Topology of matrix product states with onsite symmetry

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KS and Takahiro Morimoto, in preparation.

Outline

Introduction

- Thouless pump
- Kitaev's proposal: invertible phases are described by a generalized cohomology theory

Main part

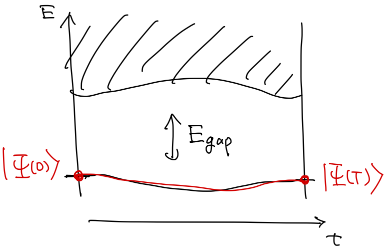
- Adiabatic cycle of $\mathbb{Z}_2$ symmetric spin chain
- Matrix product state
- homotopy class
- Cluster model
- Generic finite group onsite symmetry

Adiabatic cycle in unique gapped ground states

- Consider a cycle of Hamiltonians with a unique gapped ground state in a many-body system.

$$H(t), \quad t \in [0, T], \quad H(T) = H(0).$$

- We want to study the "homotopy class" of the cycle $|\Psi(t)\rangle$ of the instantaneous ground state of the Hamiltonian defined by $H(t)\,|\Psi(t)\rangle = E_{\text{GS}}\,|\Psi(t)\rangle$.
- We are interested in the "topology" of the space of unique gapped ground states that is not rigorously defined yet.
- In the rest of this talk, I simply refer to an adiabatic cycle of unique gapped ground state as an adiabatic cycle.



Thouless pump PRB **27**, 6083 (1983)

- An adiabatic cycle in 1D systems with $U(1)$ symmetry.

$$[N, H] = 0, \quad N = \sum_j n_j.$$

- In this cases, adiabatic cycles are known to be classified by $\mathbb{Z}$.
- If an adiabatic cycle is "topologically non-trivial", a $U(1)$ charge is pumped by an integer from left to right when the system is on a finite chain (open boundary condition).
- The number of the pumped $U(1)$ charge is given by an integer-valued topological invariant

$$\nu := \frac{1}{2\pi i} \oint_0^T dt \frac{d}{dt} \arg Z(t) \in \mathbb{Z},$$

where $Z(t) := \langle \Psi(t) | e^{\sum_{j=1}^L \frac{2\pi i n_j}{L}} | \Psi(t) \rangle \in \mathbb{C}$ is the electric polarization of the ground state $|\Psi(t)\rangle$ of the Hamiltonian $H(t)$.

- For a closed chain this is the same as the many-body Chern number

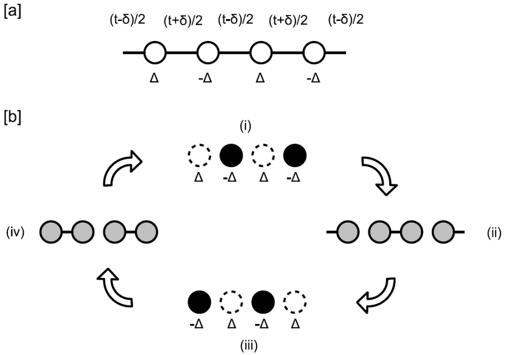
$$\nu = \frac{1}{2\pi i} \oint_0^T dt \oint_0^{2\pi} d\Phi F_{t\Phi}(t, \Phi), \quad F = \langle d\Psi(t,\Phi) | d\Psi(t,\Phi) \rangle,$$

where Φ is the twisted boundary condition.

The Rice-Mele model PRL **49**, 1455 (1982).

- The Thouless pump may be well illustrated by the Rice-Mele model.
- A free fermion model with the nearest neighbor hopping with the staggered amplitude $t+\delta, t-\delta$ and the staggered potential Δ.

$$H = \sum_j \left(\frac{t}{2} + (-1)^j \frac{\delta}{2}\right)(a_j^\dagger a_j + h.c.) + \Delta(-1)^j a_j^\dagger a_j.$$



Adiabatic cycle in generic 1D systems

- In general, one can think of an adiabatic cycle in gapped 1D systems in arbitrary setting:
  - fermion or boson (spin systems)
  - with onsite symmetry (time-reversal, $\mathbb{Z}_2$ Ising, $U(1)$, etc.)

- We want to address the following questions:
  - Do non-trivial adiabatic cycles exist?
  - If there is, how are they classified?
  - Can a topological invariant be constructed from a given cycle of Hamiltonian $H(t)$ or a given cycle of pure state $|\Psi(t)\rangle$?

- For free fermions, the answer is known.[Teo-Kane 10]

Theorem

For free fermionic systems, the classification of adiabatic cycles in 1D is the same as the classification of gapped unique ground states in 0D.

