Topological Non-Hermitian Physics: An Introduction

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Oct. 16-18, 2023 @Nagoya Univ.

Outline

1. Introduction
   — Overview of one-particle non-Hermitian systems

2. Gap condition and topology
   — Equivalence relation in general
   — Topology for matrices

3. Symmetry classes
   — 38 classes in non-Hermitian systems

4. Topological classification
   — Hermitianization and flattening
   — Classifying space
   — Dimensional reduction

5. Intrinsic non-Hermitian topology
   — Line gap implies point gap
   — Examples

Introduction: Overview of one-particle non-Hermitian systems

- Examples of non-Hermitian systems
- Some basic properties of non-Hermitian systems
- 1D hopping models

Non-Hermitian Systems

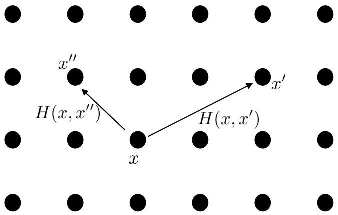
- Non-Hermitian Hamiltonians and matrices often appear in various physical systems.
- These include Photonics, Mechanics, Electrical Circuits, Biological Physics, Optomechanics, Hydrodynamics, Open Quantum Systems, and Non-unitary Conformal Field Theories.
- For more details on where non-Hermiticity shows up, see the review by, for example, [Ashida=Gong=Ueda, 2006.01837].

One-particle non-Hermitian Systems

- In this lecture, I will provide a brief introduction to the topological aspects of *one-particle* non-Hermitian systems. Specifically, we'll delve into the topological nature of matrices

$$H = \{H_{\sigma\sigma'}(x, x')\}_{x,x' \in \Lambda, \sigma, \sigma'=1,\dots,N}$$

  defined over a d-dimensional lattice, Λ, with internal degrees of freedom given by $\sigma = 1, \dots, N$.
- We'll assume the hopping range is local, i.e., $||H(x, x')|| < e^{-|x-x'|/\xi}$. (Otherwise, the concept of "dimension" would be meaningless.)
- Each physical system might possess intrinsic internal symmetries (which do not affect spatial positions).
- We may be interested in the physics robust against the disorder effect, which is compatible only with the internal symmetry.



Example: Wilson Dirac Operator

- In lattice gauge theory, we examine the lattice Dirac operator on the Euclidean cubic lattice. The Wilson Dirac operator is defined as:

$$D_W[U] = I - \kappa \sum_{\nu=1}^{3} [(I+\gamma_\nu)T_{\nu+} + (I-\gamma_\nu)T_{\nu-}] - \kappa \left[e^{\mu}(I+\gamma_4)T_{4+} + e^{-\mu}(I-\gamma_4)T_{4-}\right],$$

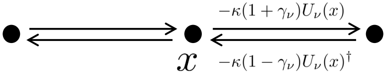
  where:

$$[T_{\nu+}]_{x,y} = U_\nu(x)\delta_{x+\hat{\nu},y}, \quad [T_{\nu-}]_{x,y} = U_\nu(y)^\dagger \delta_{x-\hat{\nu},y}.$$

  Here, $U_\mu(x) \in U(N)$ represents the $U(N)$ gauge field, and μ denotes the chemical potential.

- When the chemical potential μ is absent (i.e., $\mu = 0$), D_W satisfies the γ_5-Hermiticity condition:

$$\gamma_5 D_W[U]^\dagger \gamma_5 = D_W[U].$$



Ex. Mechanical Metamaterials

- Consider a mass-spring model with the equation of motion:

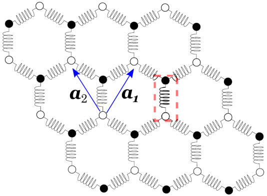
$$\ddot{\boldsymbol{u}} = -D\boldsymbol{u} + \Gamma\dot{\boldsymbol{u}},$$

  where $\boldsymbol{u} = \{u_i(x)\}_{x,i}$ denotes the displacement vector components at site x.
- The matrices D and Γ are real with D being positive semi-definite for system stability.
- Without friction, Γ is skew-symmetric (i.e., $\Gamma^T = -\Gamma$). However, this isn't generally the case.
- Using the variable $\tilde{\boldsymbol{u}} = (\sqrt{D}\boldsymbol{u}, i\dot{\boldsymbol{u}})^T$, the dynamics follows a Schrödinger-type equation [Kane=Lubensky 1308.0554, Süsstrunk=Huber 1604.01033.]:

$$i\frac{d}{dt}\tilde{\boldsymbol{u}} = H\tilde{\boldsymbol{u}}, \quad H = \begin{pmatrix} O & \sqrt{D} \\ \sqrt{D} & i\Gamma \end{pmatrix}.$$

- The Hamiltonian H inherently exhibits particle-hole symmetry:

$$\sigma_z H^* \sigma_z = -H.$$



[Figure from Yoshida=Hatsugai, PRB **100**, 054109 (2019)]

Some characteristics of Non-Hermitian Matrices

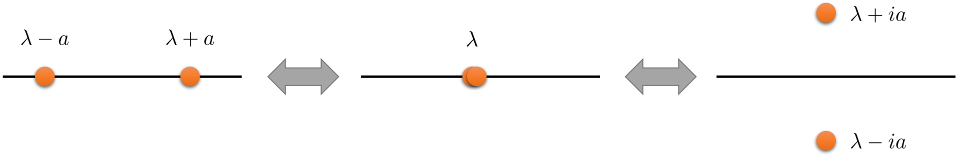
- Eigenvalues can be complex.
- Exceptional Points: These occur when the dimension of the Jordan block is 2 or more, making the matrix H non-diagonalizable. Example matrices include:

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

- Non-Hermitian Skin Effect [Yao=Wang 1803.01876]: The matrix behavior is sensitive to different boundary conditions, such as periodic boundary condition (PBC), open boundary condition (OBC), and semi-infinite boundary condition, among others.

PT Symmetry Breaking Bender=Boettcher physics/9712001

- For matrices with PT-symmetry, represented by $H^* = H$, eigenvalues either appear as an isolated real value, $E^* = E$, or as a conjugate pair, (E, E^*).
- PT-symmetry breaking refers to the transition where two real eigenvalues merge to form a complex conjugate pair (E, E^*), or vice versa. Such transitions occur at an exceptional point.



$$H = \begin{bmatrix} \lambda & a \\ a & \lambda \end{bmatrix} \qquad H = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix} \qquad H = \begin{bmatrix} \lambda & a \\ -a & \lambda \end{bmatrix}$$

数値実験

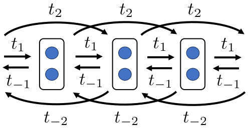
- ２つの$n \times n$複素行列H_0, H_1をランダムに生成する.
- H_0, H_1を線形に繋ぐハミルトニアン

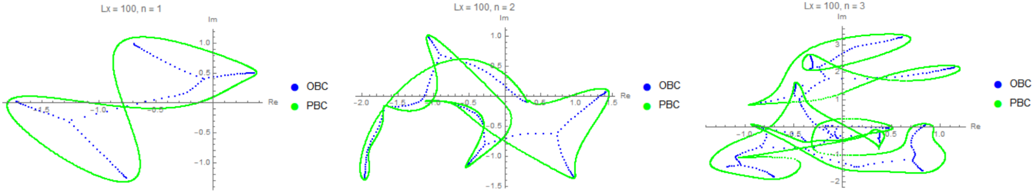
$$H_t = (1 - t)H_0 + tH_1$$

を考える. H_tのtを変化させたときの固有値の変化を見る.
- 複素行列ではなく, H_0, H_1をランダムな実行列とした場合に何が起こるかを見る.

PBC vs OBC

Here are some spectra of 1-dimensional non-Hermitian models.



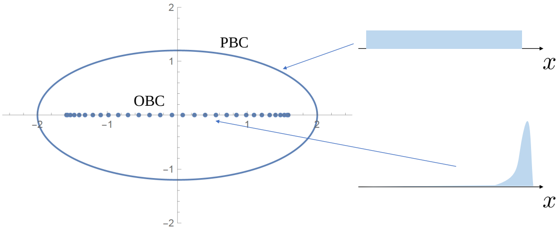


Non-Hermitian Skin effect Yao=Wang 1803.01876

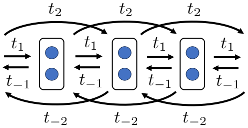
- PBC $\neq$ OBC for spectra. Extreme sensitivity against the boundary condition.
- In OBC, $O(L)$ modes are localized at an edge.
- A prime example is the Hatano-Nelson model, a one-dimensional model with non-reciprocal hopping.
- Non-Hermitian Skin effect has a topological origin. [Zhang=Yang=Fang 1910.01131, Okuma=Kawabata=KS=Sato 1910.02878] (will be explained in last Section)

$$H = \sum_{x\in\mathbb{Z}} te^{g} f^{\dagger}_{x+1} f_x + te^{-g} f^{\dagger}_x f_{x+1} \quad \overset{\text{PBC}}{\Longrightarrow} \quad H_{\text{PBC}} = \sum_k f^{\dagger}_k (te^{g} e^{-ik} + te^{-g} e^{ik}) f_k,$$

$$\overset{\text{OBC}}{\Longrightarrow} \quad H_{\text{OBC}} = \sum_{x=1}^{L} t\tilde{f}^{\dagger}_{x+1} \tilde{f}_x + t\tilde{f}^{\dagger}_x \tilde{f}_{x+1}, \quad \tilde{f}^{\dagger}_x = e^{gx} f^{\dagger}_x$$



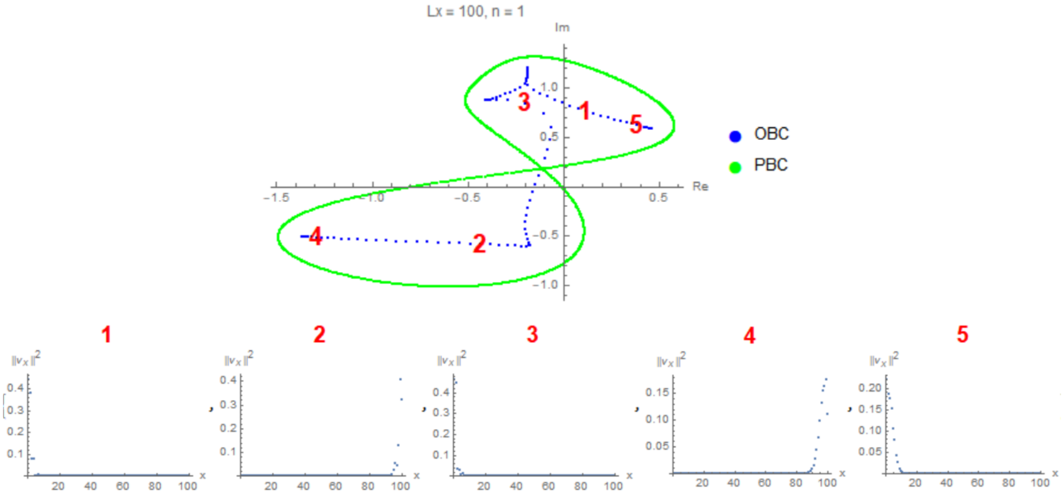
数値実験：１次元ホッピング模型



- １次元格子上のホッピング模型を考える．各サイトにn個の内部自由度がいるものとする．
- xから$x+p$サイトへの飛び移り行列をt_pとする．短距離条件として，$|p| \leq r$までの飛び移り項を考える．
- $(2r+1)$個の$n \times n$複素行列$\{t_p\}_{p=-r,\dots,r}$をランダムに与える．（$\xi > 0$を適当に選んでさらに$t_p \mapsto t_p e^{-|p|/\xi}$などと短距離性を持たせると良い．）
- 系のサイズL_xを適当に固定して，周期境界条件（PBC）と開放端境界条件（OBC）のハミルトニアンを構成する．

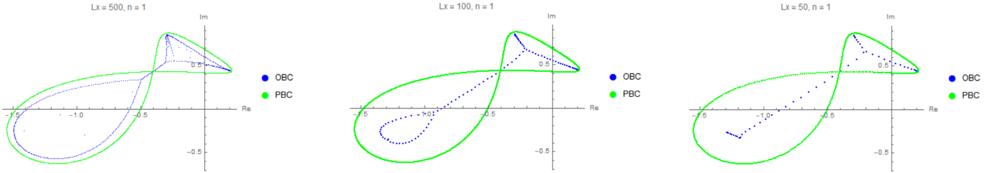
$$H_{\mathrm{PBC}} = \begin{pmatrix} t_0 & t_{-1} & & & & t_1 \\ t_1 & t_0 & t_{-1} & & & \\ & t_1 & \ddots & & & \\ & & & \ddots & t_{-1} & \\ & & & t_1 & t_0 & t_{-1} \\ t_{-1} & & & & t_1 & t_0 \end{pmatrix}, \quad H_{\mathrm{OBC}} = \begin{pmatrix} t_0 & t_{-1} & & & & O \\ t_1 & t_0 & t_{-1} & & & \\ & t_1 & \ddots & & & \\ & & & \ddots & t_{-1} & \\ & & & t_1 & t_0 & t_{-1} \\ O & & & & t_1 & t_0 \end{pmatrix}.$$

Example: No symmetry



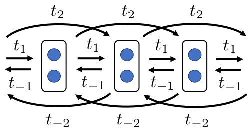
Numerical Rounding Error is not Negligible

- In computational calculation, *rounding error* refers to the small differences between the actual real number and its nearest representable value in the computer. (丸め誤差)
- Since $O(L)$ skin modes are exponentially localized at an edge, these small differences can significantly affect the results.



