Deviation from Alday-Maldacena Duality for Wavy Circle

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GIMM arXiv : 0812.4702
IMM2 arXiv : 0803.1547

see also IMM arXiv : 0712.0159
IM8 arXiv : 0712.2316

I) Introduction

http://www2.yukawa.kyoto-u.ac.jp/~tetsuji/OPS.html

- Gauge / gravity correspondence continue to be the central themes of string theory

- ① BDS's (Bern, Dixon, Smirnov) conjectured exponentiation a la Sudakov of the all order planar n- gluon amplitudes for perturbative N=4 SYM,

$$\Rightarrow \quad e^{\kappa D_{\sqcap}}$$

which is now known to be slightly violated at n=6, L=2 loop level.

BDKKRSVV 08031465
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② A new version of gauge-string duality by Alday-Maldacena : computation at strong coupling by the minimal surface of an AdS_5 string

$$\Rightarrow \quad e^{A_{\sqcap}}$$

- ① and ② ⇒ A fruitful assessment of the issue $A_{\sqcap} \overset{?}{\approx} \kappa D_{\sqcap}$ and its reformulation are called for today.

Hints in the strong coupling limit would be supplied by IMM2

- Note that $A_{\sqcap} = \kappa D_{\sqcap}$, if it were true, would give a resolution to the AdS-minimal surface problem.

Contents

II) symbolically

$$\text{“}\ \mathcal{A}_n(\mathbf{p}_1,\ldots,\mathbf{p}_n|\lambda) \;=\; \mathcal{A}_{\text{tree}}\ \mathcal{A}_{\text{IR}}\ \mathcal{A}_{\text{finite}}\ \text{”}$$

factorizes & exponentiates

To be more precise,

$$\mathcal{A}_n = g^{n-2}\sum_{L=0}^{\infty} a^L \sum_{\rho} \mathrm{Tr}(T^{a_{\rho(1)}}\cdots T^{a_{\rho(n)}})A_n^{(L)}(\rho(1),\ldots,\rho(n))$$

$$\lambda = g^2 N$$

$$D = 4 - 2\epsilon$$

$$\rho : \text{ noncyclic perm}$$

$$a = \frac{\lambda\mu^{2\epsilon}}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$$

Define, with the help of MHV & N=4 SUSY,

$$M_n^{(L)}(\epsilon) \;\equiv\; \text{“}\ A_n^{(L)}(\epsilon)/A_n^{(0)}\ \text{”} \;:\; \text{scalar function}$$

$$\mathcal{A}_n = \mathcal{A}_n^{(0)} \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon)$$

$$\stackrel{?}{=} \mathcal{A}_n^{(0)} \exp\left[\sum_{\ell=1}^{\infty} a^\ell \left(f^{(\ell)}(\epsilon) M_n^{(1)}(\ell\epsilon) + c^{(\ell)} + o(\epsilon)\right)\right]$$

BDS conjecture

$$f^{(\ell)}(\epsilon) = f_0^{(\ell)} + \epsilon f_1^{(\ell)} + \epsilon^2 f_2^{(\ell)}$$

known

BES ↓

$$f(\lambda) \equiv 4 \sum_{\ell=1}^{\infty} a^\ell f_0^{(\ell)}$$ planar cusp anomalous dimension $\stackrel{\text{large}}{\sim} \sqrt{\lambda}$

$$g(\lambda) \equiv 2 \sum_{\ell=2}^{\infty} \frac{a^\ell}{\ell} f_1^{(\ell)}$$ planar collinear anomalous dimension

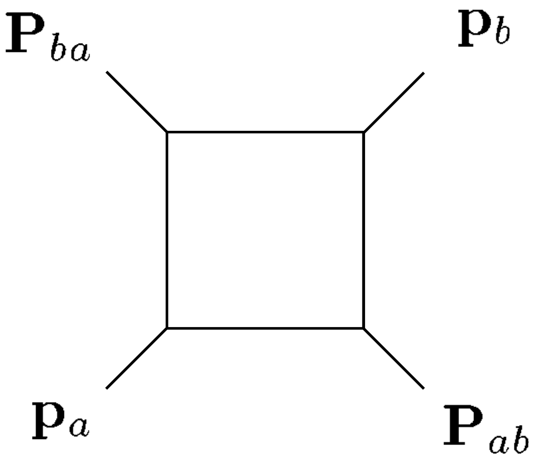
$$k(\lambda) \equiv -\frac{1}{2} \sum_{\ell=2}^{\infty} \frac{a^\ell}{\ell^2} f_2^{(\ell)}$$

