超離散プリュッカー 関係式を用いた ソリトン解の証明について

On a proof of soliton solutions by the ultradiscrete Plücker relation

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We suggest a new proof of soliton solution to the ultradiscrete 2D Toda and the ultradiscrete KP equations by using the ultradiscrete Plücker relation.

Outline

1-1 Ultradiscretizing

Ultradiscrete soliton equations are obtained by ultradiscretizing discrete soliton equations.

Ex.) The discrete KdV equation(bilinear form)

$$F_{i+1}^{n+1} F_i^{n-1} = (1-\delta) F_{i+1}^n F_i^n + \delta F_{i+1}^{n-1} F_i^{n+1}.$$

(R. Hirota, J. Phys. Soc. Jpn. **43** (1977))
Transforming $F_i^n = e^{f_i^n/\varepsilon}$, $\delta = e^{-1/\varepsilon}$, and using key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{a/\varepsilon} + e^{b/\varepsilon}) = \max(a, b),$$

we obtain the ultradiscrete KdV(uKdV) equation.

$$f_{i+1}^{n+1} + f_i^{n-1} = \max(f_{i+1}^n + f_i^n,\ f_{i+1}^{n-1} + f_i^{n+1} - 1).$$

(S. Tsujimoto and R. Hirota, J. Phys. Soc. Jpn. **67** (1998))

The solution to the ultradiscrete KdV

Ultradiscrete soliton solutions are also obtained by ultradiscretizing discrete soliton solutions.

Ex.) N soliton solution to the discrete KdV equation is expressed by

$$F_i^n = \sum_{\mu_j=0,1} \exp\Big(\sum_{j=1}^{N} \mu_j S_j(n,i) + \sum_{1\le j<j'\le N} \mu_j \mu_{j'} A_{jj'}\Big).$$

N soliton solution to the uKdV equation is expressed by

$$f_i^n = \max_{\mu_j=0,1} (\sum_{j=1}^{N} \mu_j s_j(n,i) - \sum_{1\le j<j'\le N} \mu_j \mu_{j'} a_{jj'})$$

$$s_j(n,i) = p_j n - q_j i + c_j,$$

$$q_j = \frac{1}{2}(|p_j+1| - |p_j-1|),$$

$$a_{jj'} = |p_j + p_{j'}| - |p_j - p_{j'}|.$$

In this talk, the above solution is called "perturbation form solution（摂動形式解）"

Determinant solution

On the other hand, discrete soliton equations have another expression of the solution, determinant solution, which has revealed its algebraic structure.

Ex.) N soliton solution (determinant form) to the discrete KdV equation

$$F_i^n = \begin{vmatrix} \eta_1(n,i) & \eta_1(n+2,i) & \dots & \eta_1(n+2(N-1),i) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_N(n,i) & \eta_N(n+2,i) & \dots & \eta_N(n+2(N-1),i) \end{vmatrix}$$

Determinant solution

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$\rightarrow$ **Determinant solution cannot be ultradiscretized by reason of "negative problem".**

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{a/\varepsilon} - e^{b/\varepsilon}) = ?$$

Analogue

However, Takahashi and Hirota showed an analogue of determinant solution on ultradiscrete system. It is called ultradiscrete permanent(UP) (D Takahashi, R Hirota, “Ultradiscrete Soliton Solution of Permanent Type", J. Phys. Soc. Jpn. **76** (2007)).

Discrete soliton equation	→	Ultradiscrete soliton equation
Perturbation form solution	→	Perturbation form solution
Determinant form solution		UP form solution

The permanent of a matrix $(A_{ij})_{1 \le i,j \le N}$ is defined by signature-free determinant,

$$\mathrm{perm}[A_{ij}] = \sum_{\pi_i} A_{1\pi_1} A_{2\pi_2} \dots A_{N\pi_N}$$

Applying ultradiscretization for $N = 2$,

$$\lim_{\varepsilon \to +0} \epsilon \log \left(e^{(a_{11}+a_{22})/\epsilon} + e^{(a_{12}+a_{21})/\epsilon} \right)$$
$$= \max\left(a_{11} + a_{22}, a_{12} + a_{21}\right).$$

Then, let us define "**ultradiscrete permanent (UP)**" by

$$\max[a_{ij}] = \max_{\pi_i}\left(a_{1\pi_1} + a_{2\pi_2} + \cdots + a_{N\pi_N}\right).$$

Example of UP for $N = 2$

For 2×2 matrix,

- Determinant

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Permanent

$$\mathrm{perm} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} + a_{12}a_{21}$$

- UP

$$\max \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \max\left(a_{11} + a_{22}, a_{12} + a_{21}\right)$$

UP form solution to the uKdV

The uKdV equation is expressed by

$$f_i^{n+1} + f_{i-1}^{n-1} = \max(f_i^n + f_{i-1}^n, f_{i-1}^{n+1} + f_i^{n-1} - 1).$$

The UP form solution is expressed by

$$f_i^n = \frac{1}{2} \max \begin{bmatrix} |s_1(n,i)| & |s_1(n+2,i)| & \dots & |s_1(n+2(N-1),i)| \\ \vdots & \vdots & \ddots & \vdots \\ |s_N(n,i)| & |s_N(n+2,i)| & \dots & |s_N(n+2(N-1),i)| \end{bmatrix}$$

Here s_j is the same as the perturbation form solution. (D. Takahashi, R. Hirota, J. Phys. Soc. Jpn. **76** (2007))

Other result

Some results other than uKdV are also obtained by UP form

- the ultradiscrete mKdV soliton solution
- the ultradiscrete Toda soliton solution(H. Nagai, J. Phys. Math: Gen. A 41 (2008))
- Bäcklund transformation between the generalized ultradiscrete soliton solutions

Here, the generalized soliton solution, which includes that of uKdV, is defined by

$$f_i^n = \frac{1}{2}\max\begin{bmatrix} |s_1(n,i)| & |s_1(n+k,i+l)| & \dots & |s_1(n+k(N-1),i+l(N-1))| \\ \vdots & \vdots & \ddots & \vdots \\ |s_N(n,i)| & |s_N(n+k,i+l)| & \dots & |s_N(n+k(N-1),i+l(N-1))| \end{bmatrix},$$

$$s_j(n,i) = p_j n - q_j i + c_j.$$

How to prove

These results are proved by the following procedure:

1. Transforming UP form solution into perturbation form solution by using a key formula

$$\max \begin{bmatrix} |x_1 + (-N+1)y_1| & |x_1 + (-N+3)y_1| & \ldots & |x_1 + (N-1)y_1| \\ \vdots & \vdots & \ddots & \vdots \\ |x_N + (-N+1)y_N| & |x_N + (-N+3)y_N| & \ldots & |x_N + (N-1)y_N| \end{bmatrix}$$

$$= \max_{\sigma_i = \pm 1} \left(\sum_{i=1}^{N} \sigma_i x_i + \sum_{1 \le i < j \le N} \sigma_i \sigma_j y_i \right) + \sum_{1 \le i < j \le N} y_j ,$$

where

$$0 \le y_1 \le y_2 \le \cdots \le y_N .$$

How to prove

These results are proved by the following procedure:

2. Substituting perturbation form solution into the soliton equation and simplify the equation.

Ex.) In the case of uKdV

$$g(1) = \max(g(0), g(-1) - 1),$$

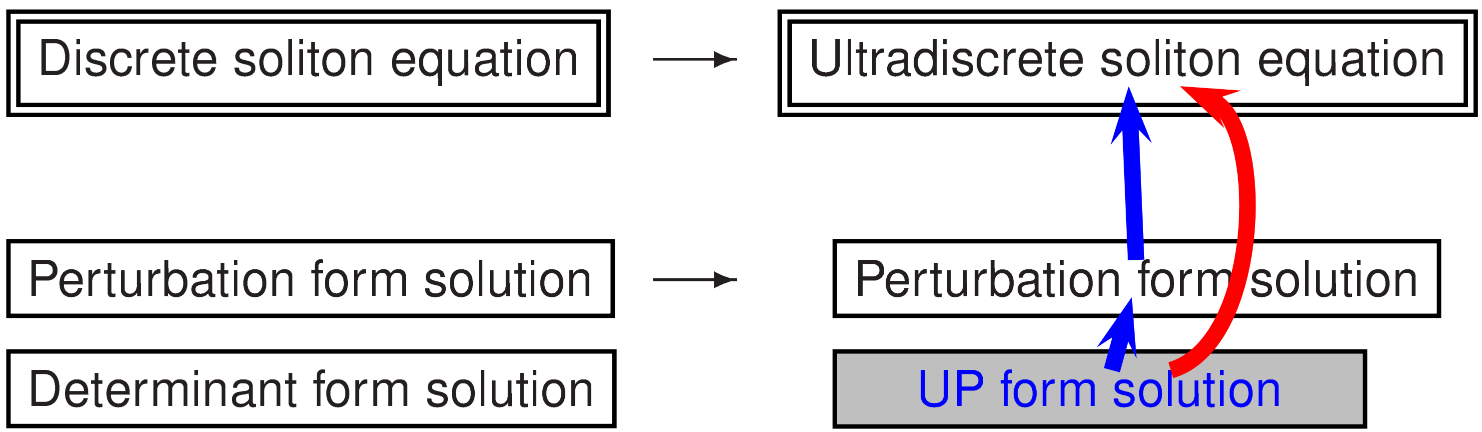
where

$$g(\alpha) = \max_{\sigma_j = \pm 1} \Big(\sum_{1 \le j \le N} \sigma_j (\alpha p_j - \frac{1}{2} q_j) - \sum_{1 \le j < j' \le N} \sigma_j \sigma_{j'} p_{j'} \Big).$$

3. Evaluating the maximum term included in $g(\alpha)$.

How to prove

The procedure of this proof is complicated and is done after transformations to the perturbation form solution.



1-3 Differences between UP and Det

Algebraic structure of UP is similar to that of determinant. For example,

<u>discrete case</u>

$$c \times \det[\boldsymbol{A_1}\ \boldsymbol{A_2}\ \ldots\ \boldsymbol{A_N}] = \det[c\boldsymbol{A_1}\ \boldsymbol{A_2}\ \ldots\ \boldsymbol{A_N}], \qquad (c : \text{const.})$$

where $\boldsymbol{A_i}$ is N dimensional column vector.

<u>ultradiscrete case</u>

$$c + \max[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}] = \max[c + \boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}]$$

where $\boldsymbol{a_i}$ is N dimensional column vector.

Differences between UP and Det

discrete case

$$\det[\boldsymbol{A_1} + \boldsymbol{B_1}\ \boldsymbol{A_2}\ \ldots\ \boldsymbol{A_N}]$$
$$= \det[\boldsymbol{A_1}\ \boldsymbol{A_2}\ \ldots\ \boldsymbol{A_N}] + \det[\boldsymbol{B_1}\ \boldsymbol{A_2}\ \ldots\ \boldsymbol{A_N}],$$

where $\boldsymbol{B_i}$ is N dimensional column vector.

ultradiscrete case

$$\max[\max(\boldsymbol{a_1},\ \boldsymbol{b_1})\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}]$$
$$= \max\Big(\max[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}],\ \max[\boldsymbol{b_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}]\Big),$$

where $\boldsymbol{b_i}$ is N dimensional column vector and $\max(\boldsymbol{a_j},\ \boldsymbol{b_j})$ denotes

$$\max(\boldsymbol{a_j},\ \boldsymbol{b_j}) = \begin{pmatrix} \max(a_{1j},\ b_{1j}) \\ \max(a_{2j},\ b_{2j}) \\ \ldots \\ \max(a_{Nj},\ b_{Nj}) \end{pmatrix}.$$

Differences between UP and Det

However, **such correspondence is not complete and there are many counter examples.**

For example,

<u>discrete case</u>

$$\det[\boldsymbol{A_1}\ \boldsymbol{A_1}\ \boldsymbol{A_3}\ \ldots\ \boldsymbol{A_N}] = 0.$$

<u>ultradiscrete case</u>

$$\max[\boldsymbol{a_1}\ \boldsymbol{a_1}\ \boldsymbol{a_3}\ \ldots\ \boldsymbol{a_N}] \neq -\infty$$

Differences between UP and Det

The Plücker relation for $n = 3$ is expressed by

$$\begin{aligned} &\det[\boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}}\ \boldsymbol{B_1}\ \boldsymbol{B_2}] \times \det[\boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}}\ \boldsymbol{B_3}\ \boldsymbol{B_4}] \\ -&\det[\boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}}\ \boldsymbol{B_1}\ \boldsymbol{B_3}] \times \det[\boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}}\ \boldsymbol{B_2}\ \boldsymbol{B_4}] \\ +&\det[\boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}}\ \boldsymbol{B_1}\ \boldsymbol{B_4}] \times \det[\boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}}\ \boldsymbol{B_3}\ \boldsymbol{B_4}] = \quad 0. \end{aligned}$$

Especially, using the notation

$$-- \equiv \boldsymbol{A_1}\ \ldots\ \boldsymbol{A_{N-2}},$$

the Plücker relation for $n = 3$ is expressed by

$$\begin{aligned} &\det[--\ \boldsymbol{B_1}\ \boldsymbol{B_3}] \times \det[--\ \boldsymbol{B_2}\ \boldsymbol{B_4}] \\ =&\det[--\ \boldsymbol{B_1}\ \boldsymbol{B_2}] \times \det[--\ \boldsymbol{B_3}\ \boldsymbol{B_4}] + \det[--\ \boldsymbol{B_1}\ \boldsymbol{B_4}] \times \det[--\ \boldsymbol{B_2}\ \boldsymbol{B_3}]. \end{aligned}$$

the Plücker relation for $n = 3$ is expressed by

$$\det[- - \ \boldsymbol{B_1}\ \boldsymbol{B_3}] \times \det[- - \boldsymbol{B_2}\ \boldsymbol{B_4}]$$
$$= \det[- - \ \boldsymbol{B_1}\ \boldsymbol{B_2}] \times \det[- - \boldsymbol{B_3}\ \boldsymbol{B_4}] + \det[- - \ \boldsymbol{B_1}\ \boldsymbol{B_4}] \times \det[- - \ \boldsymbol{B_2}\ \boldsymbol{B_3}].$$

In ultradiscrete case,

$$\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_3}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_4}]$$
$$= \max\Big(\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_2}] + \max[- - \ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_4}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_3}]\Big)$$

does not always hold .