- The "Non-Bloch band theory" is used to compute the OBC spectrum in the thermodynamic limit. Yao=Wang 1803.01876, Yokomizo=Murakami 1902.10958

数値実験：１次元ホッピング模型に対称性を入れる



- エルミート性：

$$t_{-n} = t_n^\dagger.$$

- 擬エルミート性：

$$t_{-n} = \eta t_n^\dagger \eta^\dagger, \quad \eta^2 = 1, \mathrm{tr}\,[\eta] = 0.$$

- $\mathbb{Z}_2$対称性：

$$t_n = U t_n U^\dagger, \quad U^2 = 1.$$

- 時間反転対称性：

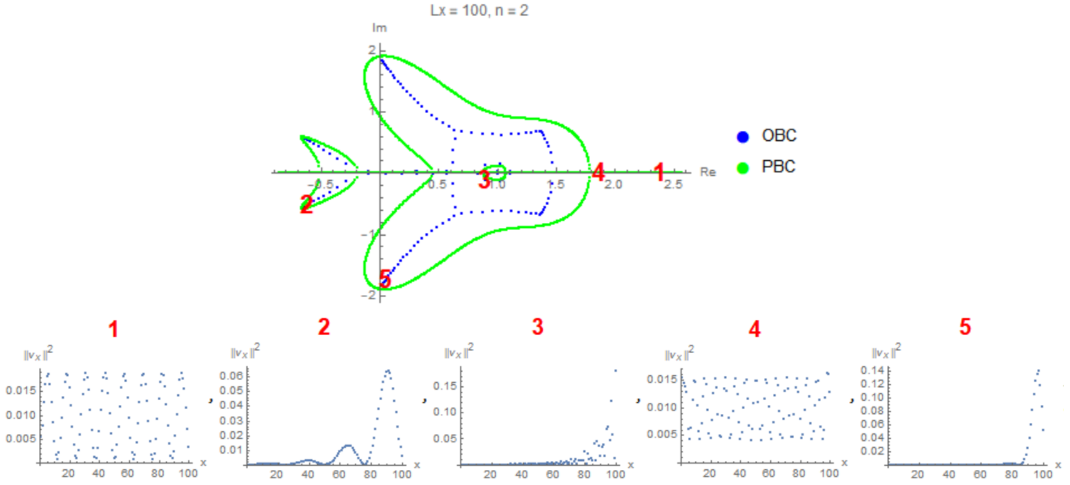
$$t_n = t_n^*, \quad \text{(class AI)},$$
$$t_n = (i\sigma_y) t_n^* (i\sigma_y)^\dagger, \quad \text{(class AII)}.$$

- 反転対称性：

$$t_{-n} = I t_n I^\dagger, \quad I^2 = 1.$$

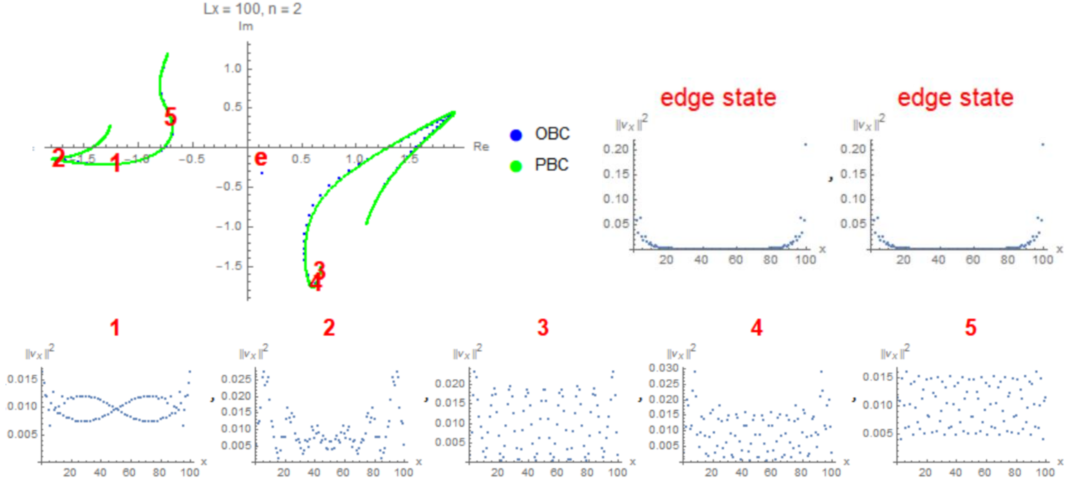
Example: Pseudo Hermiticity

$$\eta t_n^\dagger \eta^\dagger = t_{-n}, \quad \eta^2 = 1, \quad \mathrm{tr}\left[\eta\right] = 0.$$



Example: Inversion symmetry → the Non-Hermitian skin effect is suppressed

$$u t_n u^\dagger = t_{-n}, \quad u^2 = 1.$$

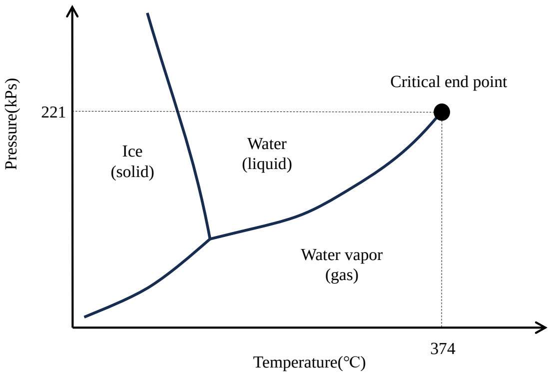


Gap Conditions and Topology

- Why gap condition?
- Hermitian systems
- Non-Hermitian systems

Equivalence Condition and Phases of Matter

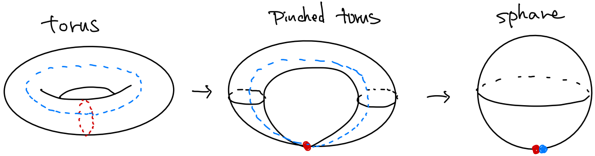
- Water Phase Diagram:



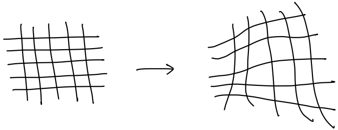
- The ice and water phases are distinct: A singularity in the thermodynamic function exists between these two phases, indicating a phase transition.
- Conversely, water and vapor can be considered the same phase since there exists a continuous path connecting them without encountering a thermodynamic singularity.

Topological Equivalence

- A torus and a sphere are considered to have distinct topologies.
- By shrinking one circle of the torus, we obtain a pinched torus. By further shrinking another circle, we ultimately transform it into a sphere.



- What exactly defines topology?
- Topological equivalence is determined by deformations that preserve the local structure of the Euclidean space.



- Given a defined equivalence relation, we can identify a set of equivalence classes.

Topology of Matrices

- What does it mean to classify matrices topologically?
- Consider two $N \times N$ matrices H_0 and H_1.
- They always can be connected to each other by a continuous path defined as:

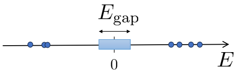
$$H_t = (1-t)H_0 + tH_1, \quad t \in [0,1].$$

$\rightarrow$ no topological classification.

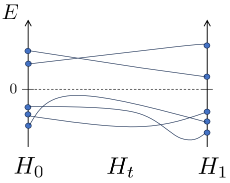
Hermitian Matrices: Gap Condition

- For meaningful classifications, we impose a gap condition.
- For Hermitian matrices H (where $H^\dagger = H$), the eigenvalues E are always real $E \in \mathbb{R}$.
- A reasonable gap condition is to impose a finite energy gap $E_{\text{gap}} > 0$ around zero (or the Fermi energy E_F) on eigenvalues of matrices:

$$E \neq 0.$$



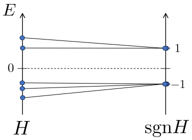
- Two Hermitian matrices H_0 and H_1 with no zero eigenvalues are considered equivalent if they can be continuously connected via a homotopy $H_{t\in[0,1]}$ provided that H_t also satisfies the gap condition throughout.



Hermitian Matrices: Gap Condition (cont.)

- We may think two H_0 and H_1 are equivalent if the numbers of negative eigenstates are the same.
- This is true: H can be flattened while keeping the gap condition.

$$H_t = \{(1-t)E_n + t\,\mathrm{sgn}(E_n)\} |n\rangle \langle n| \xrightarrow{t\to 1} \sum_{n=1}^{N} \mathrm{sgn}(E_n) |n\rangle \langle n| =: \mathrm{sgn}H.$$



- The flattened Hamiltonian $\mathrm{sgn}H$ is uniquely identified with a point of the complex Grassmaniann:

$$\mathrm{sgn}H = U \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} U^\dagger, \quad U \sim U \begin{pmatrix} V & \\ & W \end{pmatrix},$$

$$U \in U(N), V \in U(N-M), W \in U(M).$$

$$\to H \in \mathrm{Gr}_M(\mathbb{C}^N) = U(N)/U(N-M) \times U(M).$$

- No further classifications arise since the complex Grassmaniann is simply connected $\pi_0[\mathrm{Gr}_M(\mathbb{C}^N)] = 0$. For example, $\mathrm{Gr}_1(\mathbb{C}^2) \cong S^2$.

Hermitian Matrices: Example of Symmetry

- Even when two matrices have an equal number of negative (and positive) eigenvalues, certain symmetries can forbid a continuous transformation between them.
- Let's consider a Hermitian matrix H with an additional skew-symmetric constraint

$$H^T = -H, \quad H \in \mathrm{Mat}_{2N\times 2N}(\mathbb{C}).$$

- The Pfaffian $\mathrm{pf}\, H \in \mathbb{C}$ is well-defined. [1]
- Given the relationship $(\mathrm{pf}\, H)^* = \mathrm{pf}\, H^* = \mathrm{pf}\, H^T = (-1)^N \mathrm{pf}\, H$, the ratio of the Pfaffians of two matrices is always real:

$$\frac{\mathrm{pf}\, H_0}{\mathrm{pf}\, H_1} \in \mathbb{R},$$

implying that its sign is an invariant that takes on values in $\mathbb{Z}_2 = \{\pm 1\}$.
- For example, consider these two matrices:

$$H_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad H_1 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

No continuous transformation connects them while preserving the gap condition and the symmetries $H^\dagger = H$ and $H^T = -H$.

[1] $\mathrm{pf}\, H := \sum_{\sigma\in S_{2N}, \sigma(2i-1)<\sigma(2i), \sigma(1)<\sigma(3)<\cdots<\sigma(2N-1)} \mathrm{sgn}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2N-1)\sigma(2N)}$

数値実験：２つのハミルトニアンが断熱的に繋がるかどうか？

- 実は，Pfaffianのようなトポロジカル不変量が未知でもハミルトニアンの分類はできるLong=Zhang, PRL **130**, 036601 (2023).
- 行列次元の等しいエルミートなハミルトニアンH_0, H_1を考える.
- 何らかのG(群)対称性を任意に仮定する.

$$\begin{cases} u_g H u_g^\dagger = H, & \text{unitary symmetry,} \\ u_g H^* u_g^\dagger = H, & \text{time-reversal symmetry(TRS),} \\ u_g H^* u_g^\dagger = -H, & \text{particle-hole symmetry(PHS),} \\ u_g H u_g^\dagger = -H, & \text{chiral symmetry,} \end{cases} \quad g \in G.$$

- 平坦化したハミルトニアン$Q_j = \mathrm{sgn} H_j$を用いて，両者を線形に繋ぐハミルトニアン

$$Q_t = (1-t)Q_0 + tQ_1, \quad t \in [0,1]$$

を導入する． Q_tは対称性を満たす.

数値実験：２つのハミルトニアンが断熱的に繋がるかどうか？(続き)

- 以下が成立.

Long=Zhang, PRL 130, 036601 (2023)

- Q_tの固有値の構造は$|\epsilon_t| = a(t-\frac{1}{2})^2 + b$.
- $Q_{t=\frac{1}{2}}$がゼロ固有値を持たない $\Rightarrow$ H_0とH_1を断熱的に繋ぐパスが存在する.
- H_0とH_1を断熱的に繋ぐパスが存在しない $\Rightarrow$ $Q_{t=\frac{1}{2}}$のゼロ固有値は摂動に対して安定.

- したがって，与えられた同一の対称性を満たすハミルトニアンの対H_0, H_1が断熱的に繋がるかどうかは，断熱パスを探索する必要はなく，$Q_{t=\frac{1}{2}}$の固有値を計算すれば良い.
- 実装は，$\delta = 10^{-8}$などと固定して，
  - $|\lambda_{\min}| = \min_{\lambda\in\mathrm{Spec}(Q_{\frac{1}{2}})}|\lambda| > \delta$ ならばH_0とH_1は同一グループに属する.
  - $|\lambda_{\min}| = \min_{\lambda\in\mathrm{Spec}(Q_{\frac{1}{2}})}|\lambda| < \delta$のときは，摂動$H_j \mapsto H_j + \delta H_j$に対して条件$|\lambda_{\min}| < \delta$が安定であれば，$H_0$と$H_1$は異なるグループに属する.

  として，グループ分けを行う．[2] N個のグループが得られれば，N個のトポロジカル・クラスがある.

[2]断熱的に繋がるかどうかは同値関係なので，代表元のみ調べれば良いことに注意.

Hermitian Matrices: Finite Space dimensions & Translational Invariance

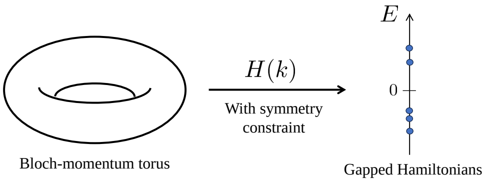
- We have discussed Hermitian matrices H without an extended space direction.
- In a d-dimensional finite space, the legs of H extend to an infinite lattice:

$$H = \{H(x, x')\}_{x,x'}, \quad x, x' \in \mathbb{Z}^d.$$

- Translational symmetry lets us define the Hamiltonian in the Bloch-momentum torus T^d:

$$H(x, x') = H(x - x') = \sum_{k \in T^d} H(k) e^{ik\cdot(x-x')}.$$

- Classification is about homotopy for matrix families $H(k)$ over the torus T^d.



- $H_0(k)$ is equivalent to $H_1(k)$ if a homotopy $H_{t\in[0,1]}(k)$ exists that bridges them while preserving the gap condition and symmetry.

数値実験：２バンド模型

- 2×2模型であって，正負のエネルギー固有状態を一つ持つハミルトニアンHはGrassmann多様体$\mathrm{Gr}_1(\mathbb{C}^2)\cong S^2$に値を取る．

$$\mathrm{sgn}H=(|\phi_+\rangle,|\phi_-\rangle)\begin{pmatrix}1 & \\ & -1\end{pmatrix}(|\phi_+\rangle,|\phi_-\rangle)^\dagger,\quad |\phi_+\rangle\sim|\phi_+\rangle e^{i\chi_+},\quad |\phi_-\rangle\sim|\phi_-\rangle e^{i\chi_-}. \qquad (1)$$

- ２バンド系の場合は$|\phi_+\rangle$を決めれば$|\phi_-\rangle$はそれに直交する状態として決まる．よってハミルトニアンHの選び方は，占有状態$|\phi_-\rangle\sim|\phi_-\rangle e^{i\chi_-}$の選び方と同じ． $\Rightarrow S^2$でパラメータ付けできる．

$$|\phi_-\rangle\sim\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}e^{i\phi}\end{pmatrix},\quad |\phi_+\rangle\sim\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix},\quad (\theta,\phi)\in S^2.$$

- このとき，単純計算より，

$$\mathrm{sgn}H=|\phi_+\rangle\langle\phi_+|-|\phi_-\rangle\langle\phi_-|=\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}=\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$

- よって，与えられたハミルトニアンを平坦化することにより，$\boldsymbol{n}\in S^2$が得られる．

$$\boldsymbol{n}=\frac{1}{2}\mathrm{tr}\left[\boldsymbol{\sigma}\,\mathrm{sgn}H\right].$$

数値実験：２バンド模型（続き）

- 2×2模型であって，正負のエネルギー固有状態を一つ持つハミルトニアンHはGrassmann多様体$\mathrm{Gr}_1(\mathbb{C}^2)\cong S^2$に値を取る．

$$\mathrm{sgn}H=(|\phi_+\rangle,|\phi_-\rangle)\begin{pmatrix}1 & \\ & -1\end{pmatrix}(|\phi_+\rangle,|\phi_-\rangle)^\dagger,\quad |\phi_+\rangle\sim|\phi_+\rangle e^{i\chi_+},\quad |\phi_-\rangle\sim|\phi_-\rangle e^{i\chi_-}. \tag{2}$$

- ２バンド系の場合は$|\phi_+\rangle$を決めれば$|\phi_-\rangle$はそれに直交する状態として決まる．よってハミルトニアンHの選び方は，占有状態$|\phi_-\rangle\sim|\phi_-\rangle e^{i\chi_-}$の選び方と同じ． $\Rightarrow S^2$でパラメータ付けできる．

$$|\phi_-\rangle\sim\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}e^{i\phi}\end{pmatrix},\quad |\phi_+\rangle\sim\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix},\quad (\theta,\phi)\in S^2.$$

- このとき，単純計算より，

$$\mathrm{sgn}H=|\phi_+\rangle\langle\phi_+|-|\phi_-\rangle\langle\phi_-|=\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}=\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$

- よって，与えられたハミルトニアンを平坦化することにより，$\boldsymbol{n}\in S^2$が得られる．

$$H\mapsto\boldsymbol{n}=\frac{1}{2}\mathrm{tr}\left[\boldsymbol{\sigma}\,\mathrm{sgn}H\right]\in S^2.$$

数値実験：２バンド模型（続き）

- ０次元: ハミルトニアンHはS^2上の１点を定める.
- １次元：ハミルトニアン$H(k_x)$は$k_x \in [-\pi, \pi]$上で定義される. $\Rightarrow$ 写像 $S^1 \to S^2$ を定める. （立体角はBerry位相を与える.）
- ２次元：ハミルトニアン$H(k_x, k_y)$は$(k_x, k_y) \in [-\pi, \pi]^{\times 2}$上で定義される. $\Rightarrow$ 写像 $T^2 \to S^2$ を定める. マップ$T^2 \to S^2$がS^2を何回覆い尽くす回数，つまり写像度（Chern数）

$$\frac{1}{8\pi}\int_{T^2} \boldsymbol{n} \cdot (d\boldsymbol{n} \times d\boldsymbol{n}) \in \mathbb{Z}$$

によってトポロジカルな分類が生じる.