Kitaev's proposal

- On the basis of the results of free fermions, Kitaev proposed a generic topological structure behind the unique gapped ground states.

Proposition (Kitaev, 11,13,15)

The spaces $\{F_d\}_{d\in\mathbb{Z}_{\geq 0}}$ of unique gapped ground states in d space dimensions form an Ω-spectrum of a generalized cohomology theory.

- I will not go into detail today. See also Gaiotto and Johnson-Freyd, JHEP (2109)

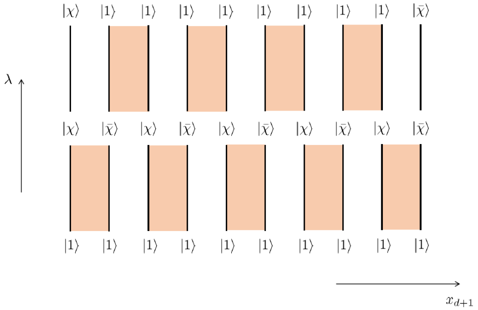
- This proposal implies the generalization about the classification of adiabatic cycles in 1D many-body systems.

Proposition

For generic many-body systems, the classification of adiabatic cycles in 1D is the same as the classification of gapped unique ground states in 0D.

Canonical adiabatic cycle

- Kitaev provided a sort of canonical adiabatic cycle to support his proposal:
- A characteristic of a unique gapped ground state is the existence of an inverse of it up to homotopy equivalence $|\chi\rangle \otimes |\bar{\chi}\rangle \sim |1\rangle \otimes |1\rangle$. Here, $|\chi\rangle$ is a unique gapped ground state in dD, and $|1\rangle$ is, for example, the tensor product state.
- Applying this homotopy to the tensor product state $\cdots \otimes |\chi\rangle \otimes |\bar{\chi}\rangle \otimes \cdots$ in $(d+1)$D, we have an adiabatic cycle with the pumped state $|\chi\rangle$ / $|\bar{\chi}\rangle$ at the left/right edge.



- I'm not sure this picture correctly describes the Ω-spectrum structure of unique gapped ground states... Continuity of adiabatic Hamiltonian?? No entanglement??

Adiabatic cycles in spin chains with $\mathbb{Z}_2$ symmetry

- For the simplest setting of the many-body problem, consider the following case.
- What is the classification of adiabatic cycles of spin chains with $\mathbb{Z}_2$ symmetry?

$$
\begin{aligned}
&H = \sum_j a S_j^z + b(S_j^z)^2 + c S_j^z S_{j+1}^z + d \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + e(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 + \cdots \, , \\
&Z = e^{i\pi \sum_j S_j^z}, \quad [Z, H] = 0.
\end{aligned}
$$

- If the Kitaev's proposal is correct, adiabatic cycles are classified by $\mathbb{Z}_2$, the classification of $\mathbb{Z}_2$ charge.

Matrix product state

- The Matrix Product State (MPS) is quite useful tool to describe unique gapped ground states in spin chains.
- For simplicity, we assume translational symmetry.
- A matrix product state defined by a collection of matrices $\{A_m\}$ is written as

$$|\psi(\{A_m\})\rangle = \sum_{\{m_j\}} \mathrm{Tr}[\cdots A_{m_j} A_{m_{j+1}} \cdots] \, |\cdots m_j m_{j+1} \cdots\rangle \, .$$

Here, $A_m = [A_m]_{\alpha\beta}$ are $D \times D$ matrices with $m = 1, \ldots, \mathrm{dim}\mathcal{H}_j$ the indices of the local Hilbert space $\mathcal{H}_j$, and $\alpha\beta$ stands for the bond Hilbert space. D measures the entanglement between two sites.

Injective MPS and uniqueness

- Unique gapped ground states are described by injective MPSs. (Short-range correlation, and no cat states.)
- The only property of injective MPS we use is the following.

Lemma

Two injective MPSs $|\Psi(\{A_m\})\rangle$ *and* $|\Psi(\{\tilde{A}_m\})\rangle$ *represent the same state iff there exists* $e^{i\chi} \in U(1)$ *and* $W \in U(D)$ *such that*

$$\tilde{A}_m = e^{i\chi} W^\dagger A_m W.$$

Here, $e^{i\chi}$ *is unique and* W *is unique up to a* $U(1)$ *phase.*

- This is a kind of gauge choice of an MPS. Physical consequences should be independent of this gauge choice.