III)

- $M_n^{(1)}$ is expressed as a sum of the so-called ``two mass easy box functions'' F^{2me} : (Bern, Dunbar, Dixon, Kosower)

$$M_n^{(1)} = \sum_{a<b} F^{2me}(\mathbf{p}_a, \mathbf{P}_{ab}, \mathbf{p}_b, \mathbf{P}_{ba}) \;, \quad \mathbf{P}_{ab} = \sum_{c=a+1}^{b-1} \mathbf{p}_c$$

massless ($\mathbf{p}_a$, $\mathbf{p}_b$) massive ($\mathbf{P}_{ab}$, $\mathbf{P}_{ba}$)



with

$$s = (p+P)^2 \;, \quad t = (p+Q)^2 \qquad P \equiv P_{ab}, \quad Q \equiv P_{ba}$$

$$a = \frac{s+t-P^2-Q^2}{st-P^2Q^2} \;, \quad \mathrm{Li}_p(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^p}$$

- dilogarithmic rep. of F^{2me}

$$F^{2me}(p,P,q,Q) = -\frac{1}{\epsilon^2}\left[\left(\frac{-s}{\mu^2}\right)^{-\epsilon} + \left(\frac{-t}{\mu^2}\right)^{-\epsilon} - \left(\frac{-P^2}{\mu^2}\right)^{-\epsilon} - \left(\frac{-Q^2}{\mu^2}\right)^{-\epsilon}\right]$$
$$+\mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)$$

- $D_{\sqcap} = \oint_{\sqcap} \oint_{\sqcap} \frac{dy^{\mu} dy'_{\mu}}{(y - y')^{2+\epsilon}}$ is known to reproduce, after computation, the rep. of $M_n^{(1)}$

  in momentum space

$\sqcap$: polygon made of light-like segment $=$ external momenta $\mathbf{p}_a$

Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini

$$= \sum (\text{pair of segments in } \sqcap)$$

$$= D_{\sqcap}^{(1)} + D_{\sqcap}^{(2)} + D_{\sqcap}^{(3)}$$

two identical adjacent nonadjacent

$$= 0 \qquad \sim \frac{1}{(-\epsilon)^2} \qquad \sim \text{dilog}$$

- $D_{\sqcap}$ $\sim$ log of abelian Wilson loop

$$< \exp\left(i \oint_{\sqcap} dy^{\mu} A_{\mu}(y)\right) > \quad \text{free field}$$

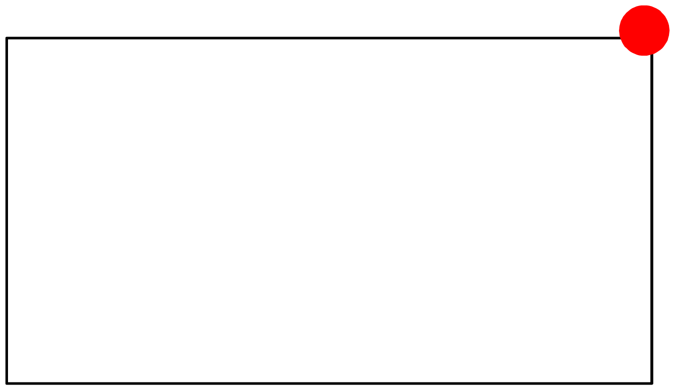
$$= \exp\left(-\frac{1}{2}\oint_{\sqcap} dy^{\mu} \oint_{\sqcap} dy'^{\nu} < A_{\mu}(y) A_{\nu}(y') > \right)$$

$$= \exp\left(\text{const } D_{\sqcap}\right)$$

$$M_n^{(1)} = (\text{const})\ D_{\sqcap}$$

- situation ; T-dualities operating

rep. on D instantons



our corner

rep. on D3 branes

weak coupling strong coupling

IV)