- 対称性が存在すると，取りうるS^2上の点に制限が課される.
- 例えば，クラスD型のPHS

$$\sigma_x H(k_x)^* \sigma_x = -H(-k_x)$$

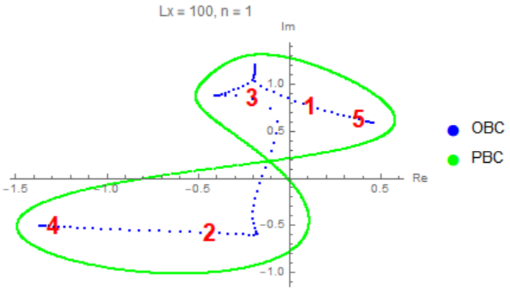
を考えると，対称点$k_x = 0, \pi$においては$\mathrm{sgn} H = \pm\sigma_z$であるので，北極か，あるいは南極に制限される. $\Rightarrow \mathbb{Z}_2$分類.

Non-Hermitian Matrices: What is the Gap Condition

- Eigenvalues of non-Hermitian matrices are complex.
- What is a meaningful gap condition?
- A characteristic feature of complex eigenvalues is that in a PBC, the phase of an eigenvalue around a reference energy $E_{\rm ref}$ may have a winding number

$$W(E_{\rm ref}) = \frac{1}{2\pi i}\oint d\log\det[H_{\rm PBC}(k) - E_{\rm ref}] \in \mathbb{Z}.$$

$\rightarrow$ the origin of the non-Hermitian skin effect [Zhang-Yang-Fang 1910.01131, Okuma-Kawabata-KS-Sato 1910.02878].



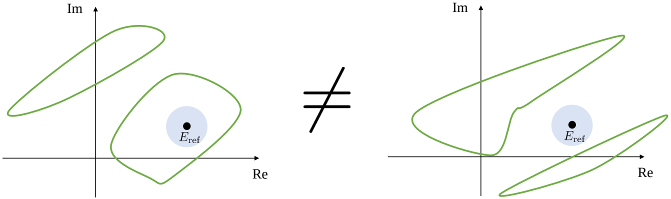
Non-Hermitian Matrices: Point Gap Gong-Ashida-Kawabata-Takasan-Higashikawa-Ueda 1802.07964

- The winding number $W(E_{\rm ref})$ is stable unless an eigenvalue touches the reference energy $E_{\rm ref}$.
- The point gap condition

$$E \neq E_{\rm ref} \quad (\det(H(k) - E_{\rm ref}) \neq 0)$$

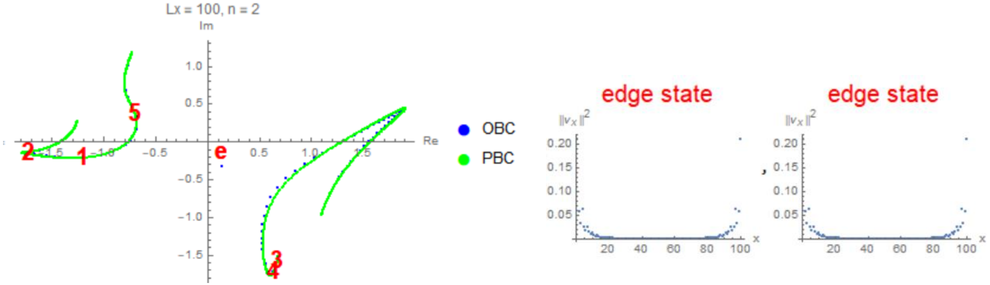
  makes sense.

- Eg: The following two Hamiltonians are in distinct point-gapped topological phases w.r.t. the reference energy $E_{\rm ref}$.



Non-Hermitian Matrices: Remnants of Hermitian edge states

- Even with non-Hermiticity, the remnant of Hermitian topological phases, the boundary states, might persist.
- A minor perturbation doesn't eliminate the edge states inherent to Hermitian topological phases. This is because the spectrum can deform continuously smoothly when perturbed slightly.

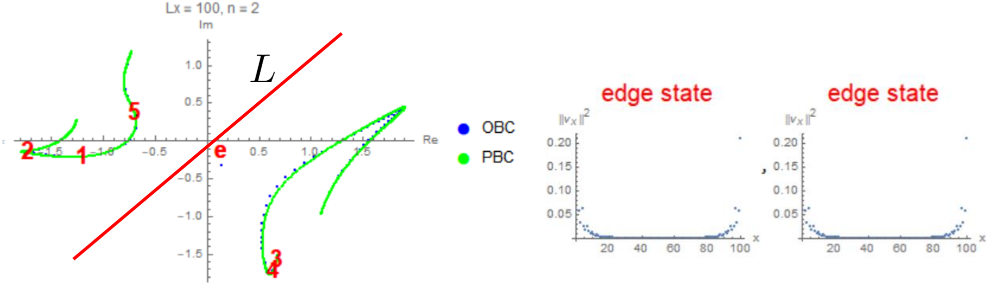


Non-Hermitian Matrices: Line Gap Kawabata-KS-Ueda-Sato 1812.09133

- To capture such remnants of Hermitian topological edge states in a non-Hermitian system, we introduce the concept of a line gap:

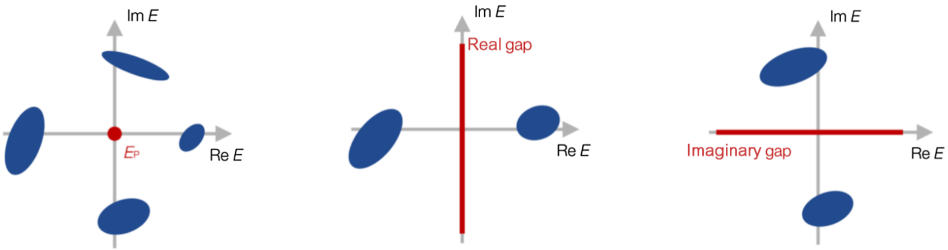
$$\mathrm{Spec}(H) \cap L = \emptyset, \quad \text{where } L \text{ is a line in the complex plane } \mathbb{C}.$$

- Hamiltonians $H_0(k)$ and $H_1(k)$ are considered to belong to the same topological phase with respect to the line gap if there exists a homotopy $H_{t\in[0,1]}(k)$ that connects them while preserving the line gap and the associated symmetry.



Non-Hermitian Matrices: Point Gap and Line gap

- It is useful to introduce two types of line gaps: real line gap and imaginary line gap. These are consistent with symmetries associating E with $-E, E^*$, or $-E^*$ (detailed later).
- P: Point-gap $E - E_{\text{ref}} \neq 0.$
- $\mathrm{L_r}$: Real line gap $\mathrm{Re}(E - E_{\text{ref}}) \neq 0.$
- $\mathrm{L_i}$: Imaginary line gap $\mathrm{Im}(E - E_{\text{ref}}) \neq 0.$



[Figure from Kawabata=KS=Ueda=Sato 1812.09133]

Symmetry in non-Hermitian systems

Symmetries in Non-Hermitian Systems

- What kind of symmetries exist in non-Hermitian systems?
- Example:
  - Time-reversal symmetry (TRS) is a fundamental symmetry.

$$U_T H^* U_T^\dagger = H.$$

  - In the mean-field approach to superconductors, the Bogoliubov–de Gennes (BdG) Hamiltonian H_{BdG} inherently possesses particle-hole symmetry (PHS).[3]

$$U_C H_{\mathrm{BdG}}^T U_C^\dagger = -H_{\mathrm{BdG}}, \quad H_{\mathrm{BdG}} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}, \quad U_C = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

  - Bosonic systems with quadratic interactions are captured by the bosonic BdG Hamiltonian $\hat{H} = \frac{1}{2}(\boldsymbol{a}^\dagger, \boldsymbol{a}) H_{\mathrm{BdG}} (\boldsymbol{a}, \boldsymbol{a}^\dagger)^T$. To maintain the bosonic commutation relation, H_{BdG} must be diagonalized using a paraunitary matrix [4], which is the same as the standard diagonalization of the effective matrix $H_{\sigma\mathrm{BdG}} = \sigma_z H_{\mathrm{BdG}}$. While $H_{\sigma\mathrm{BdG}}$ is non-Hermitian, the Hermiticity of $\hat{H}$ is encoded in its pseudo-Hermiticity:

$$\sigma_z H_{\sigma\mathrm{BdG}}^\dagger \sigma_z = H_{\sigma\mathrm{BdG}}.$$

[3] Note that $\Delta^T = -\Delta$ due to the fermion anti-commutation relation.
[4] $U\sigma_z U^\dagger = \sigma_z, U^\dagger \sigma_z U = \sigma_z$.

Symmetries in Non-Hermitian Systems (cont.)

- We consider the following 8 types of symmetries :

Symmetry in non-Hermitian systems

$$u \left\{ \begin{array}{c} H \\ H^* \\ H^T \\ H^\dagger \end{array} \right\} u^\dagger = \left\{ \begin{array}{c} H \\ -H \end{array} \right\}, \quad u \text{ is a unitary matrix.}$$

- This choice is ad hoc. In quantum mechanics, Winger's theorem tells us symmetry, a transformation that does not change the observation, is either unitary or anti-unitary. In non-Hermitan systems without specifying a physical system, we have no such guiding principles. We may consider different types of symmetry such as

$$u \left\{ \begin{array}{c} H \\ H^* \\ H^T \\ H^\dagger \end{array} \right\} v^\dagger = e^{i\phi} H, \quad u \neq v, \quad e^{i\phi} \in U(1).$$

For example, the symmetry type $uH^\dagger v^\dagger = H$ was discussed to construct the symmetry indicator in KS=Ono 2105.00677.

Symmetries in Non-Hermitian Systems (cont.)

- Let G be a group. We introduce three homomorphims [5] $\phi, \eta, c : G \to \mathbb{Z}_2 = \{\pm 1\}$ to specify the type of symmetry as

$$\left\{ \begin{array}{ll} u_g H u_g^\dagger & (\phi_g = 1, \eta_g = 1) \\ u_g H^* u_g^\dagger & (\phi_g = -1, \eta_g = 1) \\ u_g H^T u_g^\dagger & (\phi_g = -1, \eta_g = -1) \\ u_g H^\dagger u_g^\dagger & (\phi_g = 1, \eta_g = -1) \end{array} \right\} = c_g H, \quad g \in G,$$

- Comparing the transformation with two consecutive h, g transformations and the transformation with gh, we have

$$\left\{ \begin{array}{ll} u_g u_h & (\phi_g = 1) \\ u_g u_h^* & (\phi_g = -1) \end{array} \right\} = z_{g,h} u_{gh}, \quad z_{g,h} \in U(1), \quad g, h \in G.$$

- The relation $(gh)k = g(hk)$ gives the constraint relations

$$z_{h,k}^{\phi_g} z_{gh,k}^{-1} z_{g,hk} z_{g,h}^{-1} = 1, \quad g, h, k \in G.$$

(This means $z = (z_{g,h})$ is a two-cycle in $Z^2(G, U(1)_\phi)$.)

[5] Let G_0 and G_1 be groups. $f : G_0 \to G_1$ is said to be a homomorphism if $f(gh) = f(g)f(h)$ is met.

8 types of symmetries (names from Kawabata-KS-Ueda-Sato 1812.09133)

ϕ_g	η_g	c_g	Sym.	Energy constraints	Name
1	1	1	$u_g H u_g^\dagger = H$	$E \to E$	Unitary
1	-1	1	$u_g H^\dagger u_g^\dagger = H$	$E \to E^*$	Pseudo Hermiticity (PH)
-1	1	1	$u_g H^* u_g^\dagger = H$	$E \to E^*$	Time-reversal symmetry (TRS)
-1	-1	1	$u_g H^T u_g^\dagger = H$	$E \to E$	Time-reversal dagger symmetry ($\mathrm{TRS}^\dagger$)
-1	1	-1	$u_g H^* u_g^\dagger = -H$	$E \to -E^*$	Particle-hole dagger symmetry ($\mathrm{PHS}^\dagger$)
-1	-1	-1	$u_g H^T u_g^\dagger = -H$	$E \to -E$	Particle-hole symmetry (PHS)
1	1	-1	$u_g H u_g^\dagger = -H$	$E \to -E$	Sublattice symmetry (SLS)
1	-1	-1	$u_g H^\dagger u_g^\dagger = -H$	$E \to -E^*$	Chiral symmetry (CS)

and finer classifications (detailed on the next slide).

対称性の分類

- 非エルミート系における対称性の分類をしたい．
- 対称性群Gは任意だから分類できないようにも思うが，ハミルトニアンはユニタリーな部分群の既約表現のセクターにブロック対角化されるため，各ブロックにおいて実現する対称性のみを分類すれば良い．
- 結果，38通りの独立な対称性クラスに分類されることを見る．
- まずは既約表現のセクターにブロック対角化されることを確認する．

$$G_0 = \{g \in G | \phi_g = \eta_g = c_g = 1\} \subset G$$

をユニタリーな部分群とする．つまり，

$$u_g H u_g^\dagger = H, \quad g \in G_0,$$
$$u_g u_h = z_{g,h} u_{gh}, \quad g, h \in G_0.$$

Schurの補題

Schurの補題

u_g, v_gをG_0の既約なユニタリ表現とする．任意の$g \in G_0$に対して

$$u_g H = H v_g$$

とする．このとき，u, vが非等価な表現であれば$H = O$であり，$u = v$であれば$H \propto \mathbf{1}$.

この系$+\alpha$として...