Differences between UP and Det

the Plücker relation for $n = 3$ is expressed by

$$\det[- -\ \boldsymbol{B_1}\ \boldsymbol{B_3}] \times \det[- - \boldsymbol{B_2}\ \boldsymbol{B_4}]$$
$$= \det[- -\ \boldsymbol{B_1}\ \boldsymbol{B_2}] \times \det[- - \boldsymbol{B_3}\ \boldsymbol{B_4}] + \det[- -\ \boldsymbol{B_1}\ \boldsymbol{B_4}] \times \det[- -\ \boldsymbol{B_2}\ \boldsymbol{B_3}].$$

In ultradiscrete case,

$$\max\Big(\max[- -\ \boldsymbol{b_1}\ \boldsymbol{b_2}] + \max[- -\ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[- -\ \boldsymbol{b_1}\ \boldsymbol{b_3}] + \max[- -\ \boldsymbol{b_2}\ \boldsymbol{b_4}]\Big)$$
$$= \max\Big(\max[- -\ \boldsymbol{b_1}\ \boldsymbol{b_2}] + \max[- -\ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[- -\ \boldsymbol{b_1}\ \boldsymbol{b_4}] + \max[- -\ \boldsymbol{b_2}\ \boldsymbol{b_3}]\Big)$$
$$= \max\Big(\max[- -\ \boldsymbol{b_1}\ \boldsymbol{b_3}] + \max[- -\ \boldsymbol{b_2}\ \boldsymbol{b_4}],\ \max[- -\ \boldsymbol{b_1}\ \boldsymbol{b_4}] + \max[- -\ \boldsymbol{b_2}\ \boldsymbol{b_3}]\Big)$$

holds instead（長井秀友，高橋大輔，応用数理学会 (2007)）.

In this talk, we call the above relation **the ultradiscrete Plücker(uPlücker) relation**.

Differences between UP and Det

If we provide some conditions of $\boldsymbol{a_i}$ and $\boldsymbol{b_i}$ satisfy

$$\max[--\ \boldsymbol{b_1}\ \boldsymbol{b_3}]+\max[--\ \boldsymbol{b_2}\ \boldsymbol{b_3}]\geq\max[--\ \boldsymbol{b_1}\ \boldsymbol{b_2}]+\max[--\ \boldsymbol{b_3}\ \boldsymbol{b_4}]$$

or

$$\max[--\ \boldsymbol{b_1}\ \boldsymbol{b_3}]+\max[--\ \boldsymbol{b_2}\ \boldsymbol{b_4}]\geq\max[--\ \boldsymbol{b_1}\ \boldsymbol{b_4}]+\max[--\ \boldsymbol{b_2}\ \boldsymbol{b_3}],$$

the uPlücker relation is reduced to

$$\max[--\ \boldsymbol{b_1}\ \boldsymbol{b_3}]+\max[--\ \boldsymbol{b_2}\ \boldsymbol{b_4}]$$

$$=\max\Big(\max[--\ \boldsymbol{b_1}\ \boldsymbol{b_2}]+\max[--\ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[--\boldsymbol{b_1}\ \boldsymbol{b_4}]+\max[--\boldsymbol{b_2}\ \boldsymbol{b_3}]\Big).$$

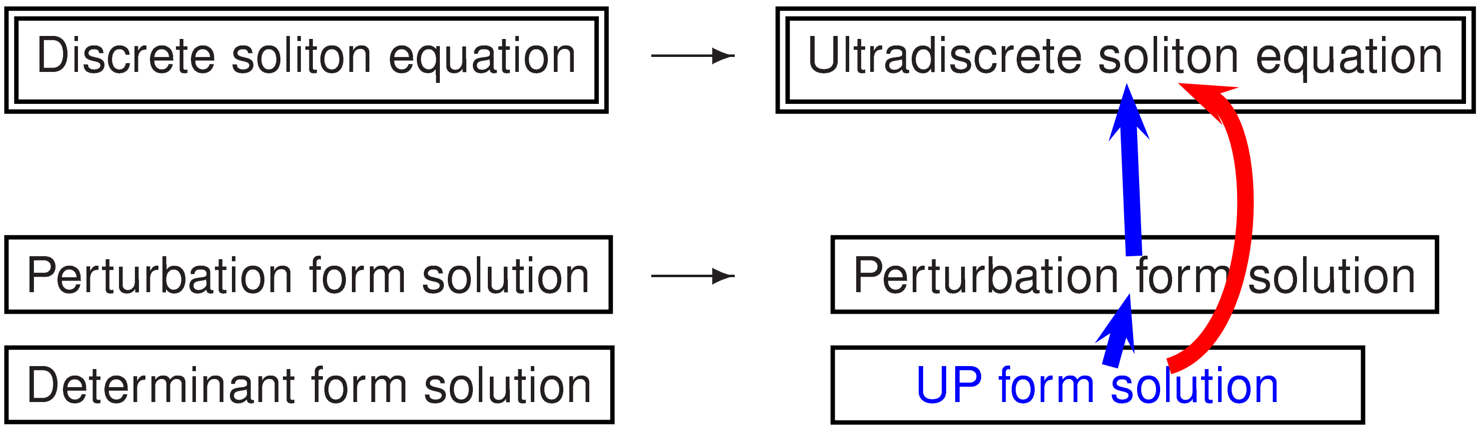
We call **the conditional uPlücker relation** in this talk.

Problem

These differences:

- UP which has the same column cannot be neglected
- The ultradiscrete Plücker relation is an implicit form

have made it difficut to prove not via perturbation form solution.



We suggest a new proof, which answer these problems, for N soliton solution to the ultradiscrete 2D Toda(u-2D Toda) and ultradiscrete KP(uKP) equation respectively. We show in the case of 3 soliton solution to u-2D Toda equation for example in this talk.

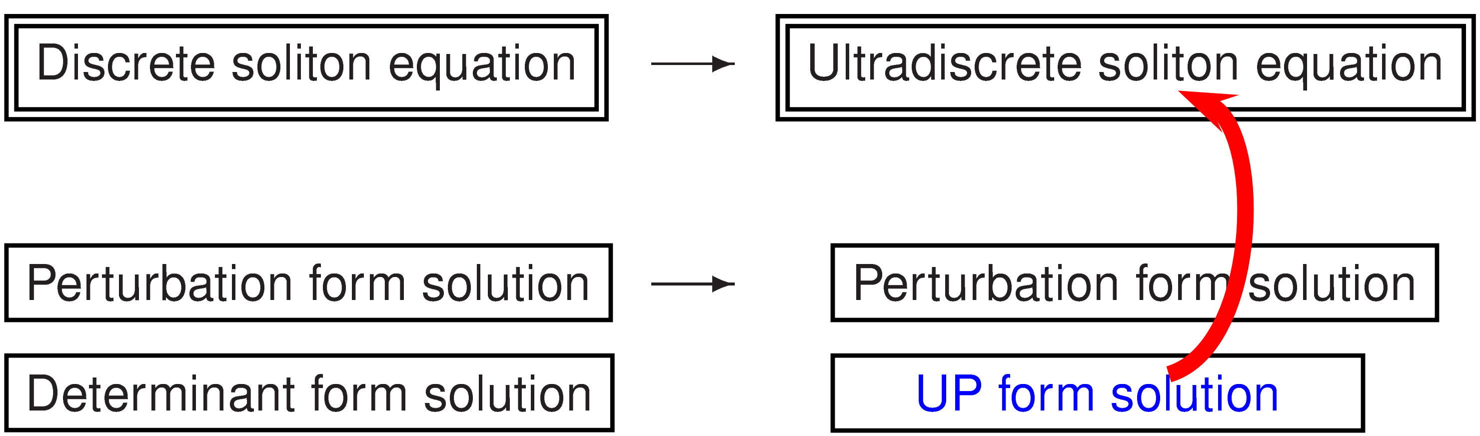
2-1 The u-2D Toda equation

2-2 N soliton solution of UP form

2-3 Reduction to the conditional uPlücker relation

2-4 In the case of the uKP equation

2-5 Conjecture



2-1 u-2D Toda equation

The discrete 2D Toda equation is expressed by

$$\tau^{dis}(l,m-1,n)\tau^{dis}(l+1,m,n)$$
$$=(1-\delta\varepsilon)\tau^{dis}(l,m,n)\tau^{dis}(l+1,m-1,n)+\delta\varepsilon\tau^{dis}(l,m-1,n+1)\tau^{dis}(l+1,m,n-1).$$

Here, blue text color denotes discrete system. (Progress of Theoretical Physics Supplement No. 94, (1988), Ryogo Hirota, Masaaki Ito and Fujio Kako).