- AdS / CFT duality ; $\sqrt{\lambda} \equiv \sqrt{g^2 N} = \frac{R^2}{\alpha'}, \ \frac{1}{N} \sim g_s$ assumed

- compute the same gluon amplitude at strong coupling using tree level semiclassical string theory

- The original AdS_5 geometry

$$-(X^{-1})^2 - (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = -R^2$$

embedding into $R^{2,4}$

$$X^{-1} + X^4 = \frac{R}{z}, \ \ X^{-1} - X^4 = R\ \frac{z^2 + x_\mu x^\mu}{z}, \ \ X^\mu = R\ \frac{x^\mu}{z}$$

$$ds^2 = dX^\mu dX_\mu = R^2\ \frac{dz^2 + dx_\mu dx^\mu}{z^2}$$

- place D3 brane at $z = z_{IR} \ \rightarrow \ \infty$ (IR regulator)

- take T-duality in $\mu = 0,1,2,3$ directions

$$\partial_a x^\mu = i\,\frac{R^2}{r^2}\,\epsilon_{ac}\,\partial_c y^\mu\ , \quad r = \frac{R^2}{z}$$

- T-dualized geometry, which is again AdS_5

$$ds^2 = R^2\,\frac{dr^2 + dy_\mu dy^\mu}{r^2}\ , \quad r_{IR} = \frac{R^2}{z_{IR}} \to 0$$

- semiclassical string amplitudes

$$\sim \ \text{(prefactor)}\ e^{-S_E[y^\mu = y^\mu_{sa}, r=r_{cl}, k^I_\mu]}$$

- Equivalence of Neuman rep. and Dirichlet rep. (AdS$_5$ σ-model)

N : $I[z, x^\mu] = S_E[z, x^\mu] - i \int_{\mathcal{D}} d^2\xi \ J_\mu(\xi; k^\mu_I) x^\mu(\xi)$

$$J_\mu(\xi; k^\mu_I) = \sum_I k^\mu_I \delta^{(2)}(\xi - \xi_I)$$

$$\left. \frac{\delta I[z, x^\mu]}{\delta_1(z, x^\mu)} \right|_{z_{cl},\ x^\mu_{cl}} = 0$$

$$I[z_{cl}, x^\mu_{cl}] = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{2} \int_{\mathcal{D}} d^2\xi \, \frac{\partial \ln z_{cl}}{\partial \xi^a} \frac{\partial \ln z_{cl}}{\partial \xi^a} + \frac{i}{2} \sum_I k^\mu_I x^{cl}_\mu(\xi_I)$$

D :

$$I[r_{cl}, y^\mu_{cl}] = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\int_{\mathcal{D}} d^2\xi\, \frac{\partial \ln r_{cl}}{\partial \xi^a}\frac{\partial \ln r_{cl}}{\partial \xi^a} + \frac{\sqrt{\lambda}}{2\pi}(-\frac{1}{2})\epsilon^{ab}\int_{\partial\mathcal{D}} d\xi^a y^\mu \frac{1}{r^2}\partial_b y_\mu$$

The 2nd term, after T-duality relation, and `` the D-instanton condition '' $y^\mu|_{\text{boundary}}(s) = \sum_I y^\mu_{0I}\Theta(s_I < s < s_{I+1})$

$$= \frac{i}{\sqrt{2}}\frac{\sqrt{\lambda}}{2\pi}\int_{\mathcal{D}} ds \frac{dy^\mu}{ds} x_\mu(\xi)$$

$$\frac{2\pi}{\sqrt{\lambda}} k^\mu_I \equiv (y^\mu_{0I} - y^\mu_{0I-1})$$

$$= \frac{i}{2}\sum_I k^\mu_I x_\mu(\xi_I)$$

$$\therefore \; I[z_{cl}, x^\mu_{cl}] \;=\; S_E[r_{cl}, y^\mu_{cl}]$$

V)

- work on the Euclidean worldsheet
- choose $\xi^1 = y_1 \ , \quad \xi^2 = y_2$
- The 1st ansatz ; $y_3 = 0 \quad \cdots$ ①

$$S_{E,NG} = \frac{\sqrt{\lambda}}{2\pi} \int dy_1 dy_2 \sqrt{\det H} \ , \quad H_{ij} = \frac{1}{r^2}(\delta_{ij} - \partial_i y_0 \partial_j y_0 + \partial_i r \partial_j r)$$

recognize this as $f(\lambda) \overset{\lambda \text{ large}}{\sim} \sqrt{\lambda}$

- The 2nd ansatz ; $1 = y_\mu y^\mu + r^2 \ \Leftrightarrow \ Y^4 = 0 \quad \cdots$ ②

IMM1, IM8

- ① and ② form AdS_3 ansatz, which contains the Alday-Maldacena rhombus solution.