表現uの既約分解を$u = \bigoplus_\alpha n_\alpha \alpha, n_\alpha \in \mathbb{Z}_{\geq 0}$とする．基底を選んで，

$$u_g = \bigoplus_\alpha u_g^\alpha \otimes \mathbf{1}_{n_\alpha}$$

とできる，この基底においてハミルトニアンは以下の形にブロック対角化される．

$$H = \bigoplus_\alpha \mathbf{1}_{\dim(\alpha)} \otimes H_\alpha, \quad H_\alpha \in \mathrm{Mat}_{n_\alpha}(\mathbb{C}).$$

数値実験：ブロック対角化

- Schurの補題の証明は，例えば英語版のWikipediaの記事で確認してください．
- ここではMathematicaに含まれる有限群G_0の群表を用いて正則表現を構成し，G_0対称性を満たすランダムなハミルトニアンの固有値が表現次元だけ縮退することを数値的に確かめます．
- まず与えられた有限群G_0と乗数系$z_{g,h}$に対して全ての既約表現を得る

正則表現

有限群G_0に対して，以下を正則表現と呼ぶ．

$$[R_g]_{hk} = z_{g,k}\delta_{h,gk}.$$

次の事実がある．

$$R = \bigoplus_\alpha \dim(\alpha)\alpha.$$

よって，正則表現Rは全ての既約表現を含み，α既約表現は$\dim(\alpha)$回出現する．[a]

[a] したがって，群表と乗数系$z_{g,h}$が与えられれば，全ての既約指標が得られます．

38 symmetry classes Kawabata-KS-Ueda-Sato 1812.09133

- What are fundamentally different symmetry classes that govern the topological nature of matrices? $\rightarrow$ We eventually reach the 38 symmetry classes. (cf. 10 Altland-Zirnbauer symmetry classes in Hermitian systems. cond-mat/9602137)

Proof

(i) The Hamiltonian H is block-diagonalized to the irreducible representations $\alpha, \beta, \gamma, \dots$ of the unitary subgroup $G_0 = \{g \in G | \phi_g = \eta_g = c_g = 1\} \subset G$.

$$H = \begin{pmatrix} H_\alpha & & & \\ & H_\beta & & \\ & & H_\gamma & \\ & & & \ddots \end{pmatrix}$$

(ii) A group element $g \in G$ in which either ϕ_g, η_g, or c_g is -1, acts on each block H_α as either

- g preserves the irreducible representation α. g is closed inside the block H_α. $\rightarrow$ g acts as a $\mathbb{Z}_2$ symmetry inside the block H_α. (cf. Wigner criteria)
- g exchanges the irreducible representations $H_\alpha \xrightarrow{g} H_\beta$. $\rightarrow$ H_β is just a copy of H_α. The topological nature is determined only in the block H_α.

補足：表現のマップについて

- まず，今の場合は$h \in G$に対して，$g \in G_0$のとき，$\phi_{h^{-1}gh} = \eta_{h^{-1}gh} = c_{h^{-1}gh} = 1$であるので，$h^{-1}G_0 h = G_0$に注意します．
- さらに，$h \in G$に対して，常に$h^2 \in G_0$にも注意します．これから，hで２回マップすると元の表現に戻ることがわかります．
- G_0の既約表現αの表現基底を$\{|i\rangle\}_{i=1}^{\dim(\alpha)}$とします．

$$\hat{g}\,|j\rangle = \sum_i |j\rangle\,[u^\alpha]_{ij}, \quad g \in G_0.$$

$h \in G$によってマップされた既約表現$h\alpha$の表現基底を形式的に$\hat{h}\,|i\rangle$として導入すると，

$$\hat{g}\hat{h}\,|j\rangle = z_{g,h}\widehat{gh}\,|j\rangle = \frac{z_{g,h}}{z_{h,h^{-1}gh}}\hat{h}\widehat{h^{-1}gh}\,|j\rangle = \frac{z_{g,h}}{z_{h,h^{-1}gh}}\hat{h}\sum_i |i\rangle\,[u^\alpha_{h^{-1}gh}]_{ij}$$

より，$\hat{h}$が反ユニタリーな場合に注意して，表現行列は

$$u^{h\alpha}_{g\in h^{-1}G_0 h} = \frac{z_{g,h}}{z_{h,h^{-1}gh}} \times \begin{cases} u^\alpha_{h^{-1}gh} & \phi_h = 1, \\ [u^\alpha_{h^{-1}gh}]^* & \phi_h = -1. \end{cases}$$

となります．
- この表式から，$h\alpha$の指標がわかるので，後は既約指標の直交関係

$$\frac{1}{|G_0|}\sum_{g\in G_0}(\chi^\alpha_g)^*\chi^\beta_g = \delta_{\alpha\beta}$$

により，αと$h\alpha$がユニタリ同値かどうかが判定できます．

補足：Wigner判定条件

- $h \in G$が反ユニタリー$\phi_h = -1$の場合でかつ既約表現αと$h\alpha$がユニタリ同値な場合は，状態$\{|i\rangle\}$と$\{\hat{h}\,|i\rangle\}$はユニタリ同値であるにもかかわらず，“直交”する場合があります．（Kramers縮退）
- 具体的には，次のWigner判定条件を用いて判断されます．

$$W_\alpha := \frac{1}{|G_0|} \sum_{g \in G_0} z_{hg,hg} \chi^{\alpha}_{(hg)^2} \in \{0, \pm 1\}.$$

$W_\alpha = 0 \quad \Rightarrow \quad \alpha$と$h\alpha$は非等価.

$W_\alpha = 1 \quad \Rightarrow \quad \alpha$と$h\alpha$はユニタリ同値であり，クラマース縮退なし.

$W_\alpha = -1 \quad \Rightarrow \quad \alpha$と$h\alpha$はユニタリ同値であり，クラマース縮退あり.

- 最も簡単な例は，自明な群$G_0 = \{e\}$の自明な表現に対して，$\mathbb{Z}_2 = \{e, T\}$の時間反転対称性が存在する場合であり，

$$\hat{T}^2 = z_{T,T} = 1 \quad \Rightarrow \quad \text{Kuramers縮退なし}, \hat{T}^2 = z_{T,T} = -1 \quad \Rightarrow \quad \text{Kuramers縮退あり}.$$

38 symmetry classes (cont.)

(iii) The problem is recast as how different symmetry actions there are in a single block H_α.

(iv) We can assume the absence of unitary symmetry (i.e., $(\phi_g, \eta_g, c_g) \neq (1,1,1)$).
$\to$ The symmetry group G realized in the single block is either one of

$$G = \mathbb{Z}_2^{\times N}, \quad N = 0, 1, 2, 3.$$

(Otherwise, there is a unitary group element.)

(v) For a group element g with $\phi_g = -1$, namely antiunitary symmetry, the square is proportional to identity (since $g^2 = e$) but its coefficient is quantized to a sign [6]

$$u_g u_g^* = \pm 1.$$

[6] The coefficient should be a sign: Set $u_g u_g^* = e^{\imath\phi}$. Then, $e^{i\phi} u_g = u_g u_g^* u_g = u_g (u_g u_g^*)^* = u_g e^{-i\phi}$. The sign ± 1 is unchanged under $u_g \mapsto e^{i\alpha} u_g$.

38 symmetry classes (cont.)

(vi) Case of $N = 0$ — Unique.

(vii) Case of $N = 1$ — Seven patterns:

$$(\phi_1, \eta_1, c_1) = (-1, 1, 1), (-1, -1, 1), (-1, 1, -1), (-1, -1, -1), (1, -1, 1), (1, 1, -1), (1, -1, -1).$$

For $\phi_1 = -1$, we have 2 cases for each, resulting in $2 \times 4 + 3 = 11$.

(viii) Case of $N = 2$ — When $\phi_g = -1$ is included, there are four patterns

$$\begin{aligned}\{(\phi_1, \eta_1, c_1), (\phi_2, \eta_2, c_2)\} =& \{(-1, 1, 1), (-1, -1, 1)\}, \{(-1, 1, 1), (-1, 1, -1)\}, \\ & \{(-1, 1, 1), (-1, -1, -1)\}, \{(-1, -1, 1), (-1, 1, -1)\},\end{aligned}$$

and choices of the signs of $u_1 u_1^* = \pm 1$ and $u_2 u_2^* = \pm 1$ for each. When $\phi_g = -1$ is not included, there is only one pattern

$$\{(\phi_1, \eta_1, c_1), (\phi_2, \eta_2, c_2)\} = \{(1, -1, 1), (1, 1, -1)\},$$

with the commutation or anticommutation relation of them $u_1 u_2 = \pm u_2 u_1$. As a result, we have $4 \times 4 + 2 = 18$.

38 symmetry classes (cont.)

(ix) Case of $N = 3$ — The set of three generators is unique

$$\{(\phi_1, \eta_1, c_1), (\phi_2, \eta_2, c_2), (\phi_3, \eta_3, c_3)\} = \{(-1, 1, 1), (-1, -1, 1), (-1, 1, -1)\}.$$

The choices of the signs of $u_1 u_1^* = \pm 1$, $u_2 u_2^* = \pm 1$, and $u_3 u_3^* = \pm 1$. We have $2 \times 2 \times 2 = 8$.

(x) In sum,

$$1 + 11 + 18 + 8 = 38 \text{ classes.} \quad \square$$

- Cf. This is contrasted to the 43-fold classes in the pioneered work by Bernard-LeClair. [cond-mat/0110649] This is due to overcounting and overlooking.

Issues in the 38 Symmetry Classes of Non-Hermitian Systems

- Having fundamental symmetry classes, several fundamental issues arise:
  - **Anderson localization problem** Hatano=Nelson cond-mat/9603165, ...
  - **Spectral statistics** (Level-spacing distribution) of random matrices Hamazaki=Kawabata=Kura=Ueda 1904.13082, ...
  - **Topological classification** w.r.t. gap conditions (point or line gap) Gong=Ashida=Kawabata=Takasan=Higashikawa=Ueda 1802.07964, Kawabata=KS=Ueda=Sato 1812.09133, Zhou=Lee 1812.10490, ...
  - **Symmetry protected exceptional points?** Kawabata=Bessho=Sato 1902.08479
  - Existence/absence of **non-Hermitian skin effect** Kawabata=KS=Ueda=Sato 1812.09133, Kawabata=Okuma=Sato 2003.07597, ...
  - **Connection to quantum many-body physics**
  - **Experimental relevance**
  - And more...

Note: This is far from the exhaustive reference list on the topics above, due to the lack of my knowledge of recent developments.

38 Symmetry Classes in Finite Space Dimensions

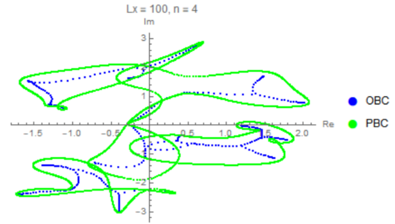
- In finite space dimensions (with $d \geq 1$), how we encode the 38 fundamental symmetries depends on the specific physical systems under consideration.
- One might focus on internal symmetries, which don't change the spatial position, as they remain compatible with the effects of the disorder.
- Here, we consider the following constraints on the hopping Hamiltonian $H(x, x')$:
  - Complex conjugation is local: $H(x, x')^* \leftrightarrow H(x, x')$.
  - Transpose exchanges the hopping direction: $H(x, x')^T \leftrightarrow H(x', x)$.

  This rule can be summarized in the table below:

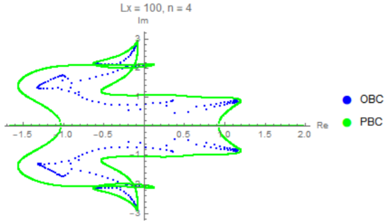
Symmetry	Symmetry in Real Space	With Translational Invariance
Unitary/SLS	$uH(x, x')u^\dagger = \pm H(x, x')$	$uH(k)u^\dagger = \pm H(k)$
TRS/PHS†	$uH(x, x')^* u^\dagger = \pm H(x, x')$	$uH(k)^* u^\dagger = \pm H(-k)$
TRS†/PHS	$uH(x, x')^T u^\dagger = \pm H(x', x)$	$uH(k)^T u^\dagger = \pm H(-k)$
PH/CS	$uH(x, x')^\dagger u^\dagger = \pm H(x', x)$	$uH(k)^\dagger u^\dagger = \pm H(k)$

A Numerical Experiment: PBC vs OBC for 38 symmetry classes

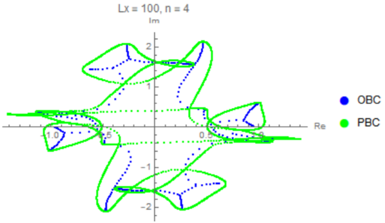
No symmetry



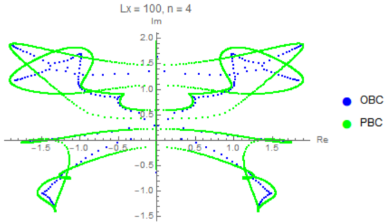
Pseudo Hermiticity $\sigma_z H(k)^\dagger \sigma_z = H(k)$



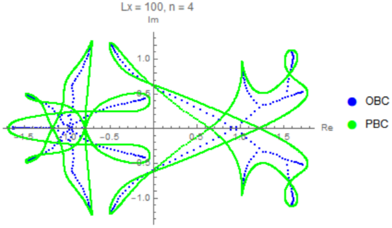
Sublattice symmetry $\sigma_z H(k) \sigma_z = -H(k)$



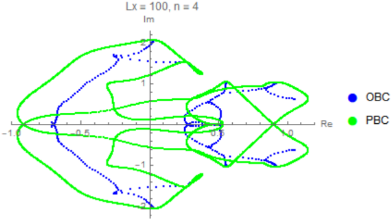
Chiral symmetry $\sigma_z H(k)^\dagger \sigma_z = -H(k)$



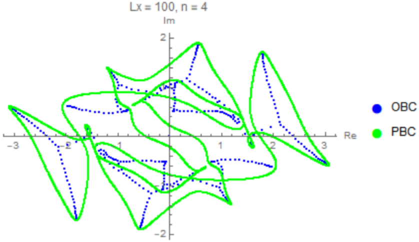
Class AI $\sigma_z H(k)^* \sigma_z = H(-k)$



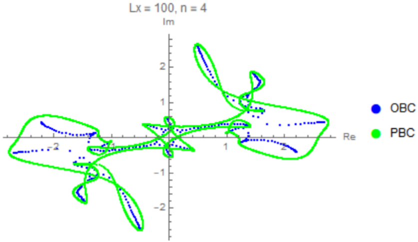
Class AII $\sigma_y H(k)^* \sigma_y = H(-k)$



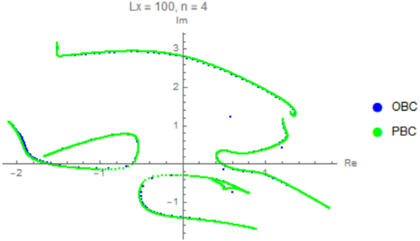
Class D $\sigma_z H(k)^T \sigma_z = -H(-k)$



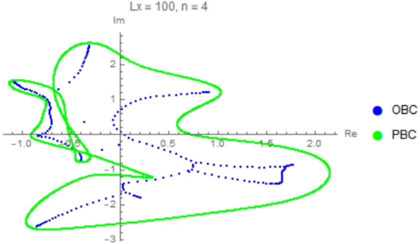
Class C $\sigma_y H(k)^T \sigma_y = -H(-k)$



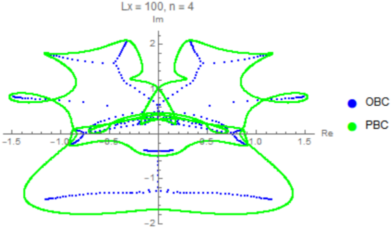
Class AI† $\sigma_z H(k)^T \sigma_z = H(-k)$



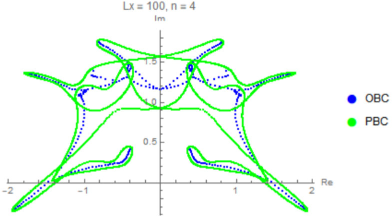
Class AII† $\sigma_y H(k)^T \sigma_y = H(-k)$



Class D† $\sigma_z H(k)^* \sigma_z = -H(-k)$



Class C† $\sigma_y H(k)^* \sigma_y = -H(-k)$



+ Other 28 classes → The PBC and OBC spectra are coincident if class AI† symmetry exists.