The u-2D Toda equation is obtained by ultradiscretizing.

$$\tau(l,m-1,n)+\tau(l+1,m,n)$$
$$=\max\Big(\tau(l,m,n)+\tau(l+1,m-1,n),\ \ \tau(l,m-1,n+1)+\tau(l+1,m,n-1)-\delta-\varepsilon\Big)$$
$$(\delta,\varepsilon\geq 0)$$

Determinant form soliton solution to the discrete 2D Toda equation is expressed by

$$\tau^{dis}(l,m,n) = \det[\phi_i^{dis}(l,m,n+j-1)]_{1\le i,j\le 3},$$

where $\phi_i^{dis}(l,m,n)$ is defined by

$$\phi_i^{dis}(l,m,n) = \alpha_i(1+\delta r_i)^l(1+\frac{\varepsilon}{r_i})^{-m} r_i^n + \beta_i(1+\delta r_i')^l(1+\frac{\varepsilon}{r_i'})^{-m} {r_i'}^n.$$

Especially, ϕ_i^{dis} satisfies the following relations.

$$\phi_i^{dis}(l+1,m,n) = \phi_i^{dis}(l,m,n) + \delta\phi_i^{dis}(l,m,n+1)$$

$$\phi_i^{dis}(l,m-1,n) = \phi_i^{dis}(l,m,n) + \varepsilon\phi_i^{dis}(l,m,n-1).$$

2-2 UP soliton solution

UP form soliton solution to the u-2D Toda equation is given by

$$\tau(l,m,n) = \max[\phi_i(l,m,n+j-1)]_{1\le i,j\le 3},$$

(cf. $\tau^{dis}(l,m,n) = \det[\phi_i^{dis}(l,m,n+j-1)]_{1\le i,j\le 3}$)

where $\phi_i(l,m,n)$ is defined by

$$\phi_i(l,m,n) = \max\Big(\max(0,\ r_i-\delta)l - \max(0,\ -r_i-\varepsilon)m + r_i n + c_i,$$
$$\max(0,\ -r_i-\delta)l - \max(0,\ r_i-\varepsilon)m - r_i n + c_i'\Big)$$

(cf. $\phi_i^{dis}(l,m,n) = \alpha_i(1+\delta r_i)^l(1+\frac{\varepsilon}{r_i})^{-m} r_i^n + \beta_i(1+\frac{\delta}{r_i})^l(1+\varepsilon r_i)^{-m}\frac{1}{r_i^n}$ $\quad (r_i' = 1/r_i)$)

Especially, $\phi_i(l,m,n)$ satisfies the following relations.

$$\phi_i(l+1,m,n) = \max(\phi_i(l,m,n),\ \phi_i(l,m,n+1)-\delta)$$
$$\phi_i(l,m-1,n) = \max(\phi_i(l,m,n),\ \phi_i(l,m,n-1)-\varepsilon)$$

2-3 Reduction to the conditional uPlc̈ker relation

Outline of the proof on the ultradiscrete system is similar to that on discrete system. We can change the shift on l of m to that on n by the dispersion relations.

Ex.) $\tau(l+1,m,n) = F(\dots,\tau(l,m,n-1),\tau(l,m,n),\tau(l,m,n+1),\dots)$

$$\tau(l,m-1,n) = G(\dots,\tau(l,m,n-1),\tau(l,m,n),\tau(l,m,n+1),\dots)$$

We shall adopt a notation of the form

$$\tau(l,m,n) = \max \begin{bmatrix} \phi_1(l,m,n) & \phi_1(l,m,n+1) & \phi_1(l,m,n+2) \\ \phi_2(l,m,n) & \phi_2(l,m,n+1) & \phi_2(l,m,n+2) \\ \phi_3(l,m,n) & \phi_3(l,m,n+1) & \phi_3(l,m,n+2) \end{bmatrix}$$

$$\equiv \max[\boldsymbol{\phi}(\mathbf{0}) \;\; \boldsymbol{\phi}(\mathbf{1}) \;\; \boldsymbol{\phi}(\mathbf{2})]$$

$$\equiv \max[0 \; 1 \; 2]$$

for simplicity.

In this notation, $\tau(l+1,m,n)$ is reduced to

$$\tau(l+1,m,n) = \max\begin{bmatrix}\phi_1(l+1,m,n) & \phi_1(l+1,m,n+1) & \phi_1(l+1,m,n+2)\\ \phi_2(l+1,m,n) & \phi_2(l+1,m,n+1) & \phi_2(l+1,m,n+2)\\ \phi_3(l+1,m,n) & \phi_3(l+1,m,n+1) & \phi_3(l+1,m,n+2)\end{bmatrix}$$

$$= \max[\max(\boldsymbol{\phi}(\mathbf{0}),\boldsymbol{\phi}(\mathbf{1})-\delta)\ \ \max(\boldsymbol{\phi}(\mathbf{1}),\boldsymbol{\phi}(\mathbf{2})-\delta)\ \ \max(\boldsymbol{\phi}(\mathbf{2}),\boldsymbol{\phi}(\mathbf{3})-\delta)],$$

by the dispersion relation,

$$\begin{pmatrix}\phi_1(l+1,m,n)\\ \phi_2(l+1,m,n)\\ \phi_3(l+1,m,n)\end{pmatrix} = \begin{pmatrix}\max(\phi_1(l,m,n),\ \phi_1(l,m,n+1)-\delta)\\ \max(\phi_2(l,m,n),\ \phi_2(l,m,n+1)-\delta)\\ \max(\phi_3(l,m,n),\ \phi_3(l,m,n+1)-\delta)\end{pmatrix}$$

$$\equiv (\max(\boldsymbol{\phi}(\mathbf{0}),\boldsymbol{\phi}(\mathbf{1})-\delta)).$$

Using UP's property,

$$\max[\max(\boldsymbol{a_1},\ \boldsymbol{b_1})\ \ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}]$$
$$=\max\left(\max[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}],\ \max[\boldsymbol{b_1}\ \boldsymbol{a_2}\ \ldots\ \boldsymbol{a_N}]\right)$$

$\tau(l+1,m,n)$ is reduced into the maximum of two UPs.