- Eq. of motion

$$\delta y_0 \quad : \quad \partial_1\left(\frac{H_{22}}{r^2\sqrt{\det H}}\partial_1 y_0\right)+\partial_2\left(\frac{H_{11}}{r^2\sqrt{\det H}}\partial_2 y_0\right) - \partial_1\left(\frac{H_{21}}{r^2\sqrt{\det H}}\partial_2 y_0\right)-\partial_2\left(\frac{H_{12}}{r^2\sqrt{\det H}}\partial_1 y_0\right)=0$$

$$\delta r \quad : \quad \partial_1\left(\frac{H_{22}}{r^2\sqrt{\det H}}\partial_1 r\right)+\partial_2\left(\frac{H_{11}}{r^2\sqrt{\det H}}\partial_2 r\right) - \partial_1\left(\frac{H_{21}}{r^2\sqrt{\det H}}\partial_2 r\right)-\partial_2\left(\frac{H_{12}}{r^2\sqrt{\det H}}\partial_1 r\right)+\frac{2\sqrt{\det H}}{r}=0$$

- linear approximation to NG

eliminate r^2 through $S_{E,NG}$ and ②, linearize w.r.t. y_0

$$\triangle_{21} y_0 = 0$$

, where $\triangle_{21} = \triangle_0 - \mathcal{D}^2 + \mathcal{D}$

$\triangle_0 = 4\partial\bar{\partial}, \ \mathcal{D} = z\partial + \bar{z}\bar{\partial}$

solution

$$y_0 = \sum_{k\geq 0} Re(\alpha_k z^k)\,{}_2F_1\!\left(\frac{k}{2},\frac{k-1}{2};k+1;z\bar{z}=x\right)$$

${}_2F_1(\ldots) \leftarrow (1+k\sqrt{1-x})(1-\sqrt{1-x})^k \ / \ x^k$

VI)

- how to deal with the issue $A_\sqcap \overset{?}{\approx} \kappa D_\sqcap$ fruitfully in the strong coupling side

- explicit examples containing ∞ ly many parameters needed

  $\Rightarrow$ an infinitesimal deformation of the unit circle into an arbitrary curve on the plane

- circle solution (formal $n = \infty$ limit of lightlike n-gon)

AdS_3 ansatz
$$\begin{cases} y_3 = 0 \\ 1 = r^2 + y_\mu y^\mu = r^2 - y_0^2 + y_i^2 \end{cases}$$

now $y_0 = 0$

- $L_{NG} = \frac{1}{r^2}\sqrt{1+(\partial_i r)^2}$

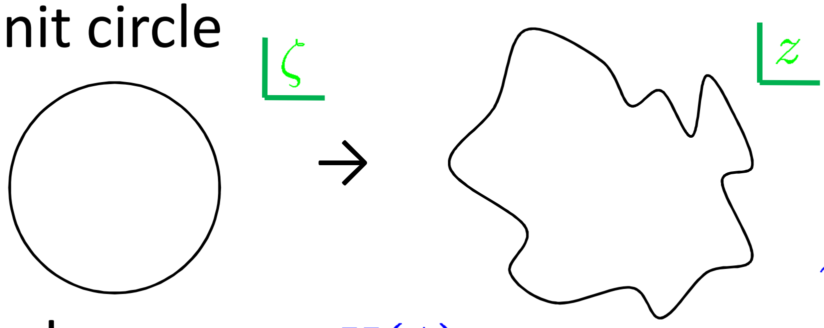
The only candidate to the solution which lie in these ansatze is

$r^2 = 1 - y_i^2$, which in fact solves IMM1

$$r(\partial_i r)(\partial_j r)(\partial_i \partial_j r) = (2 + r\partial^2 r)(1 + (\partial_i r)^2)$$