Kawabata=KS=Ueda=Sato 1812.09133, Kawabata=Okuma=Sato 2003.07597, ...

Topological Classification

- Hermitianization and flattening
- Altland-Zirnbauer symmetry class and classifying space
- Finite spacial dimension and dimensional isomorphism
- Classification of non-Hermitian topological phases

Classification table of Hermitian topological phases "Periodic Table"

Schnyder=Ryu=Furusaki=Ludwig 0803.2786, Kitaev 0901.2686

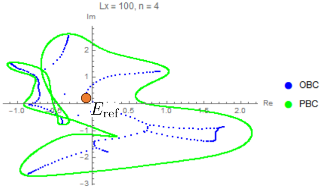
class\\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- Well-established. (The derivation is soon later.)

Point Gap and Hermitianization Gong=Ashida=Kawabata=Takasan=Higashikawa=Ueda 1802.07964

- The non-Hermitian skin effect is characterized by a nontrivial topological number with a point gap.

Class AII† $\sigma_y H(k)^T \sigma_y = H(-k)$



$$(-1)^\nu = \mathrm{sgn}\left[\frac{\mathrm{pf}[(H(0)-E_{\rm ref})\sigma_y]}{\mathrm{pf}[(H(0)-E_{\rm ref})\sigma_y]} \times \exp\frac{1}{2}\int_0^\pi d\log\det[(H(0)-E_{\rm ref})\sigma_y]\right]$$

[Okuma=Kawabata=KS=Sato 1910.02878]

- How to systematically classify such topological phases/numbers? $\to$ Use the Hermitianization trick

$$\tilde{H}(k) = \begin{pmatrix} & H(k)^\dagger \\ H(k) & \end{pmatrix}.$$

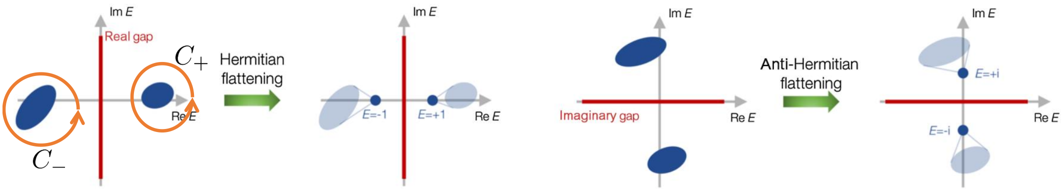
- A point gap of $\tilde{H}(k)$ implies a gap of $\tilde{H}(k)$. This is because $\mathrm{Spec}(\tilde{H}(k)) = \mathrm{Spec}(\pm\sqrt{H(k)^\dagger H(k)})$. I.e., the singular values of $H(k)$ are the same as (absolute values of) eigenvalues of $\tilde{H}(k)$.
- Classifying non-Hermitian $H(k)$ is recast as that of Hermitian Hamiltonian $\tilde{H}(k)$, which is well-established. $\to$ Done!

Line Gap and Flattening Kawabata=KS=Ueda=Sato 1812.09133

Hermitionization and Flattening

With the real/imaginary line gap, non-Hermitian Hamiltonians H can be Hermite and flattened while keeping the real/imaginary line gap.

$\rightarrow$ Done!



[Figure from Kawabata-KS-Ueda-Sato 1812.09133]

Proof (Based on App. D in Ashida=Gong=Ueda 2006.01837)

- For simplicity, from now on, we set $E_{\rm ref} = 0$.

Flattening

- Let $C_+(C_-)$ be a circle enclosing all the eigenvalues with Re $E > 0$(Re $E < 0$).
- The projector onto the eigenspace with Re $E > 0$(Re $E < 0$) is given by [7]

$$P_\pm(k) = \oint_{C_\pm} \frac{dz}{2\pi i} \frac{1}{z - H(k)}, \quad P_\pm(k)^2 = P_\pm(k).$$

- Introduce the homotopy

$$H_{t\in[0,1]}(k) = (1-t)H(k) + t[P_+(k) - P_-(k)],$$

whose eigenvalues are $(1-t)E_n(k) + t\ \mathrm{sgn}[\mathrm{Re}\ E_n(k)]$, which have a real line gap for $t \in [0,1]$.
- $H_1(k) = P_+(k) - P_-(k)$ has eigenvalues ± 1.

[7] Use the resolvent equation $(A-w)^{-1} - (A-z)^{-1} = (z-w)(A-z)^{-1}(A-w)^{-1}$ to show $[P_\pm(k)]^2 = P_\pm(k)$.

Hermitianization

- Decompose $H_1(k)$ into real and imaginary parts as

$$H_1(k) = h_1(k) + ih_2(k) = \frac{H_1(k) + H_1(k)^\dagger}{2} + i\frac{H_1(k) - H_1(k)^\dagger}{2i}.$$

- $H_1(k)^2 = P_+(k) + P_-(k) = 1$ implies that

$$h_1(k)^2 - h_2(k)^2 = 1, \quad \{h_1(k), h_2(k)\} = 0.$$

- Introduce the homotopy

$$\tilde{H}_{s\in[0,1]}(k) = (1-s)H_1(k) + sh_1(k) = h_1(k) + i(1-s)h_2(k),$$

whose square is

$$\tilde{H}_s(k)^2 = h_1(k)^2 - (1-s)^2 h_2(k)^2 = 1 + (1-(1-s)^2)h_2(k)^2 \geq 1.$$

- Thus, $\tilde{H}_s(k)$ keeps the real line gap and $H_1(k)$ is Hermitianized to $h_1(k)$.
- $h_1(k)$ is not flat. We take the flattening to $h_1(k)$ again. □

- (Remark) These flattening and Hermitianization methods are compatible with 38 symmetries. [8]

[8] Not compatible with type of symmetries $\pm u_g^\dagger H v_g = H, H^*, H^T, H^\dagger$.

Topological Classification of Hermitian Systems

- For both point and line gaps, the classification problem is recast as that for Hermitian systems, which is well-established.

$$H(k)^\dagger = H(k), \quad H(k)^2 = 1 \ \text{(after flattening)}$$

- So, in the remainder of this part, I review the classification of Hermitian topological phases.
- Strategy: Classify 0-dimensional Hamiltonians and extend to finite space dimensions.

- (Remark) The classification of non-Hermitian topological phases here is for PBC. Due to the non-Hermitian skin effect, quantitative (and possibly qualitative) properties such as edge states must be discussed using the bulk Hamiltonian in OBC. The bulk-boundary correspondence is true between the bulk OBC Hamiltonian and the edge state. Yao=Wang 1803.01876, Yao=Song=Wang 1804.04672

Altland=Zirnbauer symmetry classes

- The fundamental internal symmetries are classified into 10-fold Altland-Zirnbauer (AZ) symmetry classes. Altland=Zirnbauer cond-mat/9602137
- There are three types of symmetries: [9]

$$\begin{aligned} &\text{TRS:} && u_T H(x,x')^* u_T^\dagger = H(x,x') && u_T u_T^* = \pm 1, \\ &\text{PHS:} && u_C H(x,x')^* u_C^\dagger = -H(x,x') && u_C u_C^* = \pm 1, \\ &\text{Chiral:} && u_\Gamma H(x,x') u_\Gamma^\dagger = -H(x,x') && u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0. \end{aligned}$$

AZ class	TRS	PHS	Chiral
A	0	0	0
AIII	0	0	1
AI	1	0	0
BDI	1	1	1
D	0	1	0
DIII	-1	1	1
AII	-1	0	0
CII	-1	-1	1
C	0	-1	0
CI	1	-1	1

[9] $\mathrm{tr}\,[u_\Gamma] = 0$ is needed. Otherwise, H has zero modes.

Classifying Space

- We start with the classification of zero-dimensional Hamiltonian.

$$H^{\dagger} = H, \quad H^2 = 1 \ (\Leftrightarrow E = \pm 1) \quad + \text{AZ symmetry.}$$

- What is the "space" of such matrices?
- With "stable equivalence", such "spaces" become the *classifying spaces* in the K-theory. Kitaev 0901.2686

Example: 2×2 Hermitian matrix with $H^2 = 1$

- 2×2 Hermitian matrix H can be expanded as

$$H = d_0 + d_x \sigma_x + d_y \sigma_y + d_z \sigma_z = d_0 + \boldsymbol{d} \cdot \boldsymbol{\sigma}.$$

- Eigenvalues:

$$E = d_0 \pm |\boldsymbol{d}|.$$

- Thus, flattening implies either one of the following.
  - $d_0 = 1$ and $\boldsymbol{d} = \boldsymbol{0}$,
  - $d_0 = -1$ and $\boldsymbol{d} = \boldsymbol{0}$,
  - $d_0 = 0$ and $|\boldsymbol{d}| = 1$.
- Thus, there is a one-to-one correspondence

$$\{H \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) | H^\dagger = H, H^2 = 1\} = \underbrace{\{d_0 = 1\}}_{\text{pt}} \cup \underbrace{\{\boldsymbol{d} \in S^2\}}_{\text{Sphere}} \cup \underbrace{\{d_0 = -1\}}_{\text{pt}}.$$

Stable equivalence Kitaev 0901.2686

- Practically, the homotopy classification of Hamiltonians whose target space is a finite and fixed dimension is hard to compute.
- Even the classification is not a group.
- Example: class A 2×2 Hamiltonian in 3-space dimensions ("Hopf insulator Moore=Ran=Wen 0804.4527") [10]:

$$[T^3, S^2] = \begin{cases} \text{(i) Three Chern numbers } (n_x, n_y, n_z) \in \mathbb{Z}^{\times 3} \\ \text{(ii) Hopf invariant is classified by } \mathbb{Z}_{2 \cdot \mathrm{GCD}(n_x, n_y, n_z)} \end{cases}$$

- The "stable equivalence condition" was introduced: Two Hamiltonians $H_0(k)$ and $H_1(k)$ are said stably equivalent $H_0(k) \sim H_1(k)$ if $H_0(k) \oplus H'(k)$ and $H_1(k) \oplus H'(k)$ are homotopically equivalent. [11]
- Physical motivation: stable against hybridization of higher- and lower-energy bands and the band folding by breaking translational symmetry.
- Mathematical motivation: (relatively) easy to compute.

[10] $\pi_3(S^2) = \mathbb{Z}$, which is generated by the Hopf map $S^2 \to S^3$.

[11] We further introduce the equivalence relation to pairs of Hamiltonians with the same size $(H_0(k), H_1(k))$. Two pairs $(H_0(k), H_1(k))$ and $(H'_0(k), H'_1(k))$ are equivalent if $H_0(k) \oplus H'_1(k) \sim H'_0(k) \oplus H_1(k)$. The equivalence classes form the K-theory.

Class A: Classifying Space C_0

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$.
- H is diagonalized by a unitary matrix

$$H = U \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} U^\dagger,$$

where $M (0 \leq M \leq N)$ is the number of negative eigenvalues.

- U is not unique:

$$U \mapsto U \begin{pmatrix} V & \\ & W \end{pmatrix}, \quad V \in U(N-M), \quad W \in U(M).$$

- Thus, H is characterized by Grassmann manifolds

$$\bigcup_{M=0}^{N} \frac{U(N)}{U(N-M) \times U(M)}.$$

- With the stable equivalence [Kitaev 0901.2686], the Hamiltonian is eventually characterized by the classifying space C_0, [12]

$$C_0 = \bigcup_{k \in \mathbb{Z}} \lim_{n \to \infty} \frac{U(2n)}{U(n+k) \times U(n-k)}.$$

[12] 2つの行列の形式差(H_0, H_1)は$(H_0 \oplus (-H_1), H_1 \oplus (-H_1))$に安定同値である．$n$は$H_0, H_1$の行列次元，$k$は$H_0, H_1$の負の固有値の数の差.

Class AIII: Classifying Space C_1

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and chiral symmetry

$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0.$$

- WLOG, we can set $u_\Gamma = \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Then,

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad q \in U(N).$$

- Thus, H is characterized by the unitary group $U(N)$.
- With the stable equivalence [Kitaev, 0901.2686], the Hamiltonian is eventually characterized by the classifying space C_1,

$$C_1 = \lim_{n\to\infty} U(n).$$

Class AIII: Classifying Space C_1 (alternative)

- There is another perspective on C_1.
- Start with the diagonalization $H = U\sigma_z U^\dagger$.
- Set $u_\Gamma = \sigma_x$. The symmetry $\sigma_x H \sigma_x = -H$ implies that one can choose $\sigma_x U = U\sigma_x$. Namely,

$$U = u_+ P_+ + u_- P_- = \frac{1}{2}\begin{pmatrix} u_+ + u_- & u_+ - u_- \\ u_+ - u_- & u_+ + u_- \end{pmatrix}, \quad u_+, u_- \in U(N).$$

  where $P_\pm = \frac{1 \pm \sigma_x}{2}$ is the projection onto $\sigma_x = \pm 1$.
- The redundancy of U is $U \mapsto UV$ with $V\sigma_z V^\dagger = \sigma_z$ and $\sigma_x V = V\sigma_x$. Thus, V is a form $V = \sigma_y \otimes \tilde{V}, \tilde{V} \in U(N)$.
- We got

$$C_1 = \lim_{n\to\infty} [U(n) \times U(n)]/U(n).$$

補足：複素対称行列の分解（Autonne=高木分解）

Aを複素対称行列$A^T = A$とする．あるユニタリ行列Qと要素が非負実数の対角行列Σが存在して，

$$A = Q\Lambda Q^T.$$

(証明) [13] Q, Λの存在を示す． $A = A_1 + iA_2 = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$, $Q = Q_1 + iQ_2 = \frac{Q+Q^*}{2} + i\frac{Q-Q^*}{2i}$と実部と虚部に分解すると，方程式$A = Q\Lambda Q^T$は以下と等価.

$$A_1 = Q_1\Lambda Q_1^T - Q_2\Lambda Q_2^T, \quad A_2 = Q_1\Lambda Q_2^T + Q_2\Lambda Q_1^T.$$

これは，以下と等価.

$$\tilde{A} = \tilde{Q}\begin{pmatrix} \Lambda & \\ & -\Lambda \end{pmatrix}\tilde{Q}^T, \quad \tilde{A} = \begin{pmatrix} -A_2 & A_1 \\ A_1 & A_2 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} Q_2 & -Q_1 \\ Q_1 & Q_2 \end{pmatrix}.$$

Qがユニタリ性$QQ^\dagger = 1$は，$\tilde{Q}$の直交性$\tilde{Q}\tilde{Q}^T = 1$と等価．さて$\tilde{A}$は実対称行列であるから直交行列で対角化され，またカイラル対称性$(i\sigma_y)\tilde{A} = -\tilde{A}(i\sigma_y)$を有するから，確かに，

$$\tilde{Q} = \left(\begin{pmatrix} Q_2 \\ Q_1 \end{pmatrix}, i\sigma_y \begin{pmatrix} Q_2 \\ Q_1 \end{pmatrix}\right)$$

なる直交行列で対角化される． □

- 注意として，複素対称性行列Aは対角化可能とは限らないが，Autonne=高木分解は常に存在する．

[13] リンク先を参考にした.