$$\max[\max(\boldsymbol{\phi(0)},\boldsymbol{\phi(1)}-\delta)\ \max(\boldsymbol{\phi(1)},\boldsymbol{\phi(2)}-\delta)\ \max(\boldsymbol{\phi(2)},\boldsymbol{\phi(3)}-\delta)]$$
$$=\max\Big(\max[\boldsymbol{\phi(0)}\qquad \max(\boldsymbol{\phi(1)},\boldsymbol{\phi(2)}-\delta)\ \max(\boldsymbol{\phi(2)},\boldsymbol{\phi(3)}-\delta)],$$
$$\max[\boldsymbol{\phi(1)}-\delta\ \ \max(\boldsymbol{\phi(1)},\boldsymbol{\phi(2)}-\delta)\ \max(\boldsymbol{\phi(2)},\boldsymbol{\phi(3)}-\delta)]\Big)$$
$$=\max\Big(\max[\boldsymbol{\phi(0)}\ \max(\boldsymbol{\phi(1)},\boldsymbol{\phi(2)}-\boldsymbol{\delta})\ \max(\boldsymbol{\phi(2)},\boldsymbol{\phi(3)}-\boldsymbol{\delta})],$$
$$\max[\boldsymbol{\phi(1)}\ \max(\boldsymbol{\phi(1)},\boldsymbol{\phi(2)}-\boldsymbol{\delta})\ \max(\boldsymbol{\phi(2)},\boldsymbol{\phi(3)}-\boldsymbol{\delta})]-\delta\Big)$$

Recursively, $\tau(l+1,m,n)$ is reduced into the maximum of eight UPs.

$$\begin{aligned}&\tau(l+1,m,n)\\=&\max\Big(\max[0\ 1\ 2],\ \max[1\ 1\ 2]-\delta,\ \max[0\ 2\ 2]-\delta,\ \max[0\ 1\ 3]-\delta,\\&\max[1\ 2\ 2]-2\delta,\ \max[1\ 1\ 3]-2\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta\Big)\end{aligned}$$

where

$$\max[\boldsymbol{\phi(0)}\ \ \boldsymbol{\phi(1)}\ \ \boldsymbol{\phi(2)}]\equiv\max[0\ 1\ 2].$$

<u>Note</u> UP's property shows

$$\max[\boldsymbol{x_1}\ \boldsymbol{x_1}\ \boldsymbol{x_3}\ \ \ldots\ \ \boldsymbol{x_N}]\neq-\infty.$$

$\tau(l+1,m,n)$

$$\tau(l+1,m,n)$$
$$= \max\Big(\max[0\ 1\ 2],\ \max[1\ 1\ 2]-\delta,\ \max[0\ 2\ 2]-\delta,\ \max[0\ 1\ 3]-\delta,$$
$$\max[1\ 2\ 2]-2\delta,\ \max[1\ 1\ 3]-2\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta\Big)$$

Proposition

The following inequality holds.

$$\max[1\ 1\ 2] \le \max[0\ 2\ 2] \le \max[0\ 1\ 3],$$
$$\max[1\ 2\ 2] \le \max[1\ 1\ 3] \le \max[0\ 2\ 3],$$

$$\tau(l+1,m,n)$$
$$= \max\Big(\max[0\ 1\ 2],\ \max[1\ 1\ 2]-\delta,\ \max[0\ 2\ 2]-\delta,\ \max[0\ 1\ 3]-\delta,$$
$$\max[1\ 2\ 2]-2\delta,\ \max[1\ 1\ 3]-2\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta\Big)$$

Proposition

The following inequality holds.

$$\max[1\ 1\ 2] \le \max[0\ 2\ 2] \le \max[0\ 1\ 3],$$
$$\max[1\ 2\ 2] \le \max[1\ 1\ 3] \le \max[0\ 2\ 3],$$

This proposition means UPs which have the same column can be neglected.

Proof The definition of ϕ_i leads

$$\begin{aligned}&\phi_{i_1}(n+j)+\phi_{i_2}(n+j)\\ \le\,&\max(\phi_{i_1}(n+j-1)+\phi_{i_2}(n+j+1),\ \phi_{i_1}(n+j+1)+\phi_{i_2}(n+j-1))\end{aligned}$$

for $1 \le i_1, i_2, j \le N$.

Using this relation, for $i_1 = 2$, $i_2 = 3$, $j = 1$,

$$\phi_2(1)+\phi_3(1) \le \max(\phi_2(0)+\phi_3(2),\ \phi_2(2)+\phi_3(0)),$$

which corresponds to the relation between gray elements.

$$\max\begin{bmatrix} \phi_1(1) & \phi_1(1) & \phi_1(2) \\ \phi_2(1) & \phi_2(1) & \phi_2(2) \\ \phi_3(1) & \phi_3(1) & \phi_3(2) \end{bmatrix} \le \max\begin{bmatrix} \phi_1(0) & \phi_1(2) & \phi_1(2) \\ \phi_2(0) & \phi_2(2) & \phi_2(2) \\ \phi_3(0) & \phi_3(2) & \phi_3(2) \end{bmatrix}$$

Proof The definition of ϕ_i leads

$$\phi_{i_1}(n+j)+\phi_{i_2}(n+j)$$
$$\le \max(\phi_{i_1}(n+j-1)+\phi_{i_2}(n+j+1),\ \phi_{i_1}(n+j+1)+\phi_{i_2}(n+j-1))$$

for $1 \le i_1, i_2, j \le N$.

Using this relation, for $i_1 = 1$, $i_2 = 3$, $j = 1$,

$$\phi_1(1)+\phi_3(1) \le \max(\phi_1(0)+\phi_3(2),\ \phi_1(2)+\phi_3(0)),$$

which corresponds to the relation between gray elements.

$$\max\begin{bmatrix} \phi_1(1) & \phi_1(1) & \phi_1(2) \\ \phi_2(1) & \phi_2(1) & \phi_2(2) \\ \phi_3(1) & \phi_3(1) & \phi_3(2) \end{bmatrix} \le \max\begin{bmatrix} \phi_1(0) & \phi_1(2) & \phi_1(2) \\ \phi_2(0) & \phi_2(2) & \phi_2(2) \\ \phi_3(0) & \phi_3(2) & \phi_3(2) \end{bmatrix}$$

Proof The definition of ϕ_i leads

$$\begin{aligned} &\phi_{i_1}(n+j)+\phi_{i_2}(n+j) \\ \le &\max(\phi_{i_1}(n+j-1)+\phi_{i_2}(n+j+1),\ \phi_{i_1}(n+j+1)+\phi_{i_2}(n+j-1)) \end{aligned}$$

for $1 \le i_1, i_2, j \le N$.

Using this relation, for $i_1 = 1$, $i_2 = 2$, $j = 1$,

$$\phi_1(1)+\phi_2(1) \le \max(\phi_1(0)+\phi_2(2),\ \phi_1(2)+\phi_2(0)),$$

which corresponds to the relation between gray elements.

$$\max\begin{bmatrix} \phi_1(1) & \phi_1(1) & \phi_1(2) \\ \phi_2(1) & \phi_2(1) & \phi_2(2) \\ \phi_3(1) & \phi_3(1) & \phi_3(2) \end{bmatrix} \le \max\begin{bmatrix} \phi_1(0) & \phi_1(2) & \phi_1(2) \\ \phi_2(0) & \phi_2(2) & \phi_2(2) \\ \phi_3(0) & \phi_3(2) & \phi_3(2) \end{bmatrix}$$

Proof The definition of ϕ_i leads

$$\begin{aligned}&\phi_{i_1}(n+j)+\phi_{i_2}(n+j)\\ \leq&\max(\phi_{i_1}(n+j-1)+\phi_{i_2}(n+j+1),\ \phi_{i_1}(n+j+1)+\phi_{i_2}(n+j-1))\end{aligned}$$

for $1 \leq i_1, i_2, j \leq N$.

