Lax formalism for discrete Painlevé equations

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Integrable systems and their applications

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Aim:

Construction of a Lax formalism for discrete Painlevé equations.

[Ref: SIGMA 5(2009)42, arXiv:0811.1796]

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1. Differential Painlevé equations (P_{VI}) - short review
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5. Explicit computation - demo on Mathematica

1. Differential Painlevé equation

We will quickly recall the classical differential Painlevé equation P_{VI} and its Lax formalism.

P_{VI} equation: [P.Painlevé-B.Gambier, R.Fuchs]

$$\frac{d^2q}{dt^2} = \frac{1}{2}\Big(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\Big)\Big(\frac{dq}{dt}\Big)^2 - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\Big)\frac{dq}{dt}$$
$$+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\Big\{\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\Big\}.$$

$q = q(t)$:unknown function. $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

$$\alpha = \frac{a_1^2}{2}, \quad \beta = -\frac{a_4^2}{2}, \quad \gamma = \frac{a_3^2}{2}, \quad \delta = \frac{1-a_0^2}{2}.$$

$(a_0 + a_1 + 2a_2 + a_3 + a_4 = 1.)$

Hamiltonian form:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

Hamiltonian:

$$H = \frac{1}{t(t-1)}\Big[q(q-1)(q-t)p^2 + \big\{(a_1+2a_2)(q-1)q + a_3(t-1)q + a_4 t(q-1)\big\}p + a_2(a_1+a_2)(q-t)\Big].$$

The P_{VI} equation is a compatibility condition of the linear differential equations for $y(z,t)$.[R.Fuchs (1905)]

$$y_{zz}+\Big(\frac{1-a_4}{z}+\frac{1-a_3}{z-1}+\frac{1-a_0}{z-t}-\frac{1}{z-q}\Big)y_z$$

$$+\Big\{\frac{a_2(a_1+a_2)}{z(z-1)}-\frac{t(t-1)H}{z(z-1)(z-t)}+\frac{q(q-1)p}{z(z-1)(z-q)}\Big\}y=0,$$

$$y_t+\frac{z(z-1)(q-t)}{t(t-1)(q-z)}y_z-\frac{zp(q-1)(q-t)}{t(t-1)(q-z)}y=0.$$

Problem: Lax formalism for discrete Painlevé equations.

Jimbo, Sakai, Boalch, Murata, Arinkin, Borodin, Rains, $\cdots$

2. Discrete Painlevé equation

We will recall Sakai's theory of discrete Painlevé equations, in particular the q-P_{VI} equation and its Lax formalism.

Discrete Painlevé equations of 2nd order: [H.Sakai (2001)].

$$
\begin{array}{lccccccccccccccccc}
\text{Ell.} & E_8^{(1)} & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & \mathbb{Z} & & \\
 & & & & & & & & & & & & & \nearrow & & & & \\
\text{Mul.} & E_8^{(1)} & \rightarrow & E_7^{(1)} & \rightarrow & E_6^{(1)} & \rightarrow & D_5^{(1)} & \rightarrow & A_4^{(1)} & \rightarrow & A_{2+1}^{(1)} & \rightarrow & A_{1+1}^{(1)} & \rightarrow & A_1^{(1)} & \rightarrow & \mathcal{D}_6 \\
 & & & & & & & & & & & & & & & & & \\
\text{Add.} & E_8^{(1)} & \rightarrow & E_7^{(1)} & \rightarrow & E_6^{(1)} & \rightarrow & & & D_4^{(1)} & \rightarrow & A_3^{(1)} & \rightarrow & A_{1+1}^{(1)} & \rightarrow & A_1^{(1)} & \rightarrow & \mathbb{Z}_2 \\
 & & & & & & & & & & & & \searrow & & \searrow & & & \downarrow \\
 & & & & & & & & & & & & & A_2^{(1)} & \rightarrow & A_1^{(1)} & \rightarrow & 1
\end{array}
$$

These are characterized by point configurations.

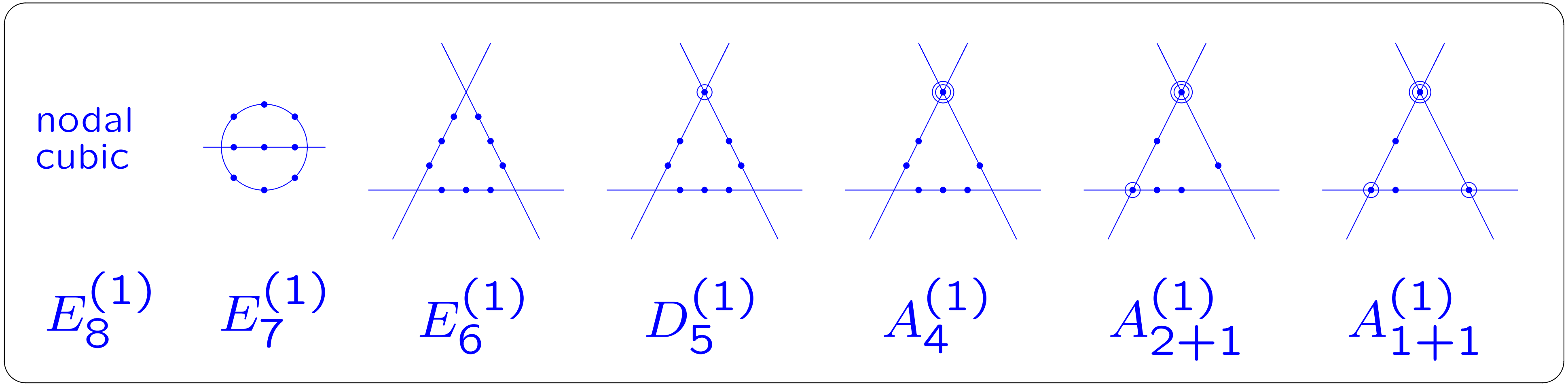
Point configurations:

9 points on $\mathbb{P}^2 \longrightarrow$ a curve of degree 3.

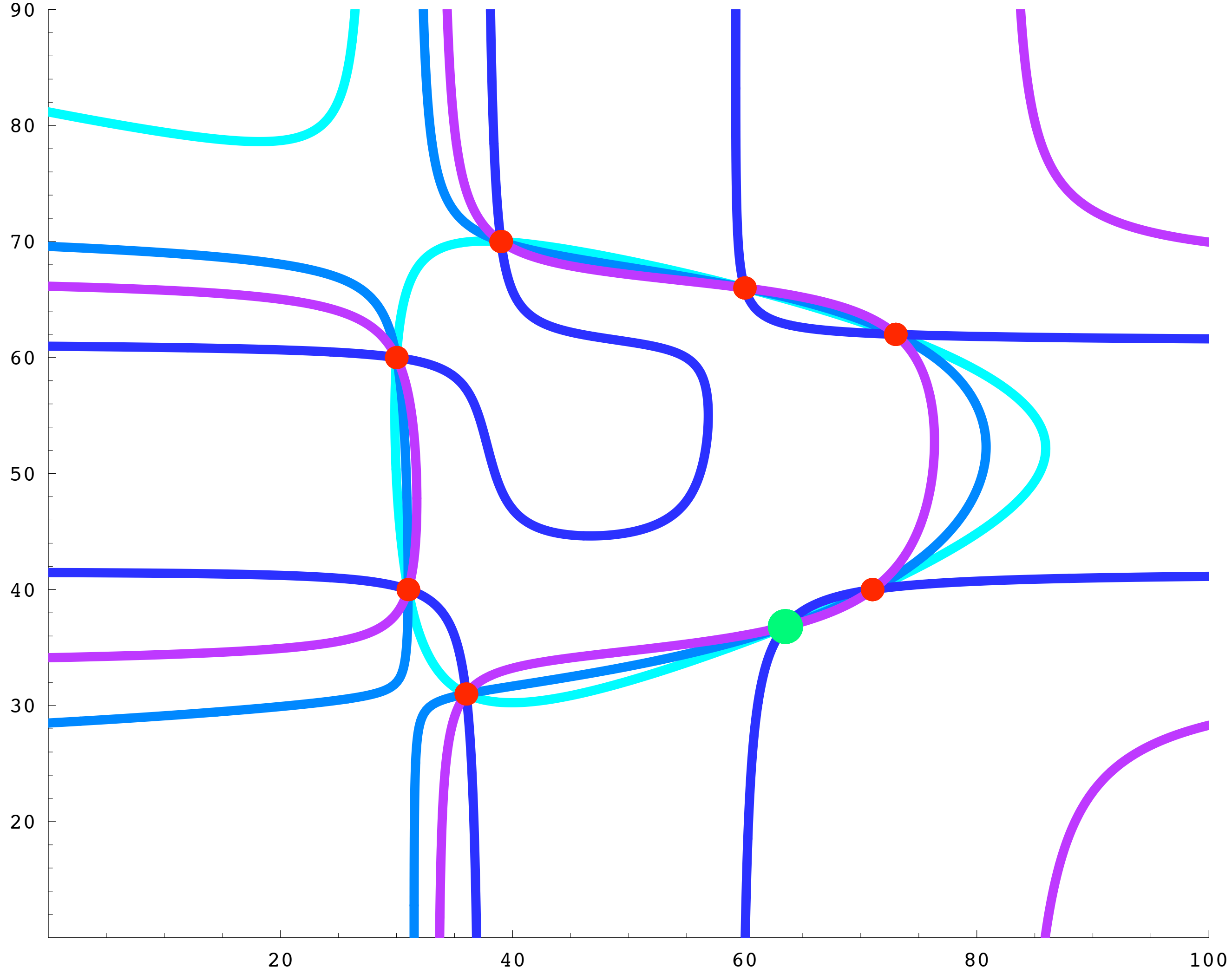
8 points on $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow$ a curve of degree (2,2).

The group structure on the curve: (Add.), (Mul.), (Ell.).