補足：複素反対称行列の分解

Aを偶数次元の複素反対称行列$A^T = -A$とする．あるユニタリ行列Qと要素が非負実数の対角行列Σが存在して，

$$A = Q\Lambda \otimes (i\sigma_y)Q^T.$$

- 証明は練習問題とする．

補足：反ユニタリな対称性の標準形

- 反ユニタリな$\mathbb{Z}_2$対称性は，２種類存在する．

$$u = \pm u^T.$$

- $u^T = u$の場合は，$u = QQ^T$なるユニタリ行列が存在する．このQを用いて基底変換をすることにより，常に

$$u \mapsto Q^\dagger u Q^* = \mathbf{1}$$

  なる基底が存在することがわかる．
- 同様に，$u^T = -u$の場合は，$u = Q(i\sigma_y)Q^T$なるユニタリ行列が存在する．このQを用いて基底変換をすることにより，常に

$$u \mapsto Q^\dagger u Q^* = i\sigma_y$$

  なる基底が存在する．

Class AI: Classifying Space R_0

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$ and class AI TRS

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1.$$

- WLOG, we can set $u_T = 1$ [14], meaning that H is diagonalized by an orthogonal matrix

$$H = O \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} O^T.$$

- The same logic as class A leads the classifying space R_0,

$$R_0 = \bigcup_{k \in \mathbb{Z}} \lim_{n \to \infty} \frac{O(2n)}{O(n+k) \times O(n-k)}.$$

[14] Every symmetric matrix $u_T^T = u_T$ can be $u_T = Q \Lambda Q^T$ with $\Lambda \geq 0$ and Q a unitary (Autonne–Takagi factorization). When u_T is unitary, $\Lambda = 1$.

Class BDI: Classifying Space R_1

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$ and class BDI symmetry

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1,$$
$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0,$$
$$u_T u_\Gamma^* = u_\Gamma u_T.$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = 1$, meaning that q is an orthogonal matrix

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad q \in O(N).$$

- We get the classifying space R_1,

$$R_1 = \lim_{n\to\infty} O(n).$$

- The $\mathbb{Z}_2$ invariant is given by $\det q \in \{\pm 1\}$.
- As for C_1, it can also be obtained as $R_1 = \lim_{n\to\infty}[O(n) \times O(n)]/O(n)$.

Class D: Classifying Space R_2

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and class D PHS

$$u_C H^* u_C^\dagger = -H, \quad u_C u_C^* = 1.$$

- We can set $u_C = 1$, meaning that iH is a real skew-symmetric matrix, which is diagonalized as

$$iH = O\left[1_N \otimes \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right] O^T, \quad O \in O(2n).$$

- O is not unique:

$$O \mapsto O\begin{pmatrix} \text{Re } U & \text{Im } U \\ -\text{Im } U & \text{Re } U \end{pmatrix}, \quad \text{Re } U = \frac{U+U^*}{2}, \quad \text{Im } U = \frac{U-U^*}{2i}, \quad U \in U(n).$$

$$\to R_2 = \lim_{n\to\infty} \frac{O(2n)}{U(n)}.$$

- The $\mathbb{Z}_2$ invariant is given by $\mathrm{pf}\,[iH] = \det O \in \{\pm 1\}$.

Class D: Classifying Space R_2 (alternative)

- Start with the diagonalization $H = U\sigma_z U^\dagger$.
- Set $u_C = 1$. Then, the symmetry constraint $H^* = -H$ implies that U can be chosen as $U^* = U\sigma_x$, which is the same as $V = Ue^{\frac{i\pi}{4}(\sigma_x - 1)}$ is real $V^* = V$.
- Then, $H = V(-\sigma_y)V^\dagger$.
- The redundancy of V is $V \mapsto VQ$ with $Q^* = Q$ and $Q\sigma_y Q^\dagger = \sigma_y$, which means $Q \in U(N)$ as before.
- We get

$$R_2 = \lim_{n\to\infty} \frac{O(2n)}{U(n)}.$$

Class DIII: Classifying Space R_3

- Let H be an $4N \times 4N$ Hermitian matrix H with $H^2 = 1$ and class CI symmetry

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1,$$
$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0,$$
$$u_T u_\Gamma^* = -u_\Gamma u_T.$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = \sigma_x \tau_y$. Then, the symmetry constraint is recast as follows.

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad \tau_y q^T \tau_y = q.$$

- The matrix $\tau_y q$ is a complex skew-symmetric and unitary, meaning that it can be a form $\tau_y q = Q(i\sigma_y)Q^T$ with $Q \in U(2N)$.
- The redundancy of Q is $Q \mapsto QV$ with $VV^\dagger = 1$ and $V(i\sigma_y)V^T = i\sigma_y$. Namely, $V \in Sp(N)$.
- We get

$$R_3 = \lim_{n\to\infty} \frac{U(2n)}{Sp(n)}.$$

Class AII: Classifying Space R_4

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and class AII TRS

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = -1.$$

- We can set $u_T = i\sigma_y = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ [15]. The eigenvectors come in Kramers pairs

$$(u_{2i-1}, u_{2i}) = (u_{2i-1}, i\sigma_y u_{2i-1}^*),$$

meaning that H is diagonalized by a compact symplectic matrix

$$H = S \begin{pmatrix} 1_{N-M} & \\ & -1_M \end{pmatrix} S^\dagger, \quad S \in Sp(N) = Sp(2N; \mathbb{C}) \cap U(2N) = \{S \in U(2N) | S^T i\sigma_y S = i\sigma_y\}.$$

$$\rightarrow R_4 = \bigcup_{k \in \mathbb{Z}} \lim_{n \to \infty} \frac{Sp(2n)}{Sp(n+k) \times Sp(k-n)}.$$

[15] Every skew-symmetric matrix $u_T^T = -u_T$ can be $u_T = Q\Lambda Q^T$ with $\Lambda = \bigoplus_i \begin{pmatrix} & \lambda_i \\ -\lambda_i & \end{pmatrix} Q^T$ with Q a unitary. When u_T is unitary, λ_is can be $\lambda_i \equiv 1$.

Class CII: Classifying Space R_5

- Let H be an $N \times N$ Hermitian matrix H with $H^2 = 1$ and class CII symmetry

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = -1,$$
$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0,$$
$$u_T u_\Gamma^* = u_\Gamma u_T.$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = \tau_y$. Then, the symmetry constraint is recast as follows.

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad \tau_y q^* \tau_y = q \Leftrightarrow q \tau_y q^T = \tau_y.$$

- We get

$$R_5 = \lim_{n\to\infty} Sp(n).$$

- As for C_1, it can be obtained as $R_5 = \lim_{n\to\infty}[Sp(n) \times Sp(n)]/Sp(n)$.

Class C: Classifying Space R_6

- Start with the diagonalization $H = U\sigma_z U^\dagger$.
- Set $u_C = \sigma_y$. Then, the symmetry constraint $\sigma_y H^* = -H\sigma_y$ implies that U can be chosen as $\sigma_y U^* = U\sigma_y$. Namely, $U \in Sp(N)$.
- The redundancy of U is $U \mapsto UV$ with $V\sigma_y V^T = \sigma_y$ and $V\sigma_z V^\dagger = \sigma_z$, which means $V = \begin{pmatrix} v & \\ & v^* \end{pmatrix}$ with $v \in U(N)$.
- We get

$$R_6 = \lim_{n\to\infty} \frac{Sp(n)}{U(n)}.$$

Class CI: Classifying Space R_7

- Let H be an $2N \times 2N$ Hermitian matrix H with $H^2 = 1$ and class CI symmetry

$$u_T H^* u_T^\dagger = H, \quad u_T u_T^* = 1,$$
$$u_\Gamma H u_\Gamma^\dagger = -H, \quad u_\Gamma^2 = 1, \quad \mathrm{tr}\,[u_\Gamma] = 0,$$
$$u_T u_\Gamma^* = -u_\Gamma u_T.$$

- We can set $u_\Gamma = \sigma_z$ and $u_T = \sigma_x$. Then, the symmetry constraint is recast as follows.

$$H = \begin{pmatrix} & q^\dagger \\ q & \end{pmatrix}, \quad q^T = q.$$

- The complex symmetric and unitary matrix can be a form $q = QQ^T$ with $Q \in U(N)$.
- The redundancy of Q is $Q \mapsto QV$ with $VV^\dagger = 1$ and $VV^T = 1$. Namely, $V \in O(N)$.
- We get

$$R_7 = \lim_{n\to\infty} \frac{U(n)}{O(n)}.$$

Classifying Space

- Eventually, we get the 10 classifying spaces and their disconnected parts. [16]

AZ class	TRS	PHS	Chiral	Classifying Space	π_0	Top. invariant
A	0	0	0	$C_0 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{U(2n)}{U(n+k)\times U(n-k)}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
AIII	0	0	1	$C_1 = \lim_{n\to\infty} U(n)$	0	
AI	1	0	0	$R_0 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{O(2n)}{O(n+k)\times O(n-k)}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
BDI	1	1	1	$R_1 = \lim_{n\to\infty} O(n)$	$\mathbb{Z}_2$	$\det q \in \pm 1$
D	0	1	0	$R_2 = \lim_{n\to\infty} \frac{O(2n)}{U(n)}$	$\mathbb{Z}_2$	$\mathrm{pf}\,[iH] \in \pm 1$
DIII	-1	1	1	$R_3 = \lim_{n\to\infty} \frac{U(2n)}{Sp(n)}$	0	
AII	-1	0	0	$R_4 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{Sp(2n)}{Sp(n+k)\times Sp(n-k)}$	$2\mathbb{Z}$	$k \in \mathbb{Z}$
CII	-1	-1	1	$R_5 = \lim_{n\to\infty} Sp(n)$	0	
C	0	-1	0	$R_6 = \lim_{n\to\infty} \frac{Sp(n)}{U(n)}$	0	
CI	1	-1	1	$R_7 = \lim_{n\to\infty} \frac{U(n)}{O(n)}$	0	

[16] $Sp(N) = Sp(2N;\mathbb{C}) \cap U(2N) = \{S \in U(2N) | S^T i\sigma_y S = i\sigma_y\}$

数値実験：ランダムなハミルトニアンのグループ分け

- 10通りのAZクラスに対して，対称性の演算子を以下のように固定して良い．[17]
- グループ分けにより，π_0が再現できるLong=Zhang, PRL **130**, 036601 (2023).

AZ class	TRS	PHS	Chiral	Classifying Space	π_0	Top. invariant
A				$C_0 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{U(2n)}{U(n+k)\times U(n-k)}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
AIII			σ_z	$C_1 = \lim_{n\to\infty} U(n)$	0	
AI	1			$R_0 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{O(2n)}{O(n+k)\times O(n-k)}$	$\mathbb{Z}$	$k \in \mathbb{Z}$
BDI	1	σ_z	σ_z	$R_1 = \lim_{n\to\infty} O(n)$	$\mathbb{Z}_2$	$\det q \in \pm 1$
D		1		$R_2 = \lim_{n\to\infty} \frac{O(2n)}{U(n)}$	$\mathbb{Z}_2$	$\mathrm{pf}\,[iH] \in \pm 1$
DIII	$i\sigma_y$	1	σ_y	$R_3 = \lim_{n\to\infty} \frac{U(2n)}{Sp(n)}$	0	
AII	$i\sigma_y$			$R_4 = \cup_{k\in\mathbb{Z}} \lim_{n\to\infty} \frac{Sp(2n)}{Sp(n+k)\times Sp(n-k)}$	$2\mathbb{Z}$	$k \in \mathbb{Z}$
CII	$i\sigma_y$	$i\tau_y$	$\sigma_y\tau_y$	$R_5 = \lim_{n\to\infty} Sp(n)$	0	
C		$i\sigma_y$		$R_6 = \lim_{n\to\infty} \frac{Sp(n)}{U(n)}$	0	
CI	1	$i\sigma_y$	σ_y	$R_7 = \lim_{n\to\infty} \frac{U(n)}{O(n)}$	0	

[17] カイラル対称性$\Gamma H + H\Gamma = 0$についてはゼロ固有値の不在より，$\mathrm{tr}\,\Gamma = 0$が必要．

Finite Space Dimensions (i) from torus to sphere

- Thanks to the stable equivalence, the topological structure from "different origins" can be discussed independently.
- For d-spatial dimensions, the Bloch-momentum space is a d-dimensional torus T^d, however, with stable equivalence, the topological classification is decomposed into that of sub-spheres S^p, $0 \leq p \leq d$, like

$$\text{“}H(\text{Skyrmion} + \text{Vortex})\text{”} \to \text{“}H(\text{Skyrmion}) \oplus H(\text{Vortex})\text{”}.$$

- We can assume the Bloch-momentum space is a d-sphere.

Finite Space Dimensions (ii) Dirac Hamiltonians

- Moreover, it is found that the representative Hamiltonian can be a form of the Dirac Hamiltonian

$$H(k) = \sum_{i=1}^{d} k_i \gamma_i + M, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \{\gamma_i, M\} = 0, \quad M^2 = 1.$$

- The topological classification of $H(k)$ is recast as the classification of the mass term M subject to the constraint by γ_is and AZ symmetry.
- Adding space dimensions $d = 1, 2, \dots$ is the same as adding gamma matrices $\gamma_1, \gamma_2, \dots$.
- The gamma matrices γ_is behave as chiral symmetries.