Using this relation,
Therefore,

$$\max\left[\begin{array}{ccc}\phi_1(1) & \phi_1(1) & \phi_1(2)\\ \phi_2(1) & \phi_2(1) & \phi_2(2)\\ \phi_3(1) & \phi_3(1) & \phi_3(2)\end{array}\right] \leq \max\left[\begin{array}{ccc}\phi_1(0) & \phi_1(2) & \phi_1(2)\\ \phi_2(0) & \phi_2(2) & \phi_2(2)\\ \phi_3(0) & \phi_3(2) & \phi_3(2)\end{array}\right]$$

$$\equiv \max[1\ 1\ 2] \leq \max[0\ 2\ 2]$$

Similarly, we can prove

$$\max[1\ 1\ 2]-\delta \le \max[0\ 2\ 2]-\delta \le \max[0\ 1\ 3]-\delta$$

$$\max[1\ 2\ 2]-2\delta \le \max[1\ 1\ 3]-2\delta \le \max[0\ 2\ 3]-2\delta \qquad \square.$$

Hence,

$$\begin{aligned} &\tau(l+1,m,n)\\ &=\max\Big(\max[0\ 1\ 2],\ \max[1\ 1\ 2]-\delta,\ \max[0\ 2\ 2]-\delta,\ \max[0\ 1\ 3]-\delta,\\ &\max[1\ 2\ 2]-2\delta,\ \max[1\ 1\ 3]-2\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta\Big). \end{aligned}$$

Similarly, we can prove

$$\max[1\ 1\ 2]-\delta \le \max[0\ 2\ 2]-\delta \le \max[0\ 1\ 3]-\delta$$

$$\max[1\ 2\ 2]-2\delta \le \max[1\ 1\ 3]-2\delta \le \max[0\ 2\ 3]-2\delta \qquad \square.$$

Hence,

$$\begin{aligned}&\tau(l+1,m,n)\\ =&\max\Big(\max[0\ 1\ 2],\ \max[0\ 1\ 3]-\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta\Big).\end{aligned}$$

That is, **UP that has the same column can be neglected for the definition of $\phi_i(l,m,n)$.**

Cf. Discrete system

In discrete system, the below reduction is obtained by elementary transformation.

$$\tau^{dis}(l+1,m,n)$$

$$= \det \begin{bmatrix} \phi_1^{dis}(l+1,m,n) & \phi_1^{dis}(l+1,m,n+1) & \phi_1^{dis}(l+1,m,n+2) \\ \phi_2^{dis}(l+1,m,n) & \phi_2^{dis}(l+1,m,n+1) & \phi_2^{dis}(l+1,m,n+2) \\ \phi_3^{dis}(l+1,m,n) & \phi_3^{dis}(l+1,m,n+1) & \phi_3^{dis}(l+1,m,n+2) \end{bmatrix}$$

$$= \det[\boldsymbol{\phi^{dis}(0)} + \delta\boldsymbol{\phi^{dis}(1)} \quad \boldsymbol{\phi^{dis}(1)} + \delta\boldsymbol{\phi^{dis}(2)} \quad \boldsymbol{\phi^{dis}(2)} + \delta\boldsymbol{\phi^{dis}(3)}]$$

$$= \det[0\ 1\ 2] + \delta \det[0\ 1\ 3] + \delta^2 \det[0\ 2\ 3] + \delta^3 \det[1\ 2\ 3]$$

$$\left(\tau(l+1,m,n) = \max\Big(\max[0\ 1\ 2],\ \max[0\ 1\ 3]-\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta\Big).\right)$$

Other arguments

Other $\tau(l+a, m+b, n+c)$ are reduced similarly.

- $\tau(l, m-1, n)$

$$= \max\Big(\max[0\ 1\ 2], \max[-1\ 1\ 2]-\varepsilon, \max[-1\ 0\ 2]-2\varepsilon, \max[-1\ 0\ 1]-3\varepsilon\Big)$$

- $\tau(l+1, m, n-1)$

$$= \max\Big(\max[-1\ 0\ 1], \max[-1\ 0\ 2]-\delta, \max[-1\ 1\ 2]-2\delta, \max[0\ 1\ 2]-3\delta\Big)$$

- $\tau(l, m-1, n+1)$

$$= \max\Big(\max[1\ 2\ 3], \max[0\ 2\ 3]-\varepsilon, \max[0\ 1\ 3]-2\varepsilon, \max[0\ 1\ 2]-3\varepsilon\Big)$$

- $\tau(l, m, n) = \max[0\ 1\ 2]$

Other arguments

$\tau(l+1, m-1, n)$ is obtained by the dispersion relations.

- $\tau(l+1, m-1, n)$

$$
\begin{aligned}
= \max\Big(&\max[0\ 1\ 2],\ \max[0\ 1\ 3]-\delta,\ \max[0\ 2\ 3]-2\delta,\ \max[1\ 2\ 3]-3\delta,\\
&\max[-1\ 1\ 2]-\varepsilon,\ \max(\max[0\ 1\ 2], \max[-1\ 1\ 3])-\delta-\varepsilon,\\
&\max(\max[0\ 1\ 3], \max[-1\ 2\ 3])-2\delta-\varepsilon,\ \max[0\ 2\ 3]-3\delta-\varepsilon,\\
&\max[-1\ 0\ 2]-2\varepsilon,\ \max(\max[-1\ 1\ 2], \max[-1\ 0\ 3])-\delta-2\varepsilon,\\
&\max(\max[0\ 1\ 2], \max[-1\ 1\ 3])-2\delta-2\varepsilon,\ \max[0\ 1\ 3]-3\delta-2\varepsilon,\\
&\max[-1\ 0\ 1]-3\varepsilon,\ \max[-1\ 0\ 2]-\delta-3\varepsilon,\\
&\max[-1\ 1\ 2]-2\delta-3\varepsilon,\ \max[0\ 1\ 2]-3\delta-3\varepsilon\Big)
\end{aligned}
$$

Substituting

In order to prove, we substitute each τ into

$$\max(\tau(l,m-1,n)+\tau(l+1,m,n),\ \underline{\tau(l,m,n)+\tau(l+1,m-1,n)-\delta-\varepsilon})$$
$$=\max(\underline{\tau(l,m,n)+\tau(l+1,m-1,n)},\ \tau(l,m-1,n+1)+\tau(l+1,m,n-1)-\delta-\varepsilon),$$

which is equivalent to the u-2D Toda equation

$$\tau(l,m-1,n)+\tau(l+1,m,n)$$
$$=\max(\tau(l,m,n)+\tau(l+1,m-1,n),\ \tau(l,m-1,n+1)+\tau(l+1,m,n-1)-\delta-\varepsilon)$$

for $\delta,\varepsilon\geq 0$. Then, it is proved when terms which have the same δ and ε of both side are equivalent respectively.

$$\tau^{dis}(l,m-1,n)\tau^{dis}(l+1,m,n)$$
$$=\underline{(1-\delta\varepsilon)\tau^{dis}(l,m,n)\tau^{dis}(l+1,m-1,n)}+\delta\varepsilon\tau^{dis}(l,m-1,n+1)\tau^{dis}(l+1,m,n-1).$$

Comparing

Ex.) $-4\delta - k_2\varepsilon \quad (k_2 = 0, 1, 2, 3)$

The terms which have $-4\delta - k_2\varepsilon$ in l.h.s are expressed by

$$\max[0\ 1\ 2] + \max\Big(\max[1\ 2\ 3], \max[0\ 2\ 3] - \varepsilon, \max[0\ 1\ 3] - 2\varepsilon, \max[0\ 1\ 2] - 3\varepsilon\Big).$$

On the other hand, that in r.h.s are expressed by

$$\max[0\ 1\ 2] + \max\Big(\max[1\ 2\ 3], \max[0\ 2\ 3] - \varepsilon, \max[0\ 1\ 3] - 2\varepsilon, \max[0\ 1\ 2] - 3\varepsilon\Big).$$

Therefore, the terms which have $-4\delta - k_2\varepsilon$ in both sides are equivalent.