(Mul.) cases $\longrightarrow$ q-Painlevé equations:

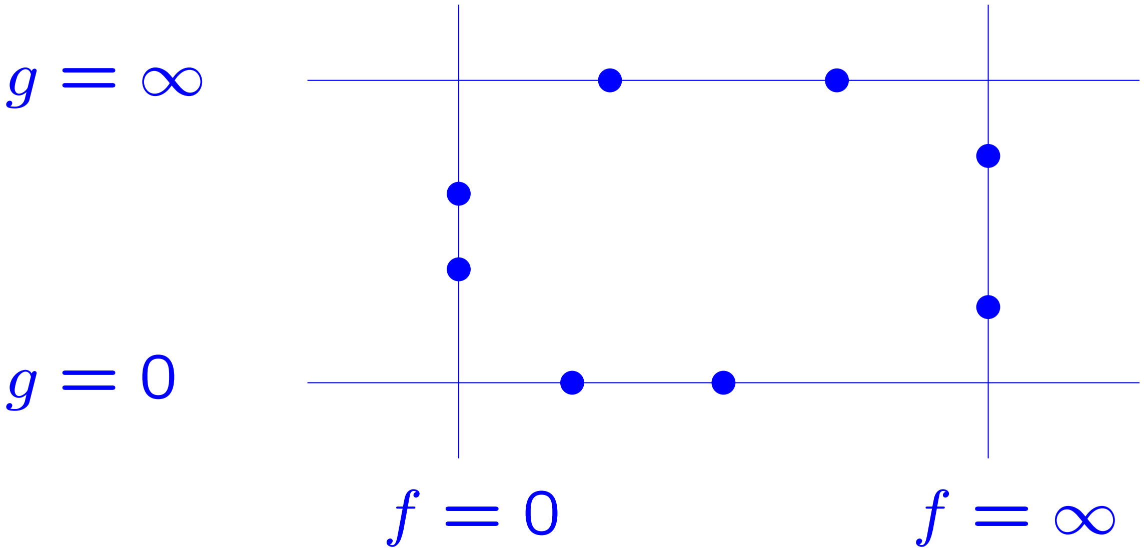


Curves of degree $(2, 2)$:



$D_5^{(1)}$-symmetry: ($\mathbb{P}^1 \times \mathbb{P}^1$ form)

$$(f,g) = (0, \frac{b_1}{q}),\ (0, \frac{b_2}{q}),\ (\infty, b_3),\ (\infty, b_4),$$
$$(a_1, 0),\ (a_2, 0),\ (a_3, \infty),\ (a_4, \infty).$$



q-P_{VI} equation: [Jimbo-Sakai (1996)]

$$T : (f, g, a_i, b_i) \mapsto (\dot{f}, \dot{g}, \dot{a_i}, \dot{b_i}),$$

$$\dot{f} f = \frac{(\dot{g} - b_1)(\dot{g} - b_2)}{(\dot{g} - b_3)(\dot{g} - b_4)} a_3 a_4,$$

$$\dot{g} g = \frac{(f - a_1)(f - a_2)}{(f - a_3)(f - a_4)} b_3 b_4,$$

$$\begin{pmatrix} \dot{a_1}, & \dot{a_2}, & \dot{a_3}, & \dot{a_4} \\ \dot{b_1}, & \dot{b_2}, & \dot{b_3}, & \dot{b_4} \end{pmatrix} = \begin{pmatrix} q a_1, & q a_2, & a_3, & a_4 \\ q b_1, & q b_2, & b_3, & b_4 \end{pmatrix},$$

$$q = \frac{a_3 a_4 b_1 b_2}{a_1 a_2 b_3 b_4}.$$

A Lax pair for q-P_{VI}:($\Leftrightarrow$ 2 $\times$ 2 Lax of Jimbo-Sakai)

$$\frac{(a_1-z)(a_2-z)}{a_1a_2(z-f)}y(qz)-\Big(c_0+c_1z+\frac{c_2z}{z-f}+\frac{c_3z}{z-qf}\Big)y(z)$$
$$+\frac{a_1a_2(z-qa_3)(z-qa_4)}{b_3b_4q^2(z-qf)}y(z/q)=0,$$

$$qg\ y(qz)-a_1a_2\ y(z)+z(z-f)\ T^{-1}y(z)=0.$$

$$c_0=-\frac{a_1a_2}{f}\Big(\frac{1}{b_1}+\frac{1}{b_2}\Big),\quad c_1=\frac{1}{q}\Big(\frac{1}{b_3}+\frac{1}{b_4}\Big),$$
$$c_2=\frac{(f-a_1)(f-a_2)}{qfg},\ \ c_3=\frac{(f-a_3)(f-a_4)g}{b_3b_4f}.$$

3. Geometry of Lax equations

For the cases with higher symmetries, the discrete Painlevé equations and their Lax equations are very complicated.

How to tame these wild beast ?

Basic idea: Characterize the Lax equation as an algebraic curve in variables (f, g), where $y(z)$ is regarded as parameter.

