{-s+1}\cdots|-\infty\rangle$$

- Fermion bilinears

$$J_k = \sum_{n\in\mathbf{Z}} {:}\psi_{-n+k}\psi_n^*{:}, \quad L_0 = \sum_{n\in\mathbf{Z}} n{:}\psi_{-n}\psi_n^*{:}, \quad W_0 = \sum_{n\in\mathbf{Z}} n^2{:}\psi_{-n}\psi_n^*{:}$$

Remark: $M_0 \leftrightarrow \frac{1}{2}\sum_{n\in\mathbf{Z}}(n-1/2)^2{:}\psi_{-n}\psi_n^*{:} = \frac{1}{2}W_0 - \frac{1}{2}L_0 + \frac{1}{8}J_0$

Tau function for double Hurwitz numbers

The special GL(∞) element

$$g = e^{\beta W_0/2} Q^{L_0}$$

determines a tau function

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \langle s| \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) e^{\beta W_0/2} Q^{L_0} \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}\right) |s\rangle$$

of the Toda hierarchy. This tau function has the Schur function expansion

$$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = e^{\beta s(s+1)(2s+1)/12} Q^{s(s+1)/2} \times \sum_{\lambda} e^{\beta \kappa_\lambda/2} (e^{\beta(2s+1)/2} Q)^{|\lambda|} s_\lambda[\boldsymbol{t}] s_\lambda[-\bar{\boldsymbol{t}}]$$

and reduces to $Z[\boldsymbol{t}, \bar{\boldsymbol{t}}]$ upon renormalizing Q and setting $s = 0$.

4. Generalized string equations

Intertwining relations of fermion bilinears

$g = e^{-\beta W_0/2} Q^{L_0}$ intertwines special fermion bilinears as

$$J_k g = g Q^k e^{-\beta k^2/2} \sum_{n\in\mathbf{Z}} e^{\beta kn} {:}\psi_{-n+k}\psi_n^*{:},$$

$$g J_{-k} = Q^k e^{\beta k^2/2} \sum_{n\in\mathbf{Z}} e^{\beta kn} {:}\psi_{-n-k}\psi_n^*{:} g$$

Remark: These relations play a role in an integrable structure of the melting crystal model as well.

Lax and Orlov-Schulman operators

$$L = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(-n+1)\partial_s}, \quad \bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \sum_{n=1}^{\infty} \bar{u}_n e^{(n-1)\partial_s},$$

$$M = \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \quad \bar{M} = -\sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n$$

Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad k = 1, 2, \cdots$$

same equations replacing $L \to M, \bar{L}, \bar{M}$

Twisted canonical commutation relations

$$[L, M] = L, \quad [\bar{L}, \bar{M}] = \bar{L}$$

Theorem

- The generalized string equations

$$L^k = Q^k e^{-\beta k^2/2} \bar{L}^k e^{\beta k \bar{M}}, \quad \bar{L}^{-k} = Q^k e^{\beta k^2/2} L^{-k} e^{\beta k M}$$

hold for $k = 1, 2, \cdots$.

- These equations reduce to the lowest ones

$$L = Q e^{-\beta/2} \bar{L} e^{\beta \bar{M}}, \quad \bar{L}^{-1} = Q e^{\beta/2} L^{-1} e^{\beta M}$$

Remark: Generalized string equations in $c = 1$ string theory

$$L = \bar{L}\bar{M} + \text{const.}\bar{L}, \quad \bar{L}^{-1} = L^{-1} M + \text{const.} L^{-1}$$

5. Classical limit of generalized string equations

Classical (= dispersionless) limit of Toda hierarchy

• Introduce a new parameter $\hbar$, allow the tau function itself to depend on $\hbar$, and assume that the rescaled tau function $\tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) = \tau(\hbar, \hbar^{-1}s, \hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}})$ behaves as

$$\log \tau_\hbar(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) \sim \hbar^{-2}\mathcal{F}(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) + O(\hbar^{-1}) \quad (\hbar \to 0).$$

$\mathcal{F}(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ is called "free energy", etc.