Similarly, the terms which have $-k_2\varepsilon$, $-k_1\delta$, and $-k_1\delta - 4\varepsilon \quad (k_1, k_2 = 0, 1, 2, 3)$ are equivalent respectively.

Comparing

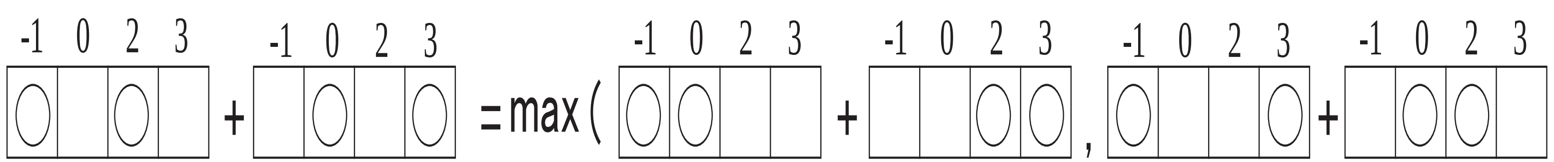
The other terms are not equivalent.
Ex.) $-\delta - \varepsilon$
Comparing both side, we have

$$\max\Big(\max[-1\ 1\ 2] + \max[0\ 1\ 3],\ 2\max[0\ 1\ 2]\Big)$$
$$= \max\Big(\max[-1\ 0\ 1] + \max[1\ 2\ 3],\ \max[0\ 1\ 2] + \max(\max[0\ 1\ 2],\ \max[-1\ 1\ 3])\Big).$$

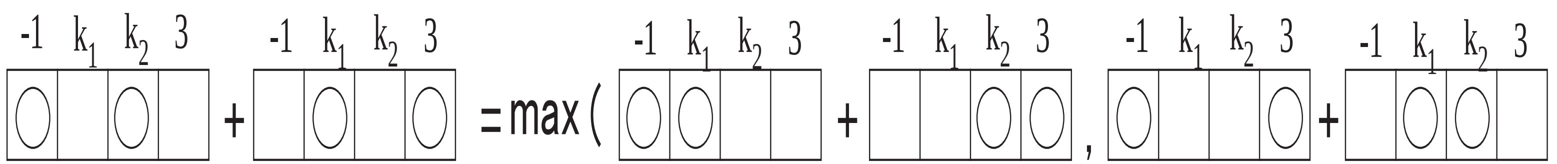
Removing the common term $2\max[0\ 1\ 2]$, we get a sufficient condition.

$$\max[-1\ 1\ 2]+\max[0\ 1\ 3] = \max\Big(\max[-1\ 0\ 1]+\max[1\ 2\ 3],\ \max[0\ 1\ 2]+\max[-1\ 1\ 3]\Big).$$

It is expressed by the Maya diagram.



Other terms are expressed as the following:



where $-1 < k_1 < k_2 < 3$. It is expressed by **the conditional uPlücker relation**,

$$\begin{aligned}
&\max[\boldsymbol{x_{-1}}\ \ldots\ \widehat{\boldsymbol{x_{k_1}}}\ \ldots\ \boldsymbol{x_2}] + \max[\boldsymbol{x_0}\ \ldots\ \widehat{\boldsymbol{x_{k_2}}}\ \ldots\ \boldsymbol{x_3}] \\
= \max\Big(&\max[\boldsymbol{x_{-1}}\ \ldots\ \widehat{\boldsymbol{x_{k_2}}}\ \ldots\ \boldsymbol{x_2}] + \max[\boldsymbol{x_0}\ \ldots\ \widehat{\boldsymbol{x_{k_1}}}\ \ldots\ \boldsymbol{x_3}], \\
&\max[\boldsymbol{x_{-1}}\ \ldots\ \widehat{\boldsymbol{x_{k_1}}}\ \ldots\ \widehat{\boldsymbol{x_{k_2}}}\ \ldots\ \boldsymbol{x_3}] + \max[\boldsymbol{x_0}\ \boldsymbol{x_1}\ \boldsymbol{x_2}]\Big).
\end{aligned}$$

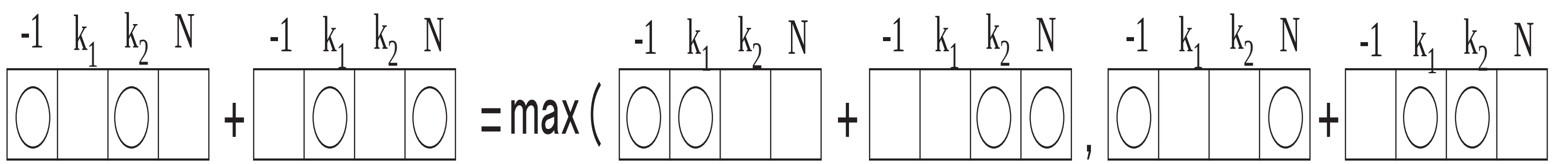
where

$$\boldsymbol{x_j} = (x_{ij})_{1\le i\le 3} = (|y_i + jr_i|)_{1\le i\le 3}.$$

by suitable transformation. And $\widehat{x_k}$ denotes the $(k+1)$-th column is deleted.

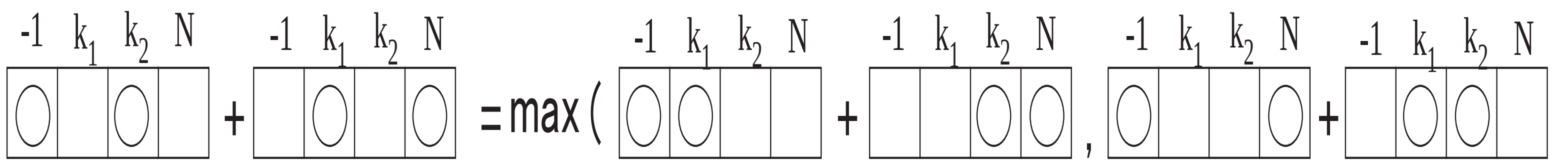
In general

The UP soliton solution to the u-2D Toda for N is reduced to the following Maya diagram($-1 < k_1 < k_2 < N$).



Note: the symbol k_i denotes $\boldsymbol{\phi(l, m, n + k_i)}$,

$$\boldsymbol{\phi(l, m, n + k_i)} = (\max(\eta_j(l, m, n) + k_i r_j,\ \eta'_j(l, m, n) - k_i r_j))_{1 \le j \le N}.$$



By suitable transformation, it is expressed by the conditional uPlücker relation.

$$
\begin{aligned}
&\max[\boldsymbol{x_0}\ \ldots\ \widehat{\boldsymbol{x_{k_1}}}\ \ldots\ \boldsymbol{x_N}] + \max[\boldsymbol{x_1}\ \ldots\ \widehat{\boldsymbol{x_{k_2}}}\ \ldots\ \boldsymbol{x_{N+1}}] \\
= \max\Big(&\max[\boldsymbol{x_0}\ \ldots\ \widehat{\boldsymbol{x_{k_2}}}\ \ldots\ \boldsymbol{x_N}] + \max[\boldsymbol{x_1}\ \ldots\ \widehat{\boldsymbol{x_{k_1}}}\ \ldots\ \boldsymbol{x_{N+1}}], \\
&\max[\boldsymbol{x_0}\ \ldots\ \widehat{\boldsymbol{x_{k_1}}}\ \ldots\ \widehat{\boldsymbol{x_{k_2}}}\ \ldots\ \boldsymbol{x_{N+1}}] + \max[\boldsymbol{x_1} \ldots\ \boldsymbol{x_N}]\Big).
\end{aligned}
$$

where

$$\boldsymbol{x_j} = (x_{ij})_{1\le i\le N} = (|y_i + jr_i|)_{1\le i\le N}.$$

However, a general proof has not been obtained yet.