Observation: The lax equation for P_{VI}:

$$\bigcirc y'' + \bigcirc y' + \bigcirc y = 0$$

is a curve of degree 4 on $\mathbb{P}^2 = \{(\frac{1}{q} : p : \frac{1}{q-1})\}$ passing through the following 9+3+2 points:

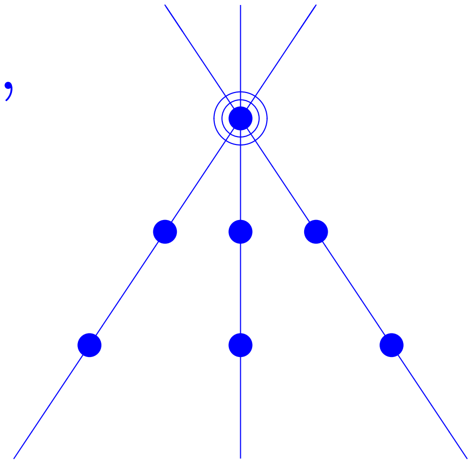
$$(0:0:1),\quad (1:-a_2:1),\quad (1:0:0),$$

$$(0:a_3:1),\quad (1:-a_1-a_2:1),\quad (1:a_4:0),$$

$$\Big((t-1)\varepsilon : 1 : t\varepsilon - a_0 t\varepsilon^2\Big)_{(\varepsilon^3=0)},$$

$$\Big((z-1)\varepsilon : 1 : z\varepsilon + z\varepsilon^2\Big)_{(\varepsilon^3=0)},$$

$$\Big(\frac{1}{z+\varepsilon} : \frac{y'(z+\varepsilon)}{y(z+\varepsilon)} : \frac{1}{z+\varepsilon-1}\Big)_{(\varepsilon^2=0)}.$$



Observation: The lax equation for q-P_{VI}:

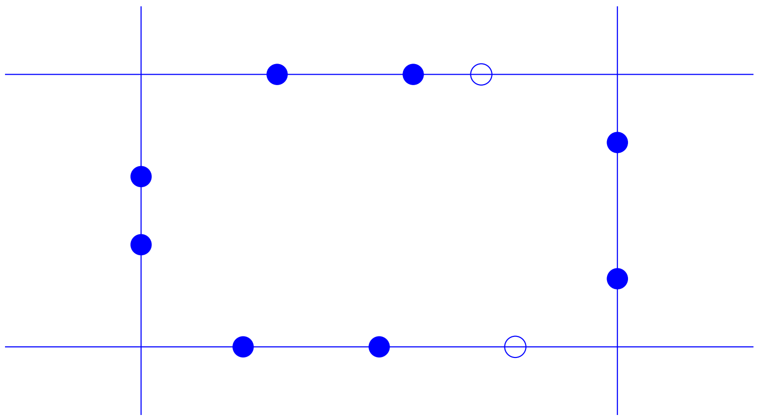
$$\bigcirc y(qz) + \bigcirc y(z) + \bigcirc y(z/q) = 0$$

is a curve of degree (3,2) on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the following 8+2+2 points:

$$(0, \frac{b_1}{q}), (0, \frac{b_2}{q}), (\infty, b_3), (\infty, b_4),$$
$$(a_1, 0), (a_2, 0), (a_3, \infty), (a_4, \infty),$$

$$(z, \infty), \ (\frac{z}{q}, 0),$$

$$\left(z, \frac{a_1 a_2}{q} \frac{y(z)}{y(qz)}\right), \ \left(\frac{z}{q}, \frac{a_1 a_2}{q} \frac{y(z/q)}{y(z)}\right).$$



It is easy to guess the general structure of Lax equations:

Curve of degree (3,2) in (f, g) variables, passing through
(1) the marked 8 points on C_0,
(2) generic point P_z on C_0,
(3) two more points depending on $y(qz), y(z), y(z/q)$.

The nontrivial problem is how to set up the condition (3).

[Hint] The condition (3) should be chosen adequately so that the Lax equation becomes linear in variables $y(qz), y(z), y(z/q)$.

4. q-$E_8^{(1)}$ Painlevé equation

We will construct a Lax formalism for the q-$E_8^{(1)}$ Painlevé equation using the geometric method.

Notation:

• $P_z = (f_z, g_z) = (z + \frac{h_1}{z}, z + \frac{h_2}{z})$: parametrization of (2,2) curve C_0.

• $P_i = (f_i, g_i) = P_{u_i}$ $(i = 1, \ldots, 8)$: 8 points on C_0.

$$q = \frac{h_1^2 h_2^2}{u_1 \cdots u_8}.$$

• (Abel's Theorem) P_{z_i} $(i = 1, \ldots, N = 2(m+n))$ are intersections of a (m,n) curve C_{mn} and the (2,2) curve C_0

$$\Leftrightarrow \quad h_1^m h_2^n = z_1 \cdots z_N.$$

Discrete time evolution: $\dot{x} = T(x)$:

• Time evolution of the parameters:

$$\dot{u_1} = u_1/q, \quad \dot{u_2} = qu_2, \quad \dot{u_i} = u_i \quad (i \neq 1, 2).$$

$\Rightarrow P_1, \dot{P}_2, P_3, \ldots, P_8$ are intersections of C_0 and another $(2,2)$ curve C. (C_0 and C form a pencil : one parameter family of elliptic curves).