- formulation

unit circle



bdd by Π

$z = y_1 + iy_2$

- consider the conformal map $z = H(\zeta)$
- find the shape of the minimal surface

$$r^2(z, \bar{z}) = 1 - \zeta\bar{\zeta} + a(\zeta, \bar{\zeta})$$

by solving the NG eq. for $a(\zeta, \bar{\zeta})$ subject to the b.c.

$$a|_{|\zeta|=1} = 0$$

action

- some simplification due to $\bar{\partial} z = 0$
- write $\partial z = 1 + \sum_{k=1}^{\infty} k h_k \zeta^{k-1} \equiv \partial H \equiv 1 + \partial h$

$$S_{NG}[a,h] = \frac{\sqrt{\lambda}}{2\pi} \int d^2\zeta \frac{1}{r^2} \sqrt{|\partial H|^2 (|\partial H|^2 + 4\partial r \bar{\partial} r)}$$
$$= \frac{\sqrt{\lambda}}{2\pi} \int d^2\zeta \frac{|1+\partial h|^2 (1 - \zeta\bar{\zeta} + a + \frac{(\partial a - \bar{\zeta})(\bar{\partial} a - \zeta)}{|1+\partial h|^2})^{1/2}}{(1 - \zeta\bar{\zeta} + a)^{3/2}}$$

- need to compute a=a(h) (at least) to the lowest order in h
- regularization needed

$$0 = \partial \left(\frac{\partial \mathcal{L}}{\partial(\partial a)} \right) + \bar{\partial} \left(\frac{\partial \mathcal{L}}{\partial(\bar{\partial} a)} \right) - \frac{\partial \mathcal{L}}{\partial a} = \frac{1/4}{(1-\zeta\bar{\zeta})^{3/2}} \triangle_{\mathrm{IMM}} (a + \bar{\zeta} h + \zeta \bar{h}) + o(h^2)$$

↑

Eq. of motion

$$\triangle_{\mathrm{IMM}}\psi = (\triangle_0 - \mathcal{D}^2 + \mathcal{D})\psi = 0,\ \triangle_0 = 4\partial\bar{\partial},\ \mathcal{D} = z\partial + \bar{z}\bar{\partial}$$

has appeared again in a-linearized problem

- The solution which satisfies the boundary condition is

$$a(\zeta, \bar{\zeta}) = 2\sum_{k=1}^{\infty} Re(h_k \zeta^{k-1}) A_k(\zeta\bar{\zeta})$$

$$A_k(x) = F_{k-1}(x) - x$$

$$F_k(x) = \frac{(1 + k\sqrt{1-x})(1 - \sqrt{1-x})^k}{x^k}$$

- remaining procedure
  - substitute the solution into the regularized area
  - and evaluate it

VII)

i) Statement of our results

$$A^{reg}_{\sqcap} = A_{\sqcap} - \frac{\pi L_{\sqcap}}{2\mu} + 2\pi = -3\pi \left(\sum_{m,n} (-)^{m+n} A^{(m|n)}_{k_1 \ldots k_m | \ell_1 \ldots \ell_n} h_{k_1+1} \cdots h_{k_m+1} \bar{h}_{\ell_1+1} \bar{h}_{\ell_n+1} \right)$$

$$D^{reg}_{\sqcap} = D_{\sqcap} - \frac{\pi L_{\sqcap}}{4\lambda} + \frac{\pi^2}{2} = -\pi^2 \left(\sum_{m,n} (-)^{m+n} D^{(m|n)}_{k_1 \ldots k_m | \ell_1 \ldots \ell_n} h_{k_1+1} \cdots h_{k_m+1} \bar{h}_{\ell_1+1} \bar{h}_{\ell_n+1} \right)$$

$$\therefore \ \kappa_{\text{smooth}} = \binom{\pi}{3}^{-1}$$

$k_j, \ \ell_j$ sums understood

$\mu, \ \lambda$; regularization parameters, $L_{\sqcap}$; the length of the contour $\sqcap$

In IMM2

$$A^{(1|1)}_{k|k} = \frac{(k+1)k(k-1)}{6} = D^{(1|1)}_{k|k}$$

$$A^{(2|1)}_{k_1,k_2|k_1+k_2} = \frac{(k_1+1)(k_2+1)}{12}(k_1^2+k_2^2+3k_1k_2-k_1-k_2) = D^{(2|1)}_{k_1,k_2|k_1+k_2}$$

- will see that the conformal inv. + polynomial assumption on indices ensure

$$A^{(m|1)}_{k_1,\ldots,k_m|k_1+\ldots+k_m} = D^{(m|1)}_{k_1,\ldots,k_m|k_1+\ldots+k_m}$$

$$A^{(1|n)}_{\ell_1+\ldots+\ell_m|\ell_1,\ldots,\ell_n} = D^{(1|n)}_{\ell_1+\ldots+\ell_m|\ell_1,\ldots,\ell_n}$$

- The conformal inv. does not control the cases $(m,n)\geq 2$ with indices $k_i+1,\ \ell_i+1>2$ and

we find $A^{(2|2)}\neq D^{(2|2)}$ ∴ the hypothesis fails.

ii) $D_{\sqcap}$ again

- Representation of a generic coeff $D^{(m|n)}$ as a multiple sum

★ $$D^{(m;n)}_{k_1,\ldots,k_m|\ell_1,\ldots,\ell_n} = \text{symmetrized}\left(\sum_{i_1=0}^{k_1}\cdots\sum_{i_m=0}^{k_m}\sum_{j_1=0}^{\ell_1}\cdots\sum_{j_n=0}^{\ell_n}(k_m-i_m)(\ell_n-j_n)\right)$$

where $$\sum_{p=1}^{m} i_p+\sum_{q=1}^{n} j_q=\sum_{p=1}^{m} k_m=\sum_{q=1}^{n}\ell_q$$

- Examples $$D^{(2)}=\sum_{k=0}^{\infty}\frac{(k+1)k(k-1)}{6}|h_{k+1}|^2$$

$$D^{(2|1)}_{k_1,k_2|k_1+k_2}=D^{(1|2)}_{k_1+k_2|k_1,k_2}=\frac{1}{12}(k_1+1)(k_2+1)(k_1^2+3k_1k_2+k_2^2-k_1-k_2)$$

- all of $D^{(m|1)}$ can be summed to give

$$D^{(\cdot|1)} \equiv \sum_{m=1}^{\infty} D^{(m|1)}_{k_1,\dots,k_m|k_1+\dots+k_m} h_{k_1+1}\cdots h_{k_m+1}\bar{h}_{k_1+\dots+k_m+1} = -\frac{1}{6}\oint \bar{h}(\bar{\zeta}) S_{\zeta}(z)\zeta^2 d\zeta$$

$z = \zeta + \sum_k h_k \zeta^k$ and $S_{\zeta}(z) = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$; Schwarzian derivative

- outline of the derivation of ★ ;

$$D_{\sqcap} = \frac{1}{2}\oint_{|\zeta|=1}\oint_{|\zeta'|=1} PQ\overline{Q}\frac{d\zeta d\bar{\zeta}'}{|\zeta-\zeta'|^2} + (\ \zeta \leftrightarrow \zeta',\ \bar{\zeta} \leftrightarrow \bar{\zeta}'\ )$$

have found $(PQ\overline{Q})_{m|n} = (-q_{1|0})^{m-1}(-q_{0|1})^{n-1}(p_{1|0}-q_{1|0})(p_{0|1}-q_{0|1})$

Here $q_{1|0} = \frac{h(\zeta)-h(\zeta')}{\zeta-\zeta'},\ p_{1|0} = \frac{\partial h(\zeta)}{\partial \zeta}$.

Integrate twice.

iii) Evaluation of Minimal Area (Perturbative in $h, \bar{h}$)

- once again, eq. of motion ;

$$0 = \partial \left(\frac{\partial \mathcal{L}}{\partial(\partial a)} \right) + \bar{\partial} \left(\frac{\partial \mathcal{L}}{\partial(\bar{\partial} a)} \right) - \frac{\partial \mathcal{L}}{\partial a} = \frac{1}{4(1-\zeta\bar{\zeta})^{3/2}} \left\{ \triangle_{\mathrm{IMM}}(a(\zeta,\bar{\zeta})) + R(a; h, \bar{h}) \right\}$$

and

$$a = a^{(1)} + a^{(2)} + \cdots$$

- Iterative construction of the solution ;

$$\triangle_{\mathrm{IMM}}(a^{(1)}(\zeta,\bar{\zeta}) + \bar{\zeta}h(\zeta) + \zeta\bar{h}(\bar{\zeta})) = 0$$

$$\triangle_{\mathrm{IMM}}(a^{(k)}(\zeta,\bar{\zeta})) = -R^{(k)}(a; h, \bar{h}) \ , \quad k \geq 2$$

$$\uparrow$$

$$a = a^{(1)} + \cdots + a^{(k-1)}$$

- $a^{(k)}$ contributes first to $A^{(2k)}$, 2k-th order in $h, \bar{h}$

- inverting $\triangle_{\mathrm{IMM}}$;

  construction of the solution to

$$\triangle_{\mathrm{IMM}}(\zeta^k F_k(\eta)) = \zeta^k m_k(\eta) \ , \qquad \eta = \zeta\bar{\zeta}$$

finite at $\eta = 0$ and $F_k(\eta = 1) = 0$,

given the complete set of harmonics

$$\triangle_{\mathrm{IMM}}(\zeta^k g_k(\eta)) = 0 \ , \qquad \triangle_{\mathrm{IMM}}(\zeta^k \tilde{g}_k(\eta)) = 0$$

$$g_k(\eta) = \frac{1 + k\sqrt{1-\eta}}{(1+\sqrt{1-\eta})^k} \ , \quad \tilde{g}_k(\eta) = \frac{1 - k\sqrt{1-\eta}}{(1-\sqrt{1-\eta})^k}$$

$\Rightarrow$ method of constant variation

$$F_k(\eta) = -\tilde{g}_k(\eta)\int_{\sqrt{1-\eta}}^{1}\frac{m_k(1-u^2)d\mu_k}{\tilde{G}_k(u)} + g_k(\eta)\left\{\int_{\sqrt{1-\eta}}^{1}\frac{m_k(1-u^2)d\mu_k}{G_k(u)} + \int_0^1\left(\frac{1}{\tilde{G}_k(u)} - \frac{1}{G_k(u)}\right)m_k(1-u^2)d\mu_k\right\}$$

$$d\mu_k = \frac{(1 - k^2u^2)du}{2k(k^2-1)u^2}$$

- coincidence

$$A_i^{(1|1)} = \frac{i(i^2-1)}{6} = D_i^{(1|1)}$$

$$A_{ij}^{(2|1)} = \frac{(i+1)(j+1)}{12}\Big(i^2+j^2+3ij-i-j\Big) = D_{ij}^{(2|1)}$$

$$A_{ijk}^{(3|1)} = \frac{(i+1)(j+1)(k+1)}{18}\Big(i^2+j^2+k^2+3(ij+jk+ik)-(i+j+k)\Big) = D_{ijk}^{(3|1)}$$

- difference

$$\begin{aligned} A_{ij|kl}^{(2|2)} = \; & \delta_{i+j,k+l}\frac{k+1}{48(i+j-1)(i+j+1)} \\ & \times\Big\{2ij(i^4+5i^3j+8i^2j^2+5ij^3+j^4)+2(i+j)^5-2(k^2-k+1)i^2j^2 \\ & +k^2(k^2+k-2)(i^2-ij+j^2)+(3k^4+3k^3-10k^2+4k-2)ij-k^2(k^2+k-2) \\ & -\frac{1}{i+j}\Big(2(k^3+k^2-2k+2)(i^4+j^4)+(7k^3+9k^2-16k+16)ij(i^2+j^2)+(9k^3+15k^2-24k+24)i^2j^2 \\ & -2(k^3+k^2-2k+1)(i^2+j^2)-(5k^3+3k^2-8k+4)ij\Big)\Big\}\times h_{i+1}h_{j+1}\bar{h}_{k+1}\bar{h}_{l+1} \end{aligned}$$

$$\begin{aligned} D_{ij|kl}^{(2|2)} = \; & \delta_{i+j,k+l}\frac{1}{24}\Big((i+1)(j+1)(k+1)(i^2+3ij+j^2-i-j) \\ & -(i+j+2)(k+2)(k+1)k(k-1)+\frac{3}{5}(k+3)(k+2)(k+1)k(k-1)\Big)h_{i+1}h_{j+1}\bar{h}_{k+1}\bar{h}_{l+1} \end{aligned}$$

iv) Conformal Invariance

With the AdS3 ansatz, reduces to SL(2).
When acting on a functional F[z(s)] of a parametrized curve