We have checked for $N = 4$.

2-4 In the case of the uKP

The uKP is obtained by ultradiscretizing the discrete KP (Hirota-Miwa) equation.

$$\tau^{l,m+1,n+1} + \tau^{l+1,m,n+1} - a_2 - a_3$$
$$= \max\Big(\tau^{l+1,m,n} + \tau^{l,m+1,n+1} - a_1 - a_3, \tau^{l,m,n+1} + \tau^{l+1,m+1,n} - a_2 - a_3\Big) \qquad (a_1 > a_2 > a_3).$$

Especially, the uKP is equivalent to

$$\max\Big(\tau^{l+1,m,n} + \tau^{l,m+1,n+1} - a_1 - a_2,$$
$$\tau^{l,m+1,n} + \tau^{l+1,m,n+1} - a_2 - a_3,\ \tau^{l,m,n+1} + \tau^{l+1,m+1,n} - a_1 - a_3\Big)$$
$$= \max\Big(\tau^{l+1,m,n} + \tau^{l,m+1,n+1} - a_1 - a_3,$$
$$\tau^{l,m+1,n} + \tau^{l+1,m,n+1} - a_1 - a_2,\ \tau^{l,m,n+1} + \tau^{l+1,m+1,n} - a_2 - a_3\Big).$$

UP solution to the uKP

UP form solution is expressed by

$$\tau(l,m,n,s) = \max[\phi_i(l,m,n,s+j-1)]_{1\le i,j\le N}.$$

Here, function $\phi_i(l,m,n)$ is defined by

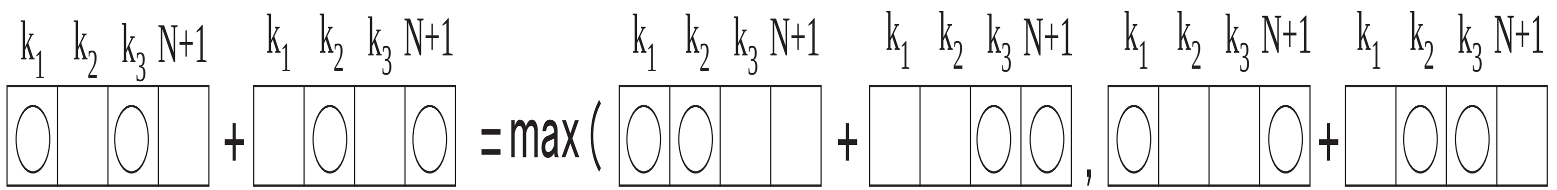
$$\begin{aligned}
&\phi_i(l,m,n,s)\\
&\quad = \max\Big(p_i s + \max(0,p_i - a_1)l + \max(0,p_i - a_2)m + \max(0,p_i - a_3)n,\\
&\qquad\qquad - p_i s + \max(0,-p_i - a_1)l + \max(0,-p_i - a_2)m + \max(0,-p_i - a_3)n\Big).
\end{aligned}$$

Function $\phi_i(l,m,n,s)$ satisfies

$$\phi_i(l+1,m,n,s) = \max(\phi_i(l,m,n,s),\ \phi_i(l,m,n,s+1) - a_1)$$
$$\phi_i(l,m+1,n,s) = \max(\phi_i(l,m,n,s),\ \phi_i(l,m,n,s+1) - a_2)$$
$$\phi_i(l,m,n+1,s) = \max(\phi_i(l,m,n,s),\ \phi_i(l,m,n,s+1) - a_3)$$

In the case of the uKP

The uKP equation reduces to the conditional uPlücker relation by the similar procedure.



Note: the symbol $0 < k_1 < k_2 < k_3 < N+1$ denotes

$$\boldsymbol{\phi(l, m, n, s + k_i)} = (\max(\eta_j(l, m, n, s) + k_i p_j,\ \eta'_j(l, m, n, s) - k_i p_j))_{1 \le j \le N}.$$

Both of the u-2D Toda and uKP equations reduces to the following conjecture.

Conjecture

$$
\begin{aligned}
&\max[\boldsymbol{x_0}\ \dots\ \widehat{\boldsymbol{x_{k_2}}}\ \dots\ \boldsymbol{x_N}] + \max[\boldsymbol{x_0}\ \dots\ \widehat{\boldsymbol{x_{k_1}}} \dots \widehat{\boldsymbol{x_{k_3}}}\ \dots\ \boldsymbol{x_{N+1}}] \\
= \max\Big(&\max[\boldsymbol{x_0}\ \dots\ \widehat{\boldsymbol{x_{k_3}}}\ \dots\ \boldsymbol{x_N}] + \max[\boldsymbol{x_0} \dots \widehat{\boldsymbol{x_{k_1}}}\ \dots\ \widehat{\boldsymbol{x_{k_2}}}\ \dots\ \boldsymbol{x_{N+1}}], \\
&\max[\boldsymbol{x_0}\ \dots\ \widehat{\boldsymbol{x_{k_2}}}\ \dots\ \widehat{\boldsymbol{x_{k_3}}}\ \dots\ \boldsymbol{x_{N+1}}] + \max[\boldsymbol{x_0}\ \dots\ \widehat{\boldsymbol{x_{k_1}}}\ \dots\ \boldsymbol{x_N}]\Big)
\end{aligned}
$$

where $0 \leq k_1 < k_2 < k_3 < N+1$ and

$$\boldsymbol{x_j} = (x_{ij})_{1 \leq i \leq N} = (|y_i + jr_i|)_{1 \leq i \leq N}$$

We have checked $N = 4$.

3 Concluding Remarks

- We suggest a new proof of UP form soliton solutions to the u-2D Toda and the uKP equation by using the conditional uPlücker relation. We have proved for $N = 4$ to be exact.
- Because of the condition of soliton solution, UP and ultradiscrete Plücker relation can be behaved like determinant.

the uPlücker relation

$$\max\Big(\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_2}] + \max[- - \ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_3}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_4}]\Big)$$
$$= \max\Big(\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_2}] + \max[- - \ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_4}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_3}]\Big)$$
$$= \max\Big(\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_3}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_4}],\ \max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_4}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_3}]\Big)$$

$\Downarrow$ Soliton form

the conditional uPlücker relation

$$\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_3}] + \max[- - \ \boldsymbol{b_2}\ \boldsymbol{b_4}]$$
$$= \max\Big(\max[- - \ \boldsymbol{b_1}\ \boldsymbol{b_2}] + \max[- - \ \boldsymbol{b_3}\ \boldsymbol{b_4}],\ \max[- - \boldsymbol{b_1}\ \boldsymbol{b_4}] + \max[- - \boldsymbol{b_2}\ \boldsymbol{b_3}]\Big).$$