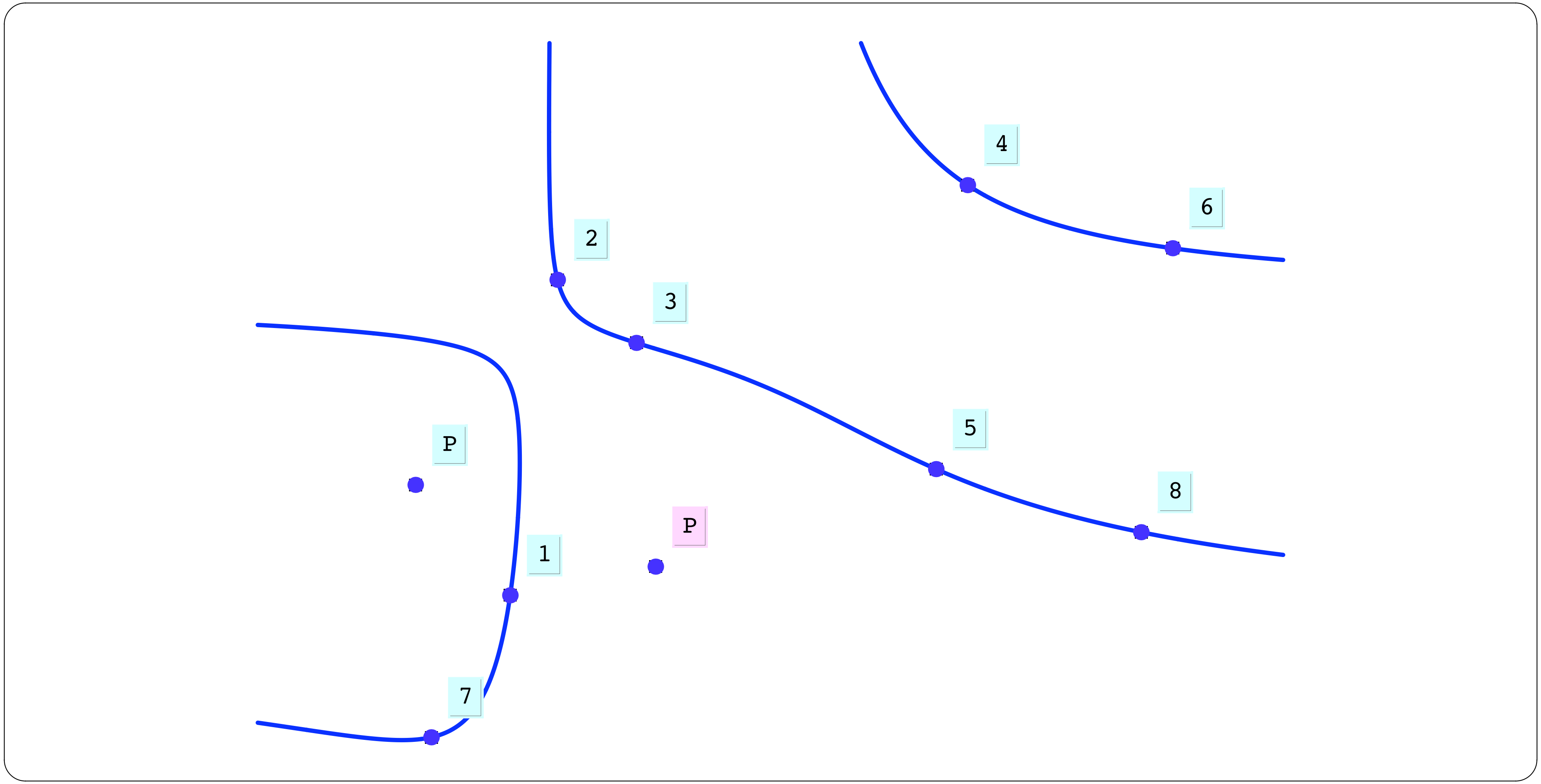
• Time evolution of the dependent variable P :