MPS with $\mathbb{Z}_2$ symmetry

- Write the Z_2-symmetry operator as $Z = \bigotimes_j U^{(j)}$, where $U^{(j)}$ is a local $\mathbb{Z}_2$ transformation at site j, namely, $U^{(j)} |n_j\rangle = \sum_{m_j} |m_j\rangle U_{m_j n_j}$ with U_{mn} a $D \times D$ unitary matrix.
- The $\mathbb{Z}_2$-symmetry action on the injective MPS $|\Psi(\{A_m\})\rangle$ is given by

$$Z |\Psi(\{A_m\})\rangle = \sum_{\{m_j\}} \mathrm{tr}\,[\cdots A_{m_j} \cdots] \,|\cdots n_j \cdots\rangle\, [\cdots U_{n_j m_j} \cdots].$$

- Therefore, from the lemma above, the $\mathbb{Z}_2$ symmetry $Z |\Psi(\{A_m\})\rangle \sim |\Psi(\{A_m\})\rangle$ implies that there exists a unitary matrix $V \in U(D)$ and phase $e^{i\theta}$ such that

$$\sum_n U_{mn} A_n = e^{i\theta} V^\dagger A_m V.$$

  Here, $e^{i\theta}$ is unique and $V \in U(D)$ is unique up to a $U(1)$ phase, again.
- Because the $\mathbb{Z}_2$ transformations twice is the same as the identity, we have $e^{i\theta} \in \pm 1$ and the V square is the identity matrix $\mathbf{1}_D$ up to a $U(1)$ phase.
- The matrix V is regarded as an element of the projective unitary group $[V] \in PU(D) := U(D)/\{z\mathbf{1}_D | z \in U(1)\}$.

Where is the topology?

- Which is the desired topological space of which the fundamental group π_1 is non-trivial?

$$\{A_m\} \qquad e^{i\theta} \qquad V \quad /\text{gauge}$$

- The space of $U(1)$ phase $e^{i\theta}$ is just two points $e^{i\theta} \in \{\pm 1\}$.
- The space to which a collection of matrices $\{A_m\}$ seems to be complected (for me). How to deal with the injectivity, which is a kind of algebraic constraint on A_ms, and the gauge freedom...
- The space to which $V \in U(D)$ belongs is simpler than one of A_ms. Let's see the detail of the space of V.

Topology of the space of V

$\underline{D=1}$

- The projective unitary group is trivial $PU(1) = \{1\}$, and no non-trivial loops exist.
- This suggests that non-trivial entanglement is necessary to obtain a non-trivial adiabatic cycle.

$\underline{D=2}$

- The projective unitary group $PU(2) = SU(2)/\{1, -1\} = SO(3)$ is identified with the group of $SO(3)$ rotation.
- Furthermore, the condition $[V]^2 = \mathrm{id}$ shows that V is either (i) identity or (ii) π-rotation around an axis $\hat{n} \in S^2$.
- For the former case, $[V]$ is constant, meaning that the fundamental group is trivial.
- For the latter case, the equivalent class $[V]$ runs over the real projective plane $RP^2 = S^2/\mathbb{Z}_2$, since $\hat{n}$ and $-\hat{n}$ represents the same π-rotation. Therefore, we obtain the $\mathbb{Z}_2$ classification $\pi_1(RP^2) = \mathbb{Z}_2$!

$\underline{D \geq 3}$

- One can show the fundamental group of the space to which V belongs is $\mathbb{Z}_2$ if $\mathrm{tr}\, V = 0$.

Homotopy class and gauge independence

- The discussion above shows the homotopy class of the matrix $V(t)$ signals a nontrivial adiabatic cycle.
- To have a periodic cycle of $V(t)$, a cycle of matrices $A_m(t)$s should be also periodic $A_m(T) = A_m(0)$. Under the periodicity of $A_m(t)$, a cycle of $V(t)$ is defined by

$$U_{mn}A_n(t) = e^{i\theta(t)}V(t)^\dagger A_m(t)V(t), \quad [V(T)] = [V(0)].$$

- We should take care about the gauge dependence of A_ms. A cycle of gauge transformation $(e^{i\chi(t)}, W(t))$ with periodicity $e^{i\chi(T)} = e^{i\chi(0)}$ and $W(T) = W(0)$ induces the gauge transformation of $V(t)$,

$$V(t) \mapsto \tilde{V}(t) = W(t)^\dagger V(t)W(t).$$

- We can see that this does not affect the homotopy class of $V(t)$:
$W(t)$ can be regarded as a cycle of the spacial unitary group $SU(D)$. Since $SU(D)$ is simply connected ($\pi_1(SU(D)) = 0$) the cycle $W(t)$ is contractible to the constant cycle $W(t) \sim \mathbf{1}_D$.