Π : s → C

The three generators are

$$\hat{J}_- F = \oint \frac{\delta F}{\delta z(s)} ds \ , \qquad \hat{J}_0 F = \oint z \frac{\delta F}{\delta z(s)} ds$$

$$\hat{J}_+ F = \oint z^2 \frac{\delta F}{\delta z(s)} ds$$

$$\hat{J}_- = \frac{\partial}{\partial h_0} \ , \qquad \hat{J}_0 = \frac{\partial}{\partial h_1} + \sum_{k=0}^{\infty} h_k \frac{\partial}{\partial h_k}$$

$$\hat{J}_+ = \frac{\partial}{\partial h_2} + 2 \sum_{k=0}^{\infty} h_k \frac{\partial}{\partial h_{k+1}} + \sum_{k,m=0}^{\infty} h_k h_m \frac{\partial}{\partial h_{k+m}}$$

- the integrand of D_Π varies by a total derivative under infinitesimal version of SL(2,R) ; $z \to \frac{az+b}{cz+d}$

- SL(2,R) is an isometry within our ansatz

- Use :

 e.g. $A^{(3|1)}_{ijk}$ $\hat{J}_-$ annih. $\Rightarrow$ vanishes if any one of the indices is minus one

 with $A^{(2|1)}$, $A^{(1|1)}$ given

$\therefore$ Let

$$A^{(3|1)}_{ijk} = \alpha(i+1)(j+1)(k+1)\Big(i^2+j^2+k^2+\beta(ij+jk+ik)+\gamma(i+j+k)+\delta\Big)$$

$\hat{J}_0$, $\hat{J}_+$ annih. $\Rightarrow$ $\alpha = \dfrac{1}{18}$, $\beta = 3$, $\gamma = -1$, $\delta = 0$

- Hence, the assumptions of the polynomial structure and of the conformal inv. of $A_{\sqcap}$ and $D_{\sqcap}$ $\Rightarrow$ $A^{(m|1)} = D^{(m|1)}$

$$A^{(1|m)} = D^{(1|m)}$$

- $A^{(n|m)}$ with min $(m,n) \geq 2$ fails to be a polynomial.

v) Nonplanar Case

relax the requirement $y^0 = 0$

- the b.c. are now

$$r(\zeta,\bar{\zeta})\Big|_{\zeta\bar{\zeta}=1} = 0$$

$$y_0(\zeta,\bar{\zeta})\Big|_{\zeta\bar{\zeta}=1} = \sum_{k=0}^{\infty} q_k \zeta^k + \sum_{k=1}^{\infty} q_{-k}\bar{\zeta}^k$$

- the iterative procedure

$r(\zeta,\bar{\zeta})$ as before

$$y_0(\zeta,\bar{\zeta}) = b^{(1)}(\zeta,\bar{\zeta}) + b^{(2)}(\zeta,\bar{\zeta}) + \cdots$$

$$b^{(1)}(\zeta,\bar{\zeta}) = \sum_{k=0}^{\infty} q_k \zeta^k g_k(\zeta\bar{\zeta}) + \sum_{k=1}^{\infty} q_{-k}\bar{\zeta}^k g_k(\zeta\bar{\zeta})$$

- A and D coincide up to the cubic order in $h,\ \bar{h}$ and q .

vi) Nonuniversality of the κ-factor

Our case $\kappa_{\text{smooth}} = \frac{3}{\pi}$

n=4 $\kappa_{\square} = 1$ Alday-Maldacena 1

n = ∞ with
Π = infinitely long strip $\kappa_{\text{strip}} = 4\frac{(2\pi)^2}{(\Gamma(1/4))^4}$ Alday-Maldacena 3

In our series representation, $h_k \sim \frac{1}{k^2}$ can provide cusps or angles to the wavy circle.

This is exactly the case in which the series fails to be convergent and the renormalization procedure becomes different.

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