$$P_1 + P = \dot{P}_2 + \dot{P}.$$

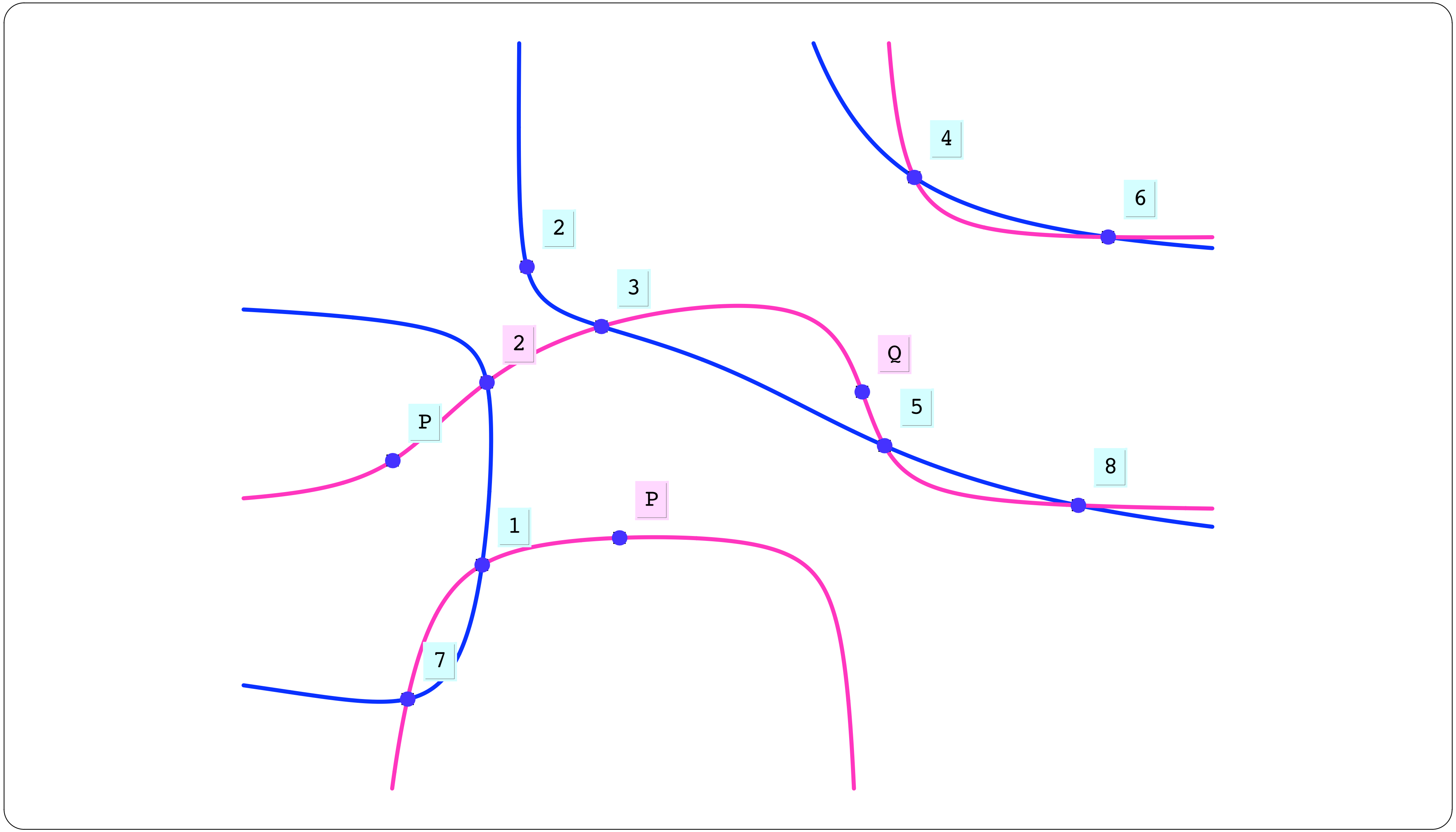
on a $(2,2)$ curve C : a member of the family.

(non-autonomous addition formula [KMNOY('03)])