Cluster model

- A simple model showing a nontrivial homotopy class of $V(t)$ is given by the cluster model [Briegel-Raussendorf, PRL **86**, 910 (2001)] with a slight modification.

$$H(\phi) = -\sum_j \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_j \tau_{j-\frac{1}{2}}^z(\phi) \sigma_j^x \tau_{j+\frac{1}{2}}^z(\phi).$$

- Here, $\boldsymbol{\sigma}_j, \boldsymbol{\tau}_{j+\frac{1}{2}}$ are spin operators at sites $j, j+\frac{1}{2} (j \in \mathbb{Z})$.
- All terms are commuted with each other.
- The spin operator $\boldsymbol{\tau}_{j+\frac{1}{2}}(\phi)$ is defined by the $\phi/2$ rotation around the x-axis of the τ spin.

$$\boldsymbol{\tau}_{j+\frac{1}{2}}(\phi) := e^{-i\frac{\phi}{2}\tau_{j+\frac{1}{2}}^x} \boldsymbol{\tau}_{j+\frac{1}{2}} e^{i\frac{\phi}{2}\tau_{j+\frac{1}{2}}^x}.$$

- The Hamiltonian is π-periodic.

$$H(\phi+\pi) = H(\phi),$$

whereas $\boldsymbol{\tau}(\phi)$ is 2π-periodic

$$\tau_{j+\frac{1}{2}}^z(\pi) = -\tau_{j+\frac{1}{2}}^z.$$

- This model enjoys the $\mathbb{Z}_2$ symmetry defined as the flip of σ spins $Z := \prod_j \sigma_j^x$.

Edge state

- The cluster Hamiltonian with the open boundary condition:

$$H^{\text{open}}(\phi) = -\sum_{j=1}^{N-1} \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_{j=1}^{N} \tau_{j-\frac{1}{2}}^z(\phi) \sigma_j^x \tau_{j+\frac{1}{2}}^z(\phi).$$

- The ground state is given by imposing

$$\sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z = \tau_{j-\frac{1}{2}}^z(\phi) \sigma_j^x \tau_{j+\frac{1}{2}}^z(\phi) = 1$$

  on the total Hilbert space.

- It is found that the ground state is four-fold degenerate, which is originated from the emergent spin 1/2 degree of freedom at each edge.
- One can read off the $\mathbb{Z}_2$-symmetry action on the ground state manifolds as

$$Z|_{\text{G.S.}} = \sigma_1^x \sigma_2^x \cdots \sigma_{N-1}^x \sigma_N^x|_{\text{G.S.}} = \tau_{\frac{1}{2}}^z(\phi) \times \tau_{N+\frac{1}{2}}^z(\phi).$$

- Remarkebely, the symmetry transformation on the edge states explicitly depend on the adiabatic parameter ϕ.

Edge state

- The $\mathbb{Z}_2$ symmetry action on the right edge $\tau^z_{N+1/2}(\phi)$ is identified with the matrix $V(\phi)$ on the bond Hilbert space for MPS. Thus, we have

$$V(\phi) = \tau^z(\phi) = \tau^z e^{i\tau^x \phi} = \begin{pmatrix} \cos\phi & i\sin\phi \\ -i\sin\phi & -\cos\phi \end{pmatrix}.$$

- (The same matrix $V(\phi)$ can be explicitly derived from the MPS representation of the ground state of the cluster Hamiltonian.)
- This belongs to a nontrivial homotopy class of $\pi_1(RP^2) = \mathbb{Z}_2$.
- The $\mathbb{Z}_2$ transformation at an edge breaks the π-periodicity.

$$\tau^z(\phi + \pi) = -\tau^z(\phi).$$

- cf. Topological Floquet phases Potter-Morimoto-Vishwanath, PRX **6**, 041001 (2016).

Spin systems with generic finite group symmetry

- The classification of adiabatic cycle above can be generalized to generic finite symmetry group G.
- Our proof is much based on the work about the homotopy type of the projective unitary group. Espinoza-Uribe, arXiv:1511.06785

Theorem

For an injective MPS with a projective representation V_g of G, the classification of adiabatic cycle is given by the subgroup of $\mathrm{Hom}(G, U(1))$ defined by

$$\{\eta \in \mathrm{Hom}(G, U(1)) | \mathrm{tr}\,[V_g] = 0 \text{ holds for any elements } g \in G \text{ with } \eta_g \neq 1\}.$$

- With a suitable equivalence relation on unique gapped ground states, it might be attributed to the group of the known classification of 0D unique gapped ground states,

$$H^1(G, U(1)) \cong \mathrm{Hom}(G, U(1)).$$

- cf. Topological Floquet phases Potter-Morimoto-Vishwanath, PRX **6**, 041001 (2016).