• In the Lax formalism, this amounts to replacing the difference operators $L, M, \bar{L}, \bar{M}$ (quantum observables) by functions $\mathcal{L}, \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}$ (classical observables):

$$\sum_n a_n(\hbar, s)e^{n\partial_s} \to \sum_n a_n^{(0)}(s)p^n, \quad a_n(\hbar, s) = a_n^{(0)}(s) + O(\hbar).$$

Lax and Orlov-Schulman functions

$$\mathcal{L} = p + \sum_{n=1}^{\infty} u_n^{(0)} p^{1-n}, \quad \bar{\mathcal{L}}^{-1} = \bar{u}_0^{(0)} p^{-1} + \sum_{n=1}^{\infty} \bar{u}_n^{(0)} p^{n-1},$$

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}, \quad \bar{\mathcal{M}} = -\sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n^{(0)} \bar{\mathcal{L}}^{n}$$

Lax equations and twisted canonical Poisson relations

$$\frac{\partial \mathcal{L}}{\partial t_k} = \{B_k, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_k} = \{\bar{B}_k, \mathcal{L}\}, \quad k = 1, 2, \cdots$$

same equations replacing $\mathcal{L} \to \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}$,

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}$$

with respect to Poisson bracket $\{F, G\} = p\frac{\partial F}{\partial p}\frac{\partial G}{\partial s} - \frac{\partial F}{\partial s} p \frac{\partial G}{\partial p}$.

Classical limit of generalized string equation

• Rescaling $s, \boldsymbol{t}, \bar{\boldsymbol{t}}$ as $s, \boldsymbol{t}, \bar{\boldsymbol{t}} \to \hbar^{-1}s, \hbar^{-1}\boldsymbol{t}, \hbar^{-1}\bar{\boldsymbol{t}}$ changes the generalized string equation as

$$L = Qe^{-\beta/2}\bar{L}e^{\beta\hbar^{-1}\bar{M}}, \quad \bar{L}^{-1} = Qe^{\beta/2}L^{-1}e^{\beta\hbar^{-1}M}$$

These equations themselves do not have a limit as $\hbar \to 0$.

• By rescaling β as $\beta \to \hbar^{-1}\beta$, one can take the classical limit as $\hbar \to 0$.

• Thus the generalized string equations

$$\mathcal{L} = Q\bar{\mathcal{L}}e^{\beta\bar{\mathcal{M}}}, \quad \bar{\mathcal{L}}^{-1} = Q\mathcal{L}^{-1}e^{\beta\mathcal{M}}$$

in the classical limit is obtained.

Theorem

There is a unique solution of the classical limit of the generalized string equations that has power series expansion with respect to $\boldsymbol{t}, \bar{\boldsymbol{t}}$:

$$\begin{aligned}\log \bar{u}_0 &= \log Q + \beta s + (\text{terms of positive orders in } \boldsymbol{t}, \bar{\boldsymbol{t}}),\\ u_n &= -\beta n \bar{t}_n \bar{u}_0^n + (\text{terms of higher orders in } \boldsymbol{t}, \bar{\boldsymbol{t}}),\\ \bar{u}_n &= \beta n t_n \bar{u}_0 + (\text{terms of higher orders in } \boldsymbol{t}, \bar{\boldsymbol{t}}).\end{aligned}$$

("$^{(0)}$" is omitted.) This solution is quasi-homogeneous with respect to rescaling

$$t_n \to c^{-n} t_n, \quad \bar{t}_n \to c^n \bar{t}_n, \quad s \to s, \quad p \to c^{-1} p,$$

$$u_n \to c^n u_n, \quad \bar{u}_n \to c^{-n} \bar{u}_n, \quad v_n \to c^n v_n, \quad \bar{v}_n \to c^{-n} \bar{v}_n.$$

Solution at special values of $\boldsymbol{t}, \bar{\boldsymbol{t}}$

When $\bar{\boldsymbol{t}}$ is specialized to $\bar{t}_k = \bar{t}_1 \delta_{k1}$ (which amounts to specializing to almost-simple Hurwitz numbers), $\mathcal{L}$ simplifies as

$$\mathcal{L} = pe^{-\beta \bar{t}_1 \bar{u}_0 p^{-1}},$$

and $\bar{u}_0 (= \partial^2 \mathcal{F}/\partial s^2)$ is determined by the single equation

$$\log \bar{u}_0 = \log Q + \beta s + \beta \sum_{k=1}^{\infty} k t_k \frac{(-k\beta \bar{t}_1 \bar{u}_0)^k}{k!}.$$

Remarkably, the functional structure of $\mathcal{L}$ essentially coincides with the defining equation $z = we^w$ of Lambert's W-function $w = W(z)$ ($\mathcal{L}^{-1} \leftrightarrow z$, $p^{-1} \leftrightarrow w$). Presumably, this will be related to the spectral curve in the sense of Eynard and Orantin.