Dimensional isomorphism

- We will show that adding gamma matrices is nothing but a shift of AZ symmetry class.

$$\cdots \to \mathrm{A} \to \mathrm{AIII} \to \mathrm{A} \to \cdots \quad \text{(without TRS and PHS)},$$

$$\cdots \mathrm{AI} \to \mathrm{CI} \to \mathrm{C} \to \mathrm{CII} \to \mathrm{AII} \to \mathrm{DIII} \to \mathrm{D} \to \mathrm{BDI} \to \mathrm{AI} \to \cdots .$$

- The key observation is that two chiral symmetries can be "solved" trivially:

$$\{\sigma_x, M\} = \{\sigma_y, M\} = 0 \quad \Rightarrow \quad M = \sigma_z \otimes \tilde{M}.$$

A $\to$ AIII $\to$ A

- Let us consider a $d=1$ class A Dirac Hamiltonian

$$H(k_1) = k_1\gamma_1 + M, \quad \{\gamma_1, M\} = 0.$$

- γ_1 behaves as chiral symmetry for M, thus,

$$(d=1,\ \text{class A}) \quad = \quad (d=0,\ \text{class AIII}).$$

- Next, let us consider a $d=1$ class AIII Dirac Hamiltonian

$$H(k_1) = k_1\gamma_2 + M, \quad \{\gamma_2, M\} = 0,$$
$$\gamma_1 H(k_1)\gamma_1^\dagger = -H(k_1).$$

- We can set $\gamma_1 = \sigma_x$ and $\gamma_2 = \sigma_z$. Then,

$$M = \sigma_y \otimes \tilde{M}.$$

- No constraints on $\tilde{M}$ exist, meaning that

$$(d=1,\ \text{class AIII}) \quad = \quad (d=0,\ \text{class A}).$$

Dimensional isomorphism with TRS or PHS

- With antiunitary symmetry, we chase the change of AZ symmetry for $\tilde{M}$.
- The symmetry constraint

$$u_T H(k)^* u_T^\dagger = H(-k),$$
$$u_T H(k)^* u_T^\dagger = -H(-k)$$

implies that

$$u_T \gamma_i^* u_T^\dagger = -\gamma_i, \quad u_T M^* u_T^\dagger = M,$$
$$u_C \gamma_i^* u_C^\dagger = \gamma_i, \quad u_C M^* u_C^\dagger = -M.$$

AI $\to$ CI

- Let us consider a $d = 1$ class AI Dirac Hamiltonian

$$H(k_1) = k_1\gamma_1 + M, \quad \{\gamma_1, M\} = 0.$$

The symmetry algebra

$$u_T\gamma_1^* u_T^\dagger = -\gamma_1, \quad u_T u_T^* = 1,$$

is solved by

$$u_T = 1, \quad \gamma_1 = \sigma_y.$$

Introducing PHS $u_C = i\gamma_1 u_T = i\sigma_y$, the constraint on the matrix M is the same as class CI:

$$u_T M^* u_T^\dagger = M, \quad u_T u_T^* = 1,$$
$$u_C M^* u_C^\dagger = -M, \quad u_C u_C^* = -1.$$

Thus,

$$(d = 1, \text{ class AI}) \quad = \quad (d = 0, \text{ class CI}).$$

CI → C

- Let us consider a $d = 1$ class CI Dirac Hamiltonian

$$H(k_1) = k_1\gamma_1 + M,$$
$$u_C\gamma_1^* u_C^\dagger = \gamma_1, \quad u_C M^* u_C^\dagger = -M, \quad u_C u_C^* = -1,$$
$$u_\Gamma\gamma_1 u_\Gamma^\dagger = -\gamma_1, \quad u_\Gamma M u_\Gamma^\dagger = -M, \quad u_\Gamma^2 = 1,$$
$$u_C u_\Gamma^* = -u_\Gamma u_C.$$

- We can set u_Γ, γ_1, and M as

$$u_\Gamma = \sigma_x, \quad \gamma_1 = \sigma_z, \quad M = \sigma_y \otimes \tilde{M}.$$

- The only remaining symmetry is u_c, which should be a form

$$u_C = \sigma_z \otimes \tilde{u}_C, \quad \tilde{u}_C\tilde{u}_C^* = -1,$$

and constrain the mass term $\tilde{M}$ as

$$\tilde{u}_C\tilde{M}^*\tilde{u}_C^\dagger = -\tilde{M}.$$

- Thus,

$$(d = 1, \text{ class CI}) \quad = \quad (d = 0, \text{ class C}).$$

Dimensional isomorphism

- In this way, we have the shift of AZ symmetry classes by adding space dimensions

$$\cdots \to \mathrm{A} \to \mathrm{AIII} \to \mathrm{A} \to \cdots \quad \text{(without TRS and PHS)},$$

$$\cdots \mathrm{AI} \to \mathrm{CI} \to \mathrm{C} \to \mathrm{CII} \to \mathrm{AII} \to \mathrm{DIII} \to \mathrm{D} \to \mathrm{BDI} \to \mathrm{AI} \to \cdots .$$

- These also show the Bott periodicity

$$C_{n-2} = C_n, \quad R_{n-8} = R_n.$$

- Eventually, the topological classification of d-dimensional Hamiltonian $H(k)$ with AZ symmetry C_n or R_n is given by

$$\pi_0[C_{n-d}] \quad \text{and} \quad \pi_0[R_{n-d}].$$

$\to$ periodic table.

Identify Mapped Symmetry: Point gap

- The remaining task is to identify how 38 non-Hermitian symmetry classes are mapped to 10 AZ Hermitian symmetry classes for each gap condition.
- For the point gap, the Hermitianized doubled Hamiltonian

$$\tilde{H}(k) = \begin{pmatrix} & H(k)^\dagger \\ H(k) & \end{pmatrix}$$

has additional chiral symmetry

$$\sigma_z \tilde{H}(k) \sigma_z = -\tilde{H}(k).$$

Other internal symmetries are mapped for a symmetry constraint of $\tilde{H}(k)$ and commutation/anticommutation relation with σ_z.

- Ex: Class AI → Class BDI

$$H(k)^* = H(-k) \quad \Rightarrow \quad \tilde{H}(k)^* = \tilde{H}(-k), \quad \sigma_z \tilde{H}(k)^* \sigma_z = -\tilde{H}(-k).$$

Identify Mapped Symmetry: Line gap

- For the real (imaginary) line gap, $H(k)$ can be (anti-)Hermite $H(k)^\dagger = H(k)$ ($H(k)^\dagger = -H(k)$) while keeping the line gap.
- The (anti-)Hermitian condition of $H(k)$ is the same as imposing an additional chiral symmetry on $\tilde{H}(k)$:

$$\sigma_y \tilde{H}(k) \sigma_y = -\tilde{H}(k) \quad \text{for real line gap} \quad \Rightarrow \quad \tilde{H}(k) = \sigma_x \otimes H'(k).$$
$$\sigma_x \tilde{H}(k) \sigma_x = -\tilde{H}(k) \quad \text{for imaginary line gap} \quad \Rightarrow \quad \tilde{H}(k) = \sigma_y \otimes H'(k).$$

  Other internal symmetries have definite commutation/anticommutation relations with σ_y (σ_x).
- Ex (real line gap): class AI $\to$ class AI

$$H(k)^* = H(-k) \quad \Rightarrow \quad H'(k)^* = H'(-k).$$

  Ex (imaginary line gap): class AI $\to$ class D

$$H(k)^* = H(-k) \quad \Rightarrow \quad H'(k)^* = -H'(-k).$$

Classification tables of non-Hermitian topological phases

Kawabata=KS=Ueda=Sato 1812.09133, cf. Gong=Ashida=Kawabata=Takasan=Higashikawa=Ueda 1802.07964, Zhou=Lee 1812.10490

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_i}$	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
BDI	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	$\mathrm{L_i}$	$\mathcal{R}_2 \times \mathcal{R}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0
D	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_r}$	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CII	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6 \times \mathcal{R}_6$	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0
C	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

+ 30 other symmetry classes. (See Kawabata=KS=Ueda=Sato 1812.09133 for the details.)

Intrinsic Non-Hermitian Topology

- Point gap vs Line gap
- Intrinsic Non-Hermitian topology
- Examples

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	P	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_i}$	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
BDI	P	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	$\mathrm{L_r}$	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	$\mathrm{L_i}$	$\mathcal{R}_2 \times \mathcal{R}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0
D	P	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	L	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	P	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_r}$	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
AII	P	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_r}$	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CII	P	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	$\mathrm{L_r}$	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	$\mathrm{L_i}$	$\mathcal{R}_6 \times \mathcal{R}_6$	0	0	$2\mathbb{Z} \oplus 2\mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	0
C	P	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	L	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	P	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	$\mathrm{L_r}$	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	$\mathrm{L_i}$	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

?

Edge Majorana zero mode

Motivating example: 1d class D non-Hermitian superconductor

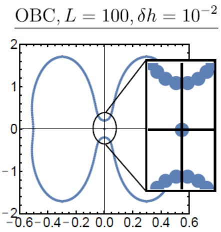
- Class D PHS symmetry:

$$\tau_x H(k_x)^T \tau_x = -H(-k_x), \quad E \to -E.$$

- Both the point gap and line gap show the $\mathbb{Z}_2$ classification.
- Non-Hermitian $\mathbb{Z}_2$ invariant:

$$(-1)^\nu = \mathrm{sgn}\left\{ \frac{\mathrm{Pf}[H(\pi)\tau_x]}{\mathrm{Pf}[H(0)\tau_x]} \times \exp\left[-\frac{1}{2}\int_0^\pi d\log\det[H(k)\tau_x] \right] \right\}$$

- If $(-1)^\nu = -1$, there is a Majorana zero mode at each edge Kawabata=KS=Ueda=Sato 1812.09133.



[Figure from Okuma=Sato 1904.06355]

- Unique to non-Hermitian systems?

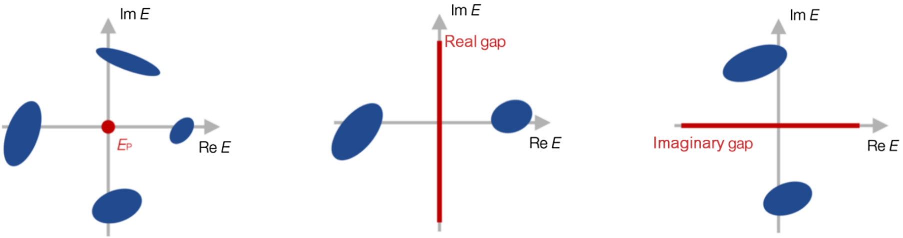
Topological phenomena unique to non-Hermitian systems

- Sometimes, we encounter topological phases which are realized only in non-Hermitian systems. Non-Hermitian skin effect, PT-symmetry breaking (exceptional point), ...
- On the other hand, there are topological phases that are remnant in non-Hermitian systems. For instance, the Chern insulator with a small non-Hermite perturbation is still characterized by the Chern number of the Bloch wave function.
- Is there any good approach to extracting topological phases realized only in the presence of non-Hermiticity?
- Our proposal [Sec.IX in Supplemental Material of Okuma=Kawabata=KS=Sato 1910.02878]: Take the cokernel of the following map

$$\text{Line-gapped topological phases} \longrightarrow \text{Point-gapped topological phases}$$

Line gap ⇒ point gap

- If a line gap is open, the point gap is also open.



[Figure from Kawabata=KS=Ueda=Sato 1812.09133]

- This implies that there exist homomorphisms f_{r} and f_{i} from the real and imaginary line-gapped topological phases to the point-gapped topological phases!

$$f_{\mathrm{r}} : \text{(Real line-gapped topological phases)} \to \text{(Point-gapped topological phases)},$$
$$f_{\mathrm{i}} : \text{(Imaginary line-gapped topological phases)} \to \text{(Point-gapped topological phases)}.$$

Intrinsic non-Hermitian Topology

- The point-gapped topological phases that are in the image

$$\text{Im } f_{\mathrm{r}} + \text{Im } f_{\mathrm{i}} \subset \text{(Point-gapped topological phases)}$$

  can be deformed into a real or imaginary line-gapped topological phase while keeping the point gap.

- Such point-gapped topological phases are also realized in Hermitian or anti-Hermitian systems.
- Importantly, their physics such as the bulk-boundary correspondence can be understood in Hermitian or anti-Hermitian systems.
- On the other hand, the quotient

$$\text{(Point-gapped topological phases)}/(\text{Im } f_r + \text{Im } f_i)$$

  represents topological phases intrinsic to non-Hermitian systems.

- Thanks to the dimensional isomorphism introduced before, it suffices to calculate the homomorphisms $f_{\mathrm{r}}, f_{\mathrm{i}}$ from line-gapped to point-gapped topological phases only for $d = 0$.

Ex: $1d$ class A with sublattice symmetry

- Sublattice symmetry (non-Hermitian SSH chain)

$$\sigma_z H(k_x)\sigma_z = -H(k_x) \qquad \Rightarrow \qquad H(k_x) = \begin{pmatrix} & h_1(k_x) \\ h_2(k_x) & \end{pmatrix}.$$

- The winding number is defined for each $h_1(k_x)$ and $h_2(k_x)$,

$$N_j = \frac{1}{2\pi i}\oint d\log\det h_j(k_x) \in \mathbb{Z} \quad (j=1,2).$$

  $\Rightarrow$ The classification of point-gapped topological phases is $\mathbb{Z}\oplus\mathbb{Z}$, which is characterized by (N_1, N_2).

- The real(imaginary)-line gap condition implies that $H(k_x)$ can be (anti-)Hermite, i.e. $h_2(k_x) = \pm h_1(k_x)^\dagger$.
  $\Rightarrow N_1 = -N_2 \Rightarrow$ Im $f_{\mathrm{r/i}} = \mathbb{Z}[(1,-1)]$.
- The classification of intrinsic Non-Hermitian topology is

$$(\mathbb{Z}\oplus\mathbb{Z})/\mathbb{Z}[(1,-1)] \cong \mathbb{Z}.$$

- Remark: The image Im $f_{\mathrm{r/i}} = \mathbb{Z}[(1,-1)] \subset \mathbb{Z}\oplus\mathbb{Z}$ does not show the non-Hermitian skin effect, since the total winding number $N_1 + N_2$ is zero.

Results: AZ class

Tables from Okuma=Kawabata=KS=Sato 1910.02878.

AZ class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	0	0	0	0	0	0
AI	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0
BDI	0	0	0	0	0	0	0	0
D	0	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
DIII	0	0	0	0	$\mathbb{Z}_2$	0	0	0
AII	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
CII	0	0	0	0	0	0	0	0
C	0	0	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CI	$\mathbb{Z}_2$	0	0	0	0	0	0	0

- $d=1$, class A: non-Hermitian skin effect.
- $d=3$, class A: non-Hermitian skin effect induced by a magnetic field. Bessho=Sato 2006.04204, Kawabata=Shiozaki=Ryu 2011.11449

AZ† class

AZ† class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI†	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI†	0	0	0	0	0	0	0	0
D†	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0
DIII†	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0
AII†	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
CII†	0	0	0	0	0	0	0	0
C†	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
CI†	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0

- $d=1,2$, class AII†: $\mathbb{Z}_2$ non-Hermitian skin effect. Okuma=Kawabata=KS=Sato 1910.02878

AZ class with sublattice symmetry or pseudo-Hermiticity

AZ class	Add. symm.	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	η	0	0	0	0	0	0	0	0
AIII	S_+, η_+	0	0	0	0	0	0	0	0
A	S	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	S_-, η_-	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
AI	η_+	0	0	0	0	0	0	0	0
BDI	S_{++}, η_{++}	0	0	0	0	0	0	0	0
D	η_+	0	0	0	0	0	0	0	0
DIII	S_{--}, η_{++}	0	0	0	0	0	0	0	0
AII	η_+	0	0	0	0	0	0	0	0
CII	S_{++}, η_{++}	0	0	0	0	0	0	0	0
C	η_+	0	0	0	0	0	0	0	0
CI	S_{--}, η_{++}	0	0	0	0	0	0	0	0

- $d=2$, class AII$+S_-$: Edge exceptional point Denner=Neupert=Schindler 2304.13743

(cont.)