Proof (see also Espinoza-Uribe, arXiv:1511.06785)

- Let $|\Psi(\{A_m\})\rangle$ be an MPS that is unique gapped and G symmetric.

$$g_{mn}A_n = e^{i\theta_g} V_g^\dagger A_m V_g.$$

- Here, V_g is a projective representation with a two-cocycle ω,

$$V_g V_h = \omega_{g,h} V_{gh}, \quad \omega_{g,h} \in U(1), \quad \omega_{h,k}\omega_{gh,k}^{-1}\omega_{gh,k}\omega_{g,h}^{-1} = 1.$$

- A homomorphism $\eta : G \to U(1)$, $\eta_g \eta_h = \eta_{gh}$ may change the ω-projective representation by $V_g \mapsto \eta_g V_g$ while keeping the two-cocycle ω.
- Therefore, a cycle $V_g(t)$ runs over the quotient space

$$\{\text{The space of } \omega\text{-projective representations}\}/\mathrm{Hom}(G, U(1)).$$

- Any ω-projective representation V_g can be written as

$$V_g = W V_g^{\mathrm{ref}} W^\dagger$$

with a reference ω-projective representation V_g^{ref}.
- The matrix W lives in the spacial unitary group $SU(D)$ that is simply connected, the fundamental group of $\{$The space of ω-projective representations$\}$ is trivial. Thus,

$$\pi_1\Big[\{\text{The space of } \omega\text{-projective representations}\}/\mathrm{Hom}(G, U(1))\Big] \cong \mathrm{Hom}(G, U(1)).$$

Summary

- Motivated by the Kitaev's proposal, we sturdy the topological classification of adiabatic cycle of unique gapped ground state of spin chains.
- Using the injective MPS, we found that the symmetry transformation V on the bond Hilbert space is essentially classified by the group $H^1(G, U(1)) \cong \mathrm{Hom}(G, U(1))$, which is the same as the classification of 0D spin systems with G symmetry.

$\mathbb{Z}_2$ charge localized at a texture

- One way to verify that the cluster model $H(\phi)$ has a nontrivial adiabatic cycle is to introduce a texture in space direction where the Hamiltonian $H(\phi)$ varies from 0 to π and measure the $\mathbb{Z}_2$ charge.
- Note that the Hamiltonian $H(\phi)$ can be written in the form of the following unitary transformation.

$$H(\phi) = U_\tau(\phi) H(\phi = 0) U_\tau(\phi)^\dagger, \quad U_\tau(\phi) = \prod_j e^{-i\frac{\phi}{2}\tau^x_{j+\frac{1}{2}}}.$$

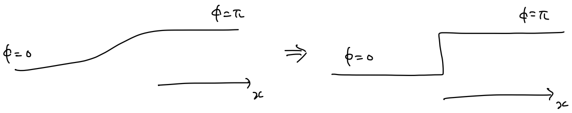
- Then the Hamiltonian H_{text} with a can be written as

$$H_{\text{text}} = U_{\text{tw}} H(\phi = 0) U_{\text{tw}}^\dagger, \quad U_{\text{tw}} = \prod_{j=1}^{N} e^{-i\frac{\pi j}{2N}\tau^x_{j+\frac{1}{2}}} \prod_{j>N} e^{-i\frac{\pi}{2}\tau^x_{j+\frac{1}{2}}} \sim \prod_{j=1}^{N} e^{-i\frac{\pi j}{2N}\tau^x_{j+\frac{1}{2}}} \prod_{j>N} \tau^x_{j+\frac{1}{2}}.$$

Cont.

- For simplicity, let us consider a kink at $j = 0$.

$$H_{\text{kink}} = U_{\text{kink}} H(\phi = 0) U_{\text{kink}}^{\dagger}, \quad U_{\text{kink}} = \prod_{j>0} \tau^{x}_{j+\frac{1}{2}}$$



- The ground state with a kink can be written as $|GS_{\text{kink}}\rangle = U_{\text{kink}} |GS(\phi = 0)\rangle$.
- Noticing $\tau^{x}_{j+\frac{1}{2}} = \sigma^{z}_{j}\sigma^{z}_{j+1}$ for the ground state manifolds, we have

$$|GS_{\text{kink}}\rangle = (\sigma^{z}_{0} \otimes \text{id} \otimes \text{id} \otimes \cdots) |GS_{\text{kink}}\rangle .$$

- Therefor, a kink has a $\mathbb{Z}_2$ charge.
- We can also confirm that the $\mathbb{Z}_2$ charge emerges at a texture varying slowly in the chain by a numerical caluclation.