1. Configuration of the 8 points and the $(2,2)$ curve C_0 passing through them:

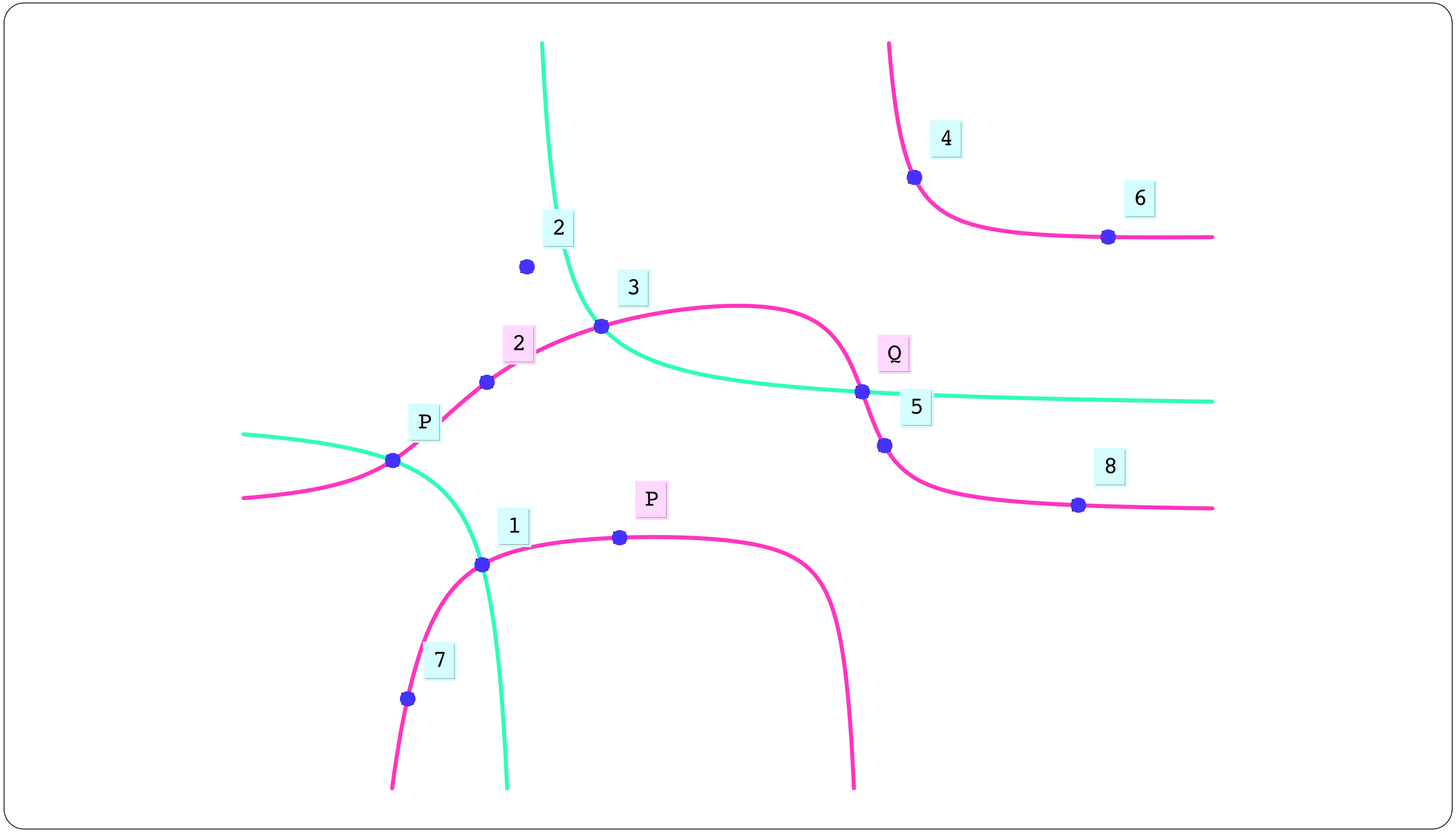


2. Another $(2,2)$ curve C passing through $P_1, P_3, \ldots, P_8$ and the initial point P:



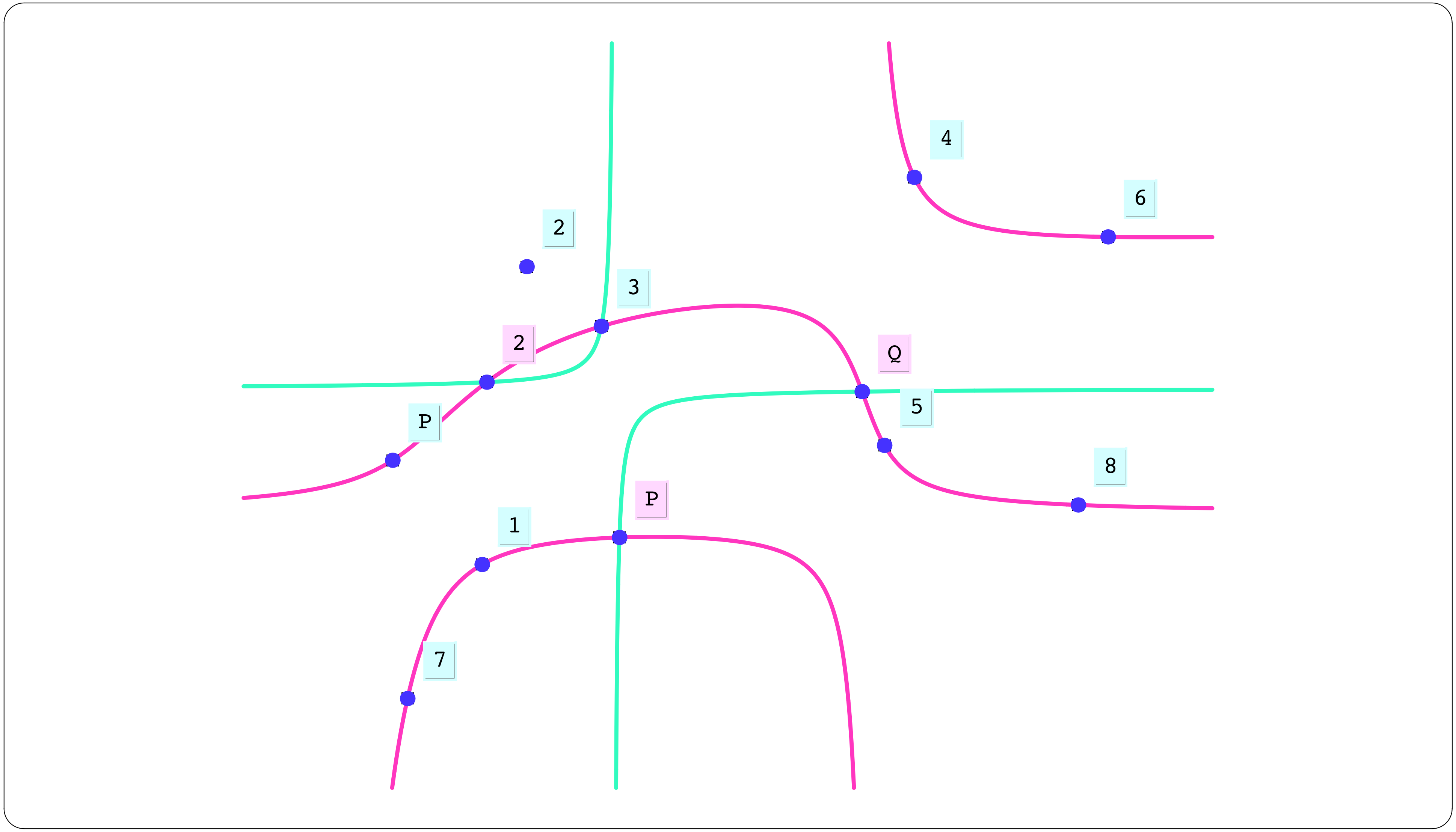
3. Intersection of C and a $(1,1)$ curve :

$$P_1 + P_3 + P + Q = 0.$$



4. Intersection of C and another $(1,1)$ curve :

$$\dot{P}_2 + P_3 + \dot{P} + Q = 0.$$



• Explicit formula for the time evolution of $P = (f, g)$:

$$\dot{f} = \frac{aF_0 + bF_1}{cF_0 + dF_1}, \qquad \dot{g} = \frac{a'G_0 + b'G_1}{c'G_0 + d'G_1}.$$

Where F_0, F_1 [or G_0, G_1] are polynomials in (f, g) of degree (5,4) [or (4,5)] , and they have zeros at $(P_1, P_2, \ldots, P_8)$ with multiplicity $(4, 0, 2, 2, \ldots, 2)$. $(30 - (10 + 6 \times 3) = 2)$

Coefficients $a, a', \ldots, d, d'$ are fixed by the condition:

$$\dot{P}_z = P_{zu_1/(qu_2)}.$$

We define Lax equations

$$L_1 = \bigcirc \overline{y} + \bigcirc y + \bigcirc \underline{y} = 0,$$
$$L_2 = \bigcirc \dot{y} + \bigcirc y + \bigcirc \underline{y} = 0,$$

as the curve of degree (3,2) on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the following 11 points:

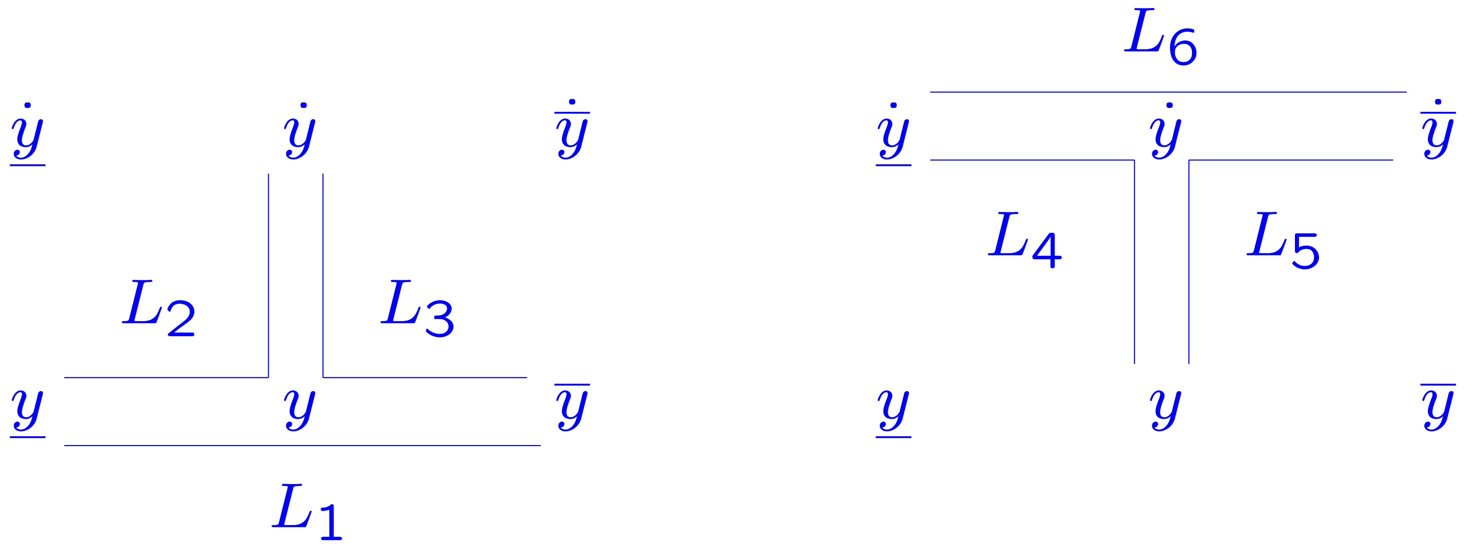
$$L_1: \; P_1, P_2, P_3, \ldots, P_8, P_z, \qquad Q_z, Q_{z/q},$$
$$L_2: \; P_1, P_3, , \ldots, P_8, P_{zu_2/u_1}, P_{h_1 q/z}, \; Q_{u_1}, Q_{z/q},$$

where

$$Q_x = (f_x, g) \quad : (g - g_x)y(x) = (g - g_{h_1/x})y(qx),$$
$$Q_{u_1} = (f_{u_1}, g) \; : (g - g_{u_1})y(z) = (g - g_{h_1/u_1})\dot{y}(z).$$

$(x = z, z/q)$

From eqs $L_1 = 0$, $L_2 = 0$, we derive $L_3 = 0, \cdots, L_6 = 0$:



The equation $L_6 = 0$ is a 3-term relation between $\overline{\dot{y}}, \dot{y}, \underline{\dot{y}}$.

Theorem (compatibility)

$$L_6(\{u_i\}, (f, g)) = 0 \Leftrightarrow L_1(\{\dot{u}_i\}, (\dot{f}, \dot{g})) = 0.$$

Summary.

We have constructed a Lax pair for discrete Painlevé equations through a geometric method.

Application.

Special solutions (determinants of ${}_{12}V_{11}$) by using the Padé interpolation. (in progress with Tsujimoto)