AZ class	Add. symm.	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	S_-	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
BDI	S_{-+}, η_{+-}	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
D	S_+	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$
DIII	S_{-+}, η_{-+}	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
AII	S_-	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0
CII	S_{-+}, η_{+-}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	0
C	S_+	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CI	S_{-+}, η_{-+}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	0
AI	η_-	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0
BDI	S_{--}, η_{--}	0	0	0	0	0	0	0	0
D	η_-	0	0	0	0	$\mathbb{Z}_2$	0	0	0
DIII	S_{++}, η_{--}	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
AII	η_-	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
CII	S_{--}, η_{--}	0	0	0	0	0	0	0	0
C	η_-	$\mathbb{Z}_2$	0	0	0	0	0	0	0
CI	S_{++}, η_{--}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	0	0	0

(cont.)

AZ class	Add. symm.	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
AI	S_+	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
BDI	S_{+-}, η_{-+}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0
D	S_-	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
DIII	S_{+-}, η_{+-}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
AII	S_+	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
CII	$S_{+-}, \eta -+$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
C	S_-	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	S_{+-}, η_{+-}	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

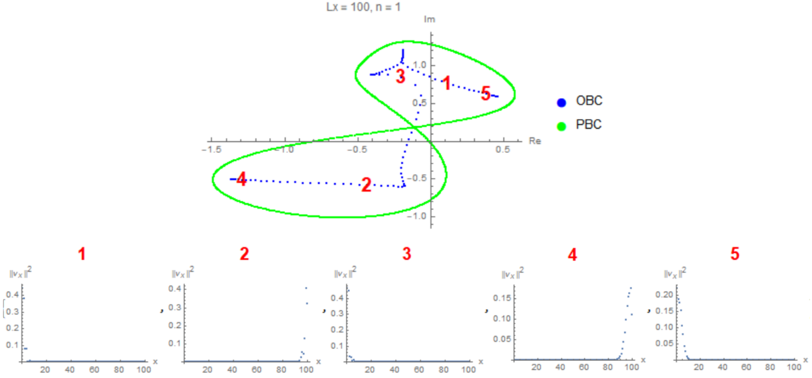
Note: I'm not familiar with the current status of the studies of intrinsic non-Hermitian topological phases. The reference list above may be very limited.

1D class A (No symmetry)

- The intrinsic non-Hermitian topology is classified by $\mathbb{Z}$.
- Topological invariant:

$$N(E_{\mathrm{ref}}) = \frac{1}{2\pi i} \oint d \log \det[H(k) - E_{\mathrm{ref}}] \in \mathbb{Z}.$$

- Nonzero winding number $N(E_{\mathrm{ref}}) \neq 0$ implies the non-Hermitian skin effect.



1D class AII†: $\mathbb{Z}_2$ non-Hermitian skin effect

- Class AII† symmetry

$$\sigma_y H(k)^T \sigma_y = H(-k) \quad \Leftrightarrow \quad \sigma_y H(x-x')^T \sigma_y = H(x'-x).$$

- The intrinsic non-Hermitian topology is classified by $\mathbb{Z}_2$!
- Topological number:

$$(-1)^{\nu(E_{\rm ref})} = \frac{\mathrm{Pf}[(H(\pi)-E_{\rm ref})\sigma_y]}{\mathrm{Pf}[(H(0)-E_{\rm ref})\sigma_y]} \times \exp\left[-\frac{1}{2}\int_0^\pi d\log\det[(H(k)-E_{\rm ref})\sigma_y]\right] \in \{\pm 1\}.$$

- Nonzero $\nu(E_{\rm ref}) \neq 0$ implies the reciprocal non-Hermitian skin effect: $O(L)$ modes localized both at left and right edges.
- Remark: Let $|E\rangle$ is an right eigenvector with eigenvalue E and $\langle\langle E|$ be the corresponding left eigenvector, i.e.,

$$H_{\rm OBC} = E\,|E\rangle\,\langle\langle E| + \cdots$$

The class AII† symmetry implies that $\sigma_y |E\rangle\rangle^*$ is also an eigenvector with eigenvalue E orthogonal to $|E\rangle$.

$$H_{\rm OBC}\,|E\rangle = E\,|E\rangle \quad \Leftrightarrow \quad H_{\rm OBC}\sigma_y|E\rangle\rangle^* = E\sigma_y|E\rangle\rangle^*,$$
$$\langle\langle E|\sigma_y|E\rangle\rangle^* = 0.$$

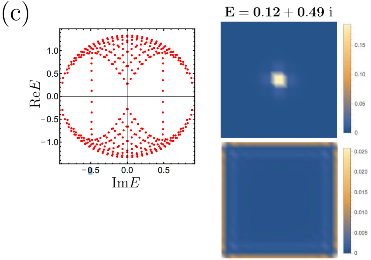
If $|E\rangle$ is localized at right, then its Kramers pair $\sigma_y|E\rangle\rangle^*$ is at left.

2D class AII†: $\mathbb{Z}_2$ non-Hermitian skin effect at π-vortex

- Class AII† symmetry

$$\sigma_y H(k_x, k_y)^T \sigma_y = H(-k_x, -k_y) \quad \Leftrightarrow \quad \sigma_y H(x - x', y - y')^T \sigma_y = H(x' - x, y' - y).$$

- The intrinsic non-Hermitian topology is still classified by $\mathbb{Z}_2$.
- When the bulk is $\mathbb{Z}_2$ nontrivial, under the π-vortex defect, the $O(L)$ non-Hermitian skin modes are localized at the π-vortex and boundary. (figure from Okuma=Kawabata=KS=Sato 1910.02878)



3D class A: non-Hermitian skin effect induced by magnetic field

- The intrinsic non-Hermitian topology is classified by $\mathbb{Z}$.
- The topological number is 3D winding number

$$W(E_{\mathrm{ref}}) = \frac{1}{24\pi^2}\int_{T^3} \mathrm{tr}\left[(H_{\boldsymbol{k}} - E_{\mathrm{ref}})^{-1} d(H_{\boldsymbol{k}} - E_{\mathrm{ref}})\right]^3 \in \mathbb{Z}.$$

- Nonzero $W(E_{\mathrm{ref}})$ implies that the non-Hermitian skin effect is induced by the magnetic field.
- Model:

$$H_{\boldsymbol{k}} = \cos k_x + \cos k_y + \cos k_z + i\gamma(\sigma_x \sin k_x + \sigma_y \sin k_y + \sigma_z \sin k_z).$$

- x,y: PBC, z: PBC/OBC. m: magnetic flux along the z direction. (figure from Kawabata=KS=Ryu 2011.11449)

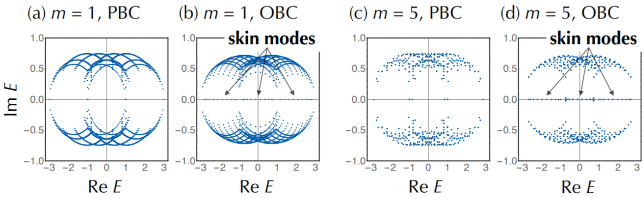


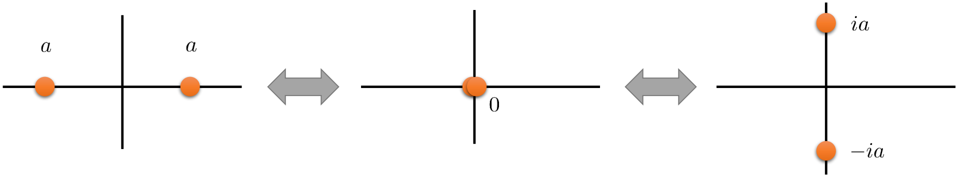
FIG. S4. Chiral magnetic skin effect. Complex spectra of the non-Hermitian Hamiltonian (S28) with the vector potential (S29) are shown. The parameters are chosen as $\gamma = 0.5$ and $L_x = L_y = L_z = 10$. The number of magnetic fluxes is $m = 1$ for (a, b), and $m = 5$ for (c, d). The periodic boundary conditions are imposed along the x and y directions. Along the z direction, the periodic boundary conditions are imposed for (a, c), and the open boundary conditions are imposed for (b, d). Skin modes appear under the open boundary conditions (b, d).

Example: Class AIII+S_- (sublattice symmetry anti-commuting with chiral symmetry)

- Symmetry:

$$\begin{cases} \sigma_z H(\boldsymbol{k})\sigma_z = -H(\boldsymbol{k}), \\ \sigma_y H(\boldsymbol{k})^\dagger \sigma_y = -H(\boldsymbol{k}). \end{cases} \quad \Rightarrow \quad H(\boldsymbol{k}) = \begin{pmatrix} & h_1(\boldsymbol{k}) \\ h_2(\boldsymbol{k}) & \end{pmatrix}, \qquad h_j(\boldsymbol{k})^\dagger = h_j(\boldsymbol{k}) \quad (j=1,2).$$

- $d=0$: (Point-gapped topological phases)/(Im $f_r \cup$ Im f_i) $= \mathbb{Z}_2$.
$\rightarrow$ is understood as the existence of the PT-symmetry breaking accompanied with an exceptional point at $E=0$:



$$H = \begin{bmatrix} & a \\ a & \end{bmatrix} \qquad H = \begin{bmatrix} & a \\ 0 & \end{bmatrix} \qquad H = \begin{bmatrix} & a \\ -a & \end{bmatrix}$$

Example: Class AIII+S_- (cont.)

- $d = 2$: (Point-gapped topological phases)/(Im $f_r \cup$ Im f_i) $= \mathbb{Z}_2$.
- There exists an intrinsic non-Hermitian topological phase.
- A model:

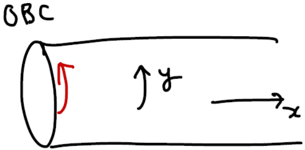
$$H(k_x, k_y) = \begin{pmatrix} & h_{\mathrm{Chern}}(k_x, k_y) \\ \mathbf{1}_{2\times 2} & \end{pmatrix},$$

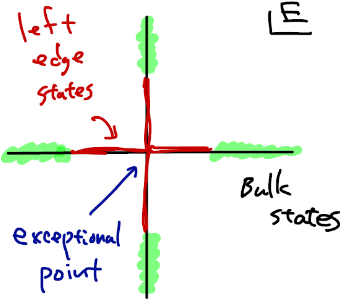
$$h_{\mathrm{Chern}}(k_x, k_y) = \sin k_x \sigma_x + \sin k_y \sigma_y + (m - t\cos k_x - t\cos k_y)\sigma_z .$$

- $H = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix} \quad \Rightarrow \quad \begin{cases} E = \pm\sqrt{\epsilon} & (\epsilon > 0) \\ E = \pm i\sqrt{-\epsilon} & (\epsilon < 0) \end{cases}$

Example: Class AIII+S_- (cont.)

- The Chern insulator $h_{\mathrm{Chern}}(k_x, k_y)$ has a chiral edge state localized at each edge.
- Therefore, the non-Hermitian Hamiltonian $H(k_x, k_y)$ has an exceptional point, the trajectory of the "PT-symmetry breaking", at each edge. Denner=Neupert=Schindler 2304.13743





Summary

In this lecture, I gave

1. Introduction
   — One-particle non-Hermitian systems
   — Exceptional point
   — Non-Hermitian skin effect
2. Gap condition and topology
   — Point gap
   — Real and imaginary line gaps
3. Symmetry classes
   — 38 classes in non-Hermitian systems
4. Topological classification
   — Point gap $\to$ doubled Hermitian Hamiltonian $\to$ Hermitian topological phases
   — Line gap $\to$ Hermitianization $\to$ Hermitian topological phases
   — Classifying spaces
   — Dimensional isomorphism
5. Intrinsic non-Hermitian topology
   — Line gap implies point gap
   — Intrinsic non-Hermitian topological phases should be interesting!

Skin effect is topological Zhang-Yang-Fang 19, Okuma-Kawabata-KS-Sato 19

- $W(H(k)) := \frac{1}{2\pi i} \oint d \log \det[H_{\mathrm{PBC}}(k)] \neq 0 \Rightarrow$ skin effect.

(Our proof)

- Let $\sigma(H_{\mathrm{PBC}})$, $\sigma(H_{\mathrm{OBC}})$ and $\sigma(H_{\mathrm{SIBC}})$ be the spectrum for PBC, OBC and the semi-infinite bdy condition, respectively. It holds that

$$\sigma(H_{\mathrm{OBC}}) \subset \sigma(H_{\mathrm{SIBC}}).$$

- The spectrum for OBC is invariant under the similarity transformation

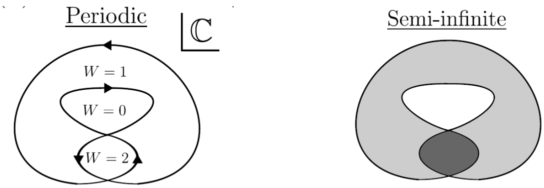
$$V_g f_x^\dagger V_g^\dagger = e^g f_x^\dagger, \quad g \in (0, \infty).$$

Therefore,

$$\sigma(H_{\mathrm{OBC}}) \subset \bigcap_{g \in (-\infty, \infty)} \sigma(V_g^{-1} H_{\mathrm{SIBC}} V_g).$$

- Toeplitz index theorem:

$$\sigma(H_{\mathrm{SIBC}}) = \sigma(H_{\mathrm{PBC}}) \cup \underbrace{\{E \in \mathbb{C} | W(H(k) - E) \neq 0\}}_{\text{dense spectrum}}.$$



This is because the bulk-boundary correspondence for the class AIII doubled Hamiltonian

$$\tilde{H}(k) = \begin{pmatrix} & H(k) - E \\ H(k)^\dagger - E^* & \end{pmatrix}.$$

If $W(H(k) - E) < 0$, there exists a zero mode $(0, |E\rangle)^T$ of $\tilde{H}$, i.e., the right eigenstate of $H(k)$ with eigenvalue E.

- Suppose that $H_{\mathrm{PBC}}(k)$ has a nonzero winding number.
- Take an arbitrary complex energy E with $W(H_{\mathrm{PBC}}(k) - E) \neq 0$. $|E\rangle$ represents an right or left eigenstate localized at the boundary.
- There exists $g \in (0, \infty)$ s.t. $|E\rangle$ such that $|E\rangle$ is a delocalized plane wave of $V_g^{-1} H_{\mathrm{SIBC}} V_g$, i.e. $E \in \sigma(V_g^{-1} H_{\mathrm{PBC}} V_g)$.
- The intersection of $\sigma(H_{\mathrm{SIBC}})$ and $\sigma(V_g^{-1} H_{\mathrm{PBC}} V_g)$ is strictly smaller than $\sigma(H_{\mathrm{SIBC}})$. This proves that $\sigma(H_{\mathrm{PBC}}) \neq \sigma(H_{\mathrm{OBC}})$.
- Furthermore, $\bigcap_{g \in (-\infty, \infty)} \sigma(V_g^{-1} H_{\mathrm{SIBC}} V_g)$ reaches a topological trivial area or curves, otherwise a contradiction arises.