On the adiabatic pump in quantum spin systems

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Outline

Introduction

- Thouless pump
- Ω-spectrum proposal by Kitaev

A 1D model with $\mathbb{Z}_2$ symmetry

- Edge state and projective representation
- Topological invariant

The group cohomology model of adiabatic cycle in any dimension (cf. Roy–Harper)

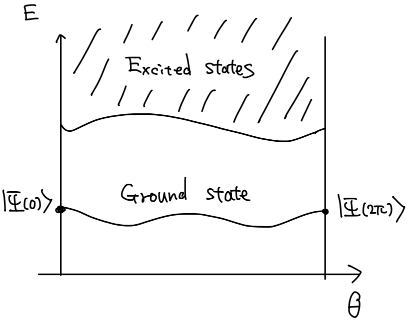
- Topological invariant from group cocycle
- Bockstein homomorphism
- Chen-Gu-Liu-Wen's construction
- Pumped SPT phase on the boundary
- Texture induced SPT phase

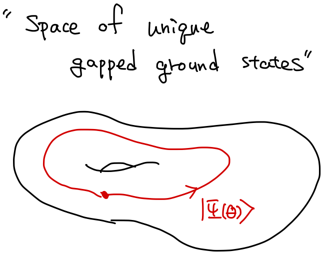
Adiabatic cycle in unique gapped ground states

- An adiabatic cycle is a periodic one-parameter family of Hamiltonians $H(\theta)$ with unique gapped ground state $|\Psi(\theta)\rangle$ in a many-body system.

$$H(\theta), \quad \theta \in [0, 2\pi], \quad H(2\pi) = H(0),$$
$$H(\theta)\,|\Psi(\theta)\rangle = E_{\mathrm{GS}}(\theta)\,|\Psi(\theta)\rangle\,.$$

- One of the motivation to study such cycles, not just a Hamiltonian, is to study the "higher-dimensional homotopy" of the "space of unique gapped ground states".





Thouless pump

- An adiabatic cycle in 1D systems with $U(1)$ symmetry

$$[N, H(\theta)] = 0, \quad N = \sum_j n_j.$$

- Given a unique gapped ground state $|\Psi\rangle$ with $U(1)$ symmetry, one can define a $U(1)$-valued quantity (polarization) [Resta]

$$e^{i\Theta_\Psi} \sim \langle \Psi \,|\, U_{\mathrm{twist}} \,|\, \Psi \rangle \in U(1), \qquad U_{\mathrm{twist}} = e^{\sum_{j=1}^{\ell} \frac{2\pi i n_j}{\ell}}.$$

- This implies that the "space of unique gapped ground states" has a "vortex" characterized by π_1.
- For a given cycle $|\Psi(\theta)\rangle$, one can define the $\mathbb{Z}$ invariant as the $U(1)$ phase winding of the polarization

$$\nu = \frac{1}{2\pi i} \oint d\Theta_{\Psi(\theta)} \in \mathbb{Z}.$$

- Physically, ν is the $U(1)$ charge pumped by a period of adiabatic cycle.

Adiabatic cycle in general

- One can think of adiabatic cycles in generic systems:
  - Generic space dimensions
  - Fermion and Boson (spin systems)
  - Any onsite symmetry (time-reversal, $\mathbb{Z}_2$ Ising, $U(1)$, etc.)

- We want to address the following questions:
  - Do non-trivial adiabatic cycles exist?
  - If there is, how are they classified?
  - Can we have a topological invariant of adiabatic cycles?

- For free fermions (with transnational invariance), the K-theory tells us [Teo-Kane]: the classification of adiabatic cycles in dD is the same as the classification of gapped unique ground states in $(d-1)$D.

$$K^{-n}(S^1 \wedge S^d) \cong K^{-(n-1)}(S^d) \cong K^{-n}(S^{d-1}).$$

Ω-spectrum proposal by Kitaev

Kitaev proposed a generic topological structure behind the unique gapped ground states. [Kitaev, 11, 13, 15]

- Let E_d be the "space of dD unique gapped ground states".
- The sequence of the spaces $\{E_d\}_{d\in\mathbb{Z}}$ forms an Ω-spectrum of the generalized cohomology theory. Namely, there is a homotopy equivalence

  $$\Omega E_{d+1} \sim E_d,$$

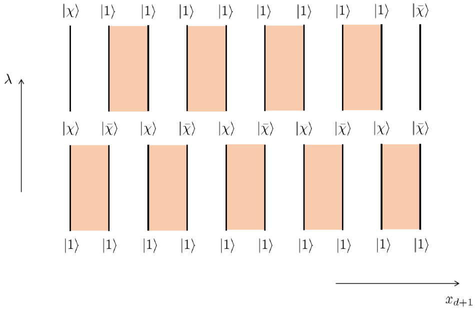
  where

  $$\Omega E_{d+1} = \{\, |\Psi(\theta)\rangle \in E_{d+1} \,|\, |\Psi(2\pi)\rangle = |\Psi(0)\rangle = |1\rangle \,\}$$

  is the (based) loop space of E_{d+1}. [Xiong, Gaiotto–Johnson-Freyd] ($|1\rangle$ is a trivial tensor product state.)
- Implication: The adiabatic cycles in $(d+1)$D are classified by the SPT phases in dD.

Canonical adiabatic cycle

- There is a canonical construction of the adiabatic cycle in $(d+1)$D for a given unique gapped ground state $|\chi\rangle$ in dD. [Kitaev]
- There should be the inverse state $|\bar{\chi}\rangle$ such that $|\chi\rangle \otimes |\bar{\chi}\rangle \sim |1\rangle \otimes |1\rangle$.
- Applying this homotopy to the tensor product state $\cdots \otimes |\chi\rangle \otimes |\bar{\chi}\rangle \otimes \cdots$ in $(d+1)$D, we have an adiabatic cycle with the pumped state $|\chi\rangle$ / $|\bar{\chi}\rangle$ at the left/right edge.



- This gives a map $E_d \to \Omega E_{d+1}$. I'm not sure an inverse map $\Omega E_{d+1} \to E_d$ is constructed and the homotopy equivalence is proven. cf. Gaiotto–Johnson-Freyd

Motivation

- Adiabatic cycles in any space dimensions and any symmetry groups.
- To give models of adiabatic cycles in many-body systems, especially, for quantum spin systems. (The canonical adiabatic cycle introduced above is too simple...)
- Is there any "geometric quantity" like the polarization $e^{i\Theta}$ for generic adiabatic cycles?

Adiabatic cycles of 1D spin systems (spin chains) with $\mathbb{Z}_2$ symmetry

- For a simple setting for the many-body problem, let me start with the spin system with onsite $\mathbb{Z}_2$ symmetry.
- What is the classification of adiabatic cycles of spin chains with $\mathbb{Z}_2$ symmetry?

$$H(\theta) = \sum_j a(\theta) S_j^x + b(\theta)(S_j^x)^2 + c(\theta) S_j^x S_{j+1}^x + d(\theta) \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + e(\theta)(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 + \cdots,$$

$$V = e^{i\pi \sum_j S_j^x}, \quad [V, H] = 0.$$

- The adiabatic cycles are supposed to be classified by $\mathbb{Z}_2$, the classification of $\mathbb{Z}_2$ charge.

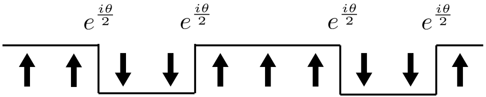
A toy model

- As a simple model of the $\mathbb{Z}_2$ spin pump, we consider the spin chain with spin $1/2$ dofs at each site and introduce the ground state parameterized by θ as in

$$|\Psi_\theta\rangle = \sum_{\{\sigma_j\}} e^{\frac{i\theta}{2}N_{\mathrm{dw}}} \left|\cdots\sigma_j\sigma_{j+1}\cdots\right\rangle, \qquad \sigma_j \in \{\uparrow,\downarrow\},$$

where N_{dw} is the number of domain walls, which is defined by

$$N_{\mathrm{dw}} = \sum_j \frac{1-\sigma^z_j\sigma^z_{j+1}}{2}.$$



- The periodicity of θ seems to be 4π, however, on the closed chain w/ PBC or APBC, the number of domain walls is even/odd, implying that the periodicity of θ is 2π.

- To be precise, the parent Hamiltonian H_θ is 2π-periodic.
- The ground state $|\Psi_\theta\rangle$ is given by the local unitary (finite time time evolution of a local Hamiltonian) which gives the $U(1)$ factor $e^{\frac{i\theta}{2}}$ to each domain wall.

$$|\Psi_\theta\rangle = U_\theta \, |\cdots \to\to \cdots\rangle \sim \sum_{\{\sigma_j\}} e^{\frac{i\theta}{2} N_{\rm dw}} \, |\cdots \sigma_j \sigma_{j+1} \cdots\rangle \,,$$

$$U_\theta = \prod_j e^{\frac{i\theta}{2} \frac{1-\sigma_j^z \sigma_{j+1}^z}{2}},$$

- The parent Hamiltonian H_θ is also given by the local unitary

$$H_\theta = U_\theta H_0 U_\theta^{-1}, \quad H_0 = -\sum_j \sigma_j^x.$$

- The parent Hamiltonian H_θ is found to be 2π-periodic, as in

$$\begin{aligned} B_j^\theta &= U_\theta \sigma_j^x U_\theta^{-1} \\ &= \frac{1+\cos\theta}{2} \sigma_j^x - \frac{1-\cos\theta}{2} \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z + \frac{1}{2} \sin\theta (\sigma_{j-1}^z \sigma_j^y + \sigma_j^y \sigma_{j+1}^z). \end{aligned}$$

- This model has a $\mathbb{Z}_2$ symmetry defined by the π-rotation around the x-axis.

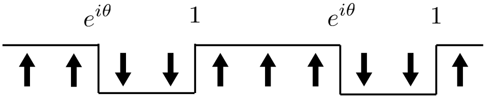
$$V_{\mathbb{Z}_2} = \prod_j \sigma_j^x.$$

Edge ambiguity of the local unitary

- The local unitary U_θ again seems not to be 2π-periodic.
- If we rewrite U_θ as

$$U_\theta = \prod_j e^{-\frac{i\theta}{4}\sigma_j^z} e^{i\theta \frac{1+\sigma_j^z}{2} \frac{1-\sigma_{j+1}^z}{2}} e^{\frac{i\theta}{4}\sigma_{j+1}^z} \sim \prod_j e^{i\theta \frac{1+\sigma_j^z}{2} \frac{1-\sigma_{j+1}^z}{2}} =: \tilde{U}_\theta,$$

  the 2π-periodicity $\tilde{U}_\theta$ is evident. $\tilde{U}_\theta$ assign the $U(1)$ phase $e^{i\theta}$ only on the configuration $\uparrow\downarrow$.



- However, $\tilde{U}_\theta$ differs from U_θ on an open chain. In other words, the local unitary U_θ has ambiguity in the local unitary near the edge.
- Later, we will see U_θ has good properties for our purpose.

Local unitary on an open chain

- Let us consider the open chain $j = 1, \ldots, N$.
- We shall define two different local unitaries

$$U_\theta = \prod_{j=1}^{N-1} e^{\frac{i\theta}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}},$$

$$\tilde{U}_\theta = \prod_{j=1}^{N-1} e^{i\theta\frac{1+\sigma_j^z}{2}\frac{1-\sigma_{j+1}^z}{2}},$$

  which are related by an edge term.
- They have different properties under $\mathbb{Z}_2$ and 2π-periodicity.
- U_θ is $\mathbb{Z}_2$ symmetric as $V_{\mathbb{Z}_2} U_\theta V_{\mathbb{Z}_2}^{-1} = U_\theta$, but breaks the 2π-periodicity at the edge $U_{2\pi} \sim \sigma_1^z \sigma_N^z$.
- $\tilde{U}_\theta$ breaks $\mathbb{Z}_2$ symmetry at the edge, but preserves the 2π-periodicity.
- Later, we use U_θ for creating the texture Hamiltonian.

Edge state

- On the open chain, the Hamiltonian is like

$$H_\theta = H_\theta^{\mathrm{bulk}} + H_\theta^{\mathrm{edge}}.$$

- H_θ^{bulk} is the sum of local Hamiltonians strictly inside the bulk, i.e.,

$$H_\theta^{\mathrm{bulk}} = -\sum_{j=2}^{N-1} B_j^\theta.$$

- H_θ^{edge} can be any local Hamiltonian near the edge, which we see some examples later.
- The ground state is four-fold degenerate from the edge free spins. The relative $U(1)$ phases are fixed, for example, as

$$|\Psi_\theta(\sigma_1, \sigma_N)\rangle = \prod_{j=2}^{N-1} \frac{1 + B_j^\theta}{2} |\sigma_1 \uparrow \cdots \uparrow \sigma_N\rangle .$$

Edge $\mathbb{Z}_2$ symmetry

- On the ground state manifold, the $\mathbb{Z}_2$ symmetry action is found to be

$$V_{\mathbb{Z}_2} |\Psi_\theta(\sigma_1, \sigma_N)\rangle = e^{i\theta \frac{\sigma_1 + \sigma_N}{2}} |\Psi_\theta(-\sigma_1, -\sigma_N)\rangle , \quad \sigma_1, \sigma_N \in \pm 1.$$

- By introducing the spin operators $\bar{\sigma}_1^\mu, \bar{\sigma}_N^\mu$ acting on the ground state manifold $|\Psi_\theta(\sigma_1, \sigma_N)\rangle$, the effective $\mathbb{Z}_2$ symmetry is written as a separated form for each edge, as for 1D SPT phases. [Pollmann–Berg–Turner–Oshikawa, Chen–Gu–Wen, Schuch–Pérez-García–Cirac]

$$P_\theta V_{\mathbb{Z}_2} P_\theta = e^{\frac{i\theta}{2} \bar{\sigma}_1^z} \bar{\sigma}_1^x \cdot e^{\frac{i\theta}{2} \bar{\sigma}_N^z} \bar{\sigma}_N^x.$$

- Let us focus on the effective $\mathbb{Z}_2$ symmetry on the left.

$$v_l^\theta \sim e^{\frac{i\theta}{2} \bar{\sigma}_1^z} \bar{\sigma}_1^x.$$

- We stress that the overall $U(1)$ phase of the left $\mathbb{Z}_2$ action v_l^θ has no physical meaning. The separated $\mathbb{Z}_2$ action should be regarded as a projective representation of $\mathbb{Z}_2$.
- As no nontrivial projective representation of $\mathbb{Z}_2$ exists $H^2(\mathbb{Z}_2, U(1)) = 0$, the effective $\mathbb{Z}_2$ action can be a linear representation for a θ. In fact, the gauge choice $v_l^\theta = e^{\frac{i\theta}{2} \bar{\sigma}_1^z} \bar{\sigma}_1^x$ is linear $(v_l^\theta)^2 = \mathbf{1}$.

The $\mathbb{Z}_2$ invariant

- However, the gauge choice $v_l^\theta = e^{\frac{i\theta}{2}\bar{\sigma}_1^z}\bar{\sigma}_1^x$ is not 2π-periodic.
- If we enforce the 2π-periodicity of v_l^θ, we realize that we can not have a linear representation of $\mathbb{Z}_2$. For example, the gauge choice

$$v_l^\theta = e^{i\theta\frac{1+\bar{\sigma}_1^z}{2}}\bar{\sigma}_1^x$$

  gives us $(v_l^\theta)^2 = e^{i\theta}\mathbf{1}$.
- We come up with the existence of the $\mathbb{Z}_2$ invariant that prevents a 2π-periodic linear representation.
- Let $\omega_\theta \in U(1)$ be the 2π-periodic two-cocycle (factor system) of the projective representation of $\mathbb{Z}_2$ defined by

$$(v_l^\theta)^2 = \omega_\theta \mathbf{1}.$$

  The $\mathbb{Z}_2$ invariant is defined by the parity of the phase winding of the two-cocycle.

$$\nu := \frac{1}{2\pi i}\oint d\log\omega_\theta \mod 2.$$

- The $\mathbb{Z}_2$-ness is because a redefinition $v_l^\theta \mapsto v_l^\theta \alpha_\theta$ with α_θ a 2π-periodic $U(1)$-valued function changes ν by an even integer.

$\mathbb{Z}_2$ charge pump

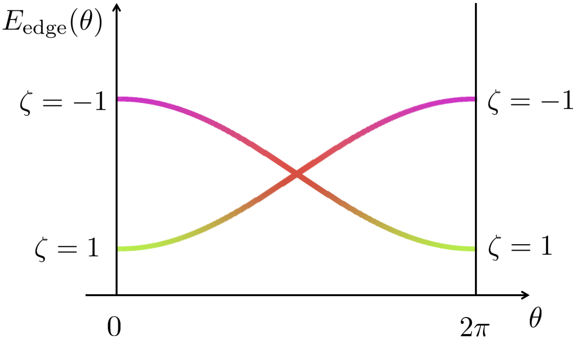
- Since any 2π-periodic one-dimensional projective representation of $\mathbb{Z}_2$ has the trivial $\mathbb{Z}_2$ invariant $\nu \equiv 0$, we conclude the following:
- If the $\mathbb{Z}_2$ invariant ν is nontrivial $\nu \equiv 1$, then the edge state can not be a unique state for all $theta$.
- To demonstrate it we consider the following edge Hamiltonian

$$H_\theta^{\mathrm{edge}} = -\lambda_1 \sigma_1^x - \lambda_N \sigma_N^x.$$

- The first-order effective edge Hamiltonian becomes

$$P_\theta H_\theta^{\mathrm{edge}} P_\theta = -\lambda_1 \cos\frac{\theta}{2} e^{-\frac{i\theta}{4}\bar{\sigma}_1^z} \bar{\sigma}_1^x e^{\frac{i\theta}{4}\bar{\sigma}_1^z} - \lambda_N \cos\frac{\theta}{2} e^{-\frac{i\theta}{4}\bar{\sigma}_N^z} \bar{\sigma}_N^x e^{\frac{i\theta}{4}\bar{\sigma}_N^z}.$$

- We have the level crossing at some θ.
- And also, the edge $\mathbb{Z}_2$ charge flips after a period of cycle.
- So it is reasonable to call this the $\mathbb{Z}_2$ charge pump.

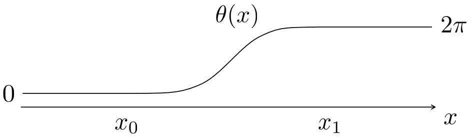


Texture induced $\mathbb{Z}_2$ charge

- Another feature of nontrivial adiabatic cycle is the texture induced $\mathbb{Z}_2$ charge.
- We modify the Hamiltonian, which is the sum of local terms B_j^θ, so that the local terms B_j^θ vary in the space as

$$H_{\text{texture}} = -\sum_j B_j^{\theta(j)},$$

with $\theta(x)$ a real function which varies from 0 to 2π in an interval.



- Interestingly, for our model, we can prove a texture indeed has the $\mathbb{Z}_2$ charge.

- Recall that our model is made with the local unitary

$$U_\theta = \prod_j e^{\frac{i\theta}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}}.$$

- We may try to introduce a kind of twist operator

$$U[\theta] = \prod_j e^{\frac{i\theta(j)}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}}$$

  to make the texture Hamiltonian by

$$H_{\rm texture} = U[\theta]H_0U[\theta]^{-1}.$$

- By the design of U_θ, this construction preserves $\mathbb{Z}_2$ symmetry.
- However, this does not work for a *closed* chain, due the absence of the 2π-periodicity of local unitary operator $e^{\frac{i\theta}{2}\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}}$.

- Let's see the detail of this point for the closed chain where the $N+1$ and 1 sites are identified.
- Let $\theta(x)$ is a function with boundaries $\theta(1)=0$ and $\theta(N)=2\pi$. Accordingly, the twist like operator is given by

$$U[\theta]=\prod_{j=1}^{N} e^{\frac{i\theta(j)}{2}\,\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}}.$$

- The local terms $B_j^{\rm tx}=U[\theta]\sigma_j^x U[\theta]^{-1}$ are found to be not smooth at $j=1$.

$$\begin{aligned}B_j^{\rm tx}=&\cos\frac{\theta(j-1)}{2}\cos\frac{\theta(j)}{2}\sigma_j^x-\sin\frac{\theta(j-1)}{2}\sin\frac{\theta(j)}{2}\sigma_{j-1}^z\sigma_j^x\sigma_{j+1}^z\\&+\sin\frac{\theta(j-1)}{2}\cos\frac{\theta(j)}{2}\sigma_{j-1}^z\sigma_j^y+\cos\frac{\theta(j-1)}{2}\sin\frac{\theta(j)}{2}\sigma_j^y\sigma_{j+1}^z\end{aligned}$$

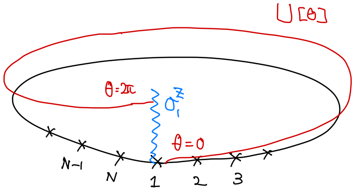
for $j=2,\dots,N$ and

$$\begin{aligned}B_1^{\rm tx}=&\cos\frac{\theta(N)}{2}\cos\frac{\theta(1)}{2}\sigma_1^x-\sin\frac{\theta(N)}{2}\sin\frac{\theta(1)}{2}\sigma_N^z\sigma_1^x\sigma_2^z\\&+\sin\frac{\theta(N)}{2}\cos\frac{\theta(1)}{2}\sigma_N^z\sigma_1^y+\cos\frac{\theta(N)}{2}\sin\frac{\theta(1)}{2}\sigma_1^y\sigma_2^z.\end{aligned}$$

- To make the texture Hamiltonian smooth, we have to insert the $\mathbb{Z}_2$ charged operator σ_1^z at site 1.
- The "true" twist operator is

$$U_{\rm twist} = \sigma_1^z U[\theta],$$

and the smooth texture Hamiltonian is $H_{\rm texture} = U_{\rm twist} H_0 U_{\rm twist}^{-1}$.



- Since $U[\theta]$ preserves the $\mathbb{Z}_2$ charge $V_{\mathbb{Z}_2} U[\theta] V_{\mathbb{Z}_2}^{-1} = U[\theta]$, we find that the twist operator $U_{\rm twist}$ has nontrivial $\mathbb{Z}_2$ charge from the inserted charged operator

$$V_{\mathbb{Z}_2} U_{\rm twist} V_{\mathbb{Z}_2}^{-1} = -U_{\rm twist}.$$

- The ground state of the texture Hamiltonian $H_{\rm texture}$ is given by $|\Psi_{\rm texture}\rangle = U_{\rm twist} |\Psi_0\rangle$, we conclude that a texture of θ has the $\mathbb{Z}_2$ charge.

Short summary and the rest plan of this talk

- We can define a $\mathbb{Z}_2$ invariant from the two-cocycle of the edge symmetry action.

  $\rightarrow$ Given a $(d+1)$-cocycle which is 2π-periodic, we can define a set of invariants taking values in $H^{d+1}(G,\mathbb{Z})$.

- Using the local unitary U_θ which is G-symmetric but is not 2π-periodic on the boundary, one can make the twist operator to introduce a spatial texture that varies from 0 to 2π.

  $\rightarrow$ The Bockstein homomorphism $H^d(G,U(1)) \rightarrow H^{d+1}(G,\mathbb{Z})$ gives us a local unitary U_θ for the adiabatic cycle in $(d+1)$D for the Chen-Gu-Liu-Wen construction, which is known constrcution by Roy–Harper (17). By using this exactly solvable model, we can show that a texture traps the SPT phase of dimension one lower.

Topological invariants of adiabatic cycles

- Given a dD non-chiral unique gapped ground state $|\Psi_\theta\rangle$, one in principle may extract the $(d+1)$-cocyele $\omega_\theta \in Z^{d+1}(G, U(1))$, by, for example, the Else–Nayak approach.
- With ω_θ, one can introduce the set of $\mathbb{Z}$ invariants

$$n(g_1, \dots, g_{d+1}) = \frac{1}{2\pi i} \oint d\omega_\theta(g_1, \dots, g_{d+1}) \in \mathbb{Z}.$$

- A part of $\mathbb{Z}$ invariants is meaningless, since ambiguity of the $(d+1)$-cocycle from the $(d+1)$-coboundary also gives the set of $\mathbb{Z}$ invaraints. Let $\alpha_\theta \in C^d(G, U(1))$ be a d-cochian. The trivialized $\mathbb{Z}$ invariants are given by the differential dm of the windings of α_θs

$$m(g_1, \dots, g_d) = \frac{1}{2\pi i} \oint d\alpha_\theta(g_1, \dots, g_d) \in \mathbb{Z}.$$

- In other words, the topological invaraint of adiabatic cycle takes a value in the group cohomology with $\mathbb{Z}$ coefficient

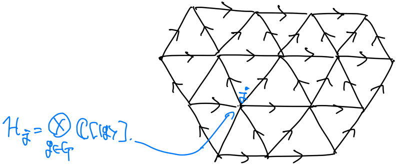
$$H^{d+1}(G, \mathbb{Z}) = Z^{d+1}(G, \mathbb{Z}) / B^{d+1}(G, \mathbb{Z}).$$

Bockstein homomorphism

- The Bockstein homomorphism
$$H^d(G, U(1)) \to H^{d+1}(G, \mathbb{Z})$$
gives us a concrete way for an exactly solvable lattice model by Chen-Gu-Liu-Wen construction, as explained below.
- Given a homogeneous d-cocyle
$$\nu(g_0, \dots, g_d) = e^{i\phi_\nu(g_0, \dots, g_d)} \in Z^d(G, U(1)),$$
which we want to pump, we introduce a lift
$$\mathbb{R}/2\pi\mathbb{Z} \ni \phi_\nu(g_0, \dots, g_d) \to \tilde{\phi}_\nu(g_0, \dots, g_d) \in \mathbb{R}.$$
- From the cocycle condition of ν, the differential $\frac{1}{2\pi} d\tilde{\phi}_\nu$ is a $(d+1)$-cocycle of $\mathbb{Z}$ coefficient
$$\frac{1}{2\pi}(d\tilde{\phi}_\nu)(g_0, \dots, g_{d+1}) \in Z^{d+1}(G, \mathbb{Z}).$$
- We introduce a 2π-periodic $(d+1)$-cocycle by
$$\nu_\theta^{(d+1)} := e^{\frac{i\theta}{2\pi}(d\tilde{\phi}_\nu)(g_0, \dots, g_{d+1})}.$$
- We apply the Chen-Gu-Liu-Wen construction to $\nu_\theta^{(d+1)}$.

Chen-Gu-Liu-Wen construction

- Let X_d be a space manifold with a triangulation and a branching structure.



- We introduce the local Hilbert space spanned by the group elements $|g \in G\rangle$ equipped with the G action $\hat{g}\,|h\rangle = |gh\rangle$ on each site.
- In this Hilbert space, we define the local unitary

$$\tilde{U}_\theta = \sum_{\{g_j\}} \prod_{\Delta^d} e^{\frac{i\theta}{2\pi} s(\Delta^d)(d\tilde{\phi}_\nu)(g_*, g_0, \dots, g_d)} \left|\{g_j\}\right\rangle \left\langle\{g_j\}\right|,$$

where the product $\prod_{\Delta^d}$ runs over all the d-simplices, $s(\Delta^d) \in \pm 1$ represents the orientation of Δ^d, and $g_* \in G$ is a reference group element.

Group cohomology construction of adiabatic cycles Roy–Harper

$$\tilde{U}_\theta = \sum_{\{g_j\}} \prod_{\Delta^d} e^{\frac{i\theta}{2\pi} s(\Delta^d)(d\tilde{\phi}_\nu)(g_*,g_0\dots,g_d)} \left|\{g_j\}\right\rangle \left\langle\{g_j\}\right|,$$

- By design, $\tilde{U}_\theta$ is 2π-periodic even in the presence of boundary.
- However, $\tilde{U}_\theta$ breaks G symmetry on the boundary of X_d, as well as local unitaries of static SPTs.
- Instead, we employ an alternative local form. The $\mathbb{R}$-valued $(d+1)$-cocycle $d\tilde{\phi}_\nu$ can be written as

$$d\tilde{\phi}_\nu(g_*, g_0, \dots, g_d) = \tilde{\phi}_\nu(g_0, g_1, \dots, g_d) - (d\alpha)(g_0, \dots, g_d)$$

  with $\alpha(g_0, \dots, g_{d-1}) = \tilde{\phi}_\nu(g_*, g_0, \dots, g_{d-1})$ a $(d-1)$-cocyle.
- The coboundary term $d\alpha$ is canceled out with adjacent d-simplices. Therefore, the local unitary

$$U_\theta = \sum_{\{g_j\}} \prod_{\Delta^d} e^{\frac{i\theta}{2\pi} s(\Delta^d)\tilde{\phi}_\nu(g_0,\dots,g_d)} \left|\{g_j\}\right\rangle \left\langle\{g_j\}\right|$$

  gives the same action for the bulk dofs as $\tilde{U}_\theta$.
- U_θ is exactly the same local unitary by Roy–Harper 17.

$$U_\theta = \sum_{\{g_j\}} \prod_{\Delta^d} e^{\frac{i\theta}{2\pi} s(\Delta^d) \tilde{\phi}_\nu(g_0,\dots,g_d)} \left|\{g_j\}\right\rangle \left\langle \{g_j\}\right|$$

- It turns out that U_θ breaks the 2π-periodicity only on the boundary, and the remaining local unitary on the boundary is that for $(d-1)$D SPT phase of $\nu \in Z^d(G, U(1))$

$$U_{2\pi} = U_{\text{bdy}}(\nu) = \sum_{\{g_{n\in\partial X_d}\}} \prod_{\Delta^{d-1}\in\partial X_d} \nu(g_*, g_0, \dots, g_{d-1})^{s(\Delta^{d-1})} \left|\{g_n\}\right\rangle \left\langle \{g_n\}\right|.$$

- In this sense, the local unitary U_θ pumps the $(d-1)$D SPT phase on the boundary. Potter–Morimoto, Roy–Harper
- U_θ is G symmetric even in the presence of boundary

$$\hat{g} U_\theta \hat{g}^{-1} = U_\theta.$$

- Moreover, for an arbitrary function $\theta : \{\Delta^d\} \to \mathbb{R}$, the space-dependent local unitary

