LSM定理の計算メモ

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Abstract

LSM定理におけるエネルギーの評価の計算のメモ．

[1]の6.2節にしたがって，計算を追う．

１次元の反強磁性鎖のハミルトニアンを考える．

$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}. \tag{0.1}$$

スピンの大きさSは一般とする．基底状態を$|\psi\rangle$とする．ねじれ演算子

$$U_{\mathrm{tw}} = e^{-i\sum_j \theta_j S^z_j} \tag{0.2}$$

を導入する．ここで，θ_jは

$$\theta_j = \begin{cases} 0 & (j < 0), \\ \frac{2\pi j}{L} & (j = 1, \dots, L), \\ 2\pi = 0 & (j \geq L), \end{cases} \tag{0.3}$$

とする．試行状態

$$|\psi_{\mathrm{tw}}\rangle = U_{\mathrm{tw}} |\psi\rangle \tag{0.4}$$

のエネルギー

$$\Delta E = \langle \psi_{\mathrm{tw}}|H|\psi_{\mathrm{tw}}\rangle - E_0 = \langle \psi_{\mathrm{tw}}|H|\psi_{\mathrm{tw}}\rangle - \langle \psi|H|\psi\rangle = \langle \psi|U^\dagger_{\mathrm{tw}} H U_{\mathrm{tw}} - H|\psi\rangle \tag{0.5}$$

を見積もる．スピン演算子の交換関係 $[S^\mu, S^\nu] = i\epsilon_{\mu\nu\rho} S^\rho$のみから従う

$$e^{i\theta S^z} S^{\pm} e^{-i\theta S^z} = e^{\pm i\theta} S^{\pm}, \quad S^{\pm} = S^x \pm i S^y, \tag{0.6}$$

に注意する．これから，

$$U^\dagger_{\mathrm{tw}} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} U_{\mathrm{tw}} = U^\dagger_{\mathrm{tw}} \left(\frac{S^+_j S^-_{j+1} + S^-_j S^+_{j+1}}{2} + S^z_j S^z_{j+1} \right) U_{\mathrm{tw}} \tag{0.7}$$

$$= \frac{e^{i(\theta_j - \theta_{j+1})} S^+_j S^-_{j+1} + e^{-i(\theta_j - \theta_{j+1})} S^-_j S^+_{j+1}}{2} + S^z_j S^z_{j+1} \tag{0.8}$$

$$= \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{2} \left[(e^{i(\theta_j - \theta_{j+1})} - 1) S^+_j S^-_{j+1} + (e^{-i(\theta_j - \theta_{j+1})} - 1) S^-_j S^+_{j+1} \right]. \tag{0.9}$$

よって，

$$\Delta E = \frac{1}{2} \sum_j \langle \psi | (e^{i(\theta_j - \theta_{j+1})} - 1) S^+_j S^-_{j+1} + (e^{-i(\theta_j - \theta_{j+1})} - 1) S^-_j S^+_{j+1} | \psi \rangle . \tag{0.10}$$

ここでは[1]に従って，

$$\Delta E = \langle\psi|U_{\mathrm{tw}}^\dagger H U_{\mathrm{tw}} - H|\psi\rangle \leq \langle\psi|U_{\mathrm{tw}}^\dagger H U_{\mathrm{tw}} - H|\psi\rangle + \langle\psi|U_{\mathrm{tw}} H U_{\mathrm{tw}}^\dagger - H|\psi\rangle \tag{0.11}$$

の右辺を評価すると，

$$\begin{aligned}
\Delta E &\leq \sum_j \langle\psi|(\cos(\theta_j - \theta_{j+1}) - 1)S_j^+ S_{j+1}^- + (\cos(\theta_j - \theta_{j+1}) - 1)S_j^- S_{j+1}^+|\psi\rangle && (0.12)\\
&= \sum_{j=0}^{L-1} (\cos(\theta_j - \theta_{j+1}) - 1) \langle\psi|S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+|\psi\rangle && (0.13)\\
&\leq \sum_{j=0}^{L-1} 2(1 - \cos(\theta_j - \theta_{j+1}))| \langle\psi|S_j^x S_{j+1}^x + S_j^y S_{j+1}^y|\psi\rangle | && (0.14)\\
&\leq \sum_{j=0}^{L-1} 2(1 - \cos(\theta_j - \theta_{j+1}))\|S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\| && (0.15)\\
&\leq 2(1 - \cos\frac{2\pi}{L}) \times 2S^2 L && (0.16)\\
&\leq (\frac{2\pi}{L})^2 \times 2S^2 L = \frac{8\pi^2 S^2}{L}. && (0.17)
\end{aligned}$$

References

[1] Tasaki, Hal. Physics and mathematics of quantum many-body systems. Vol. 66. Berlin: Springer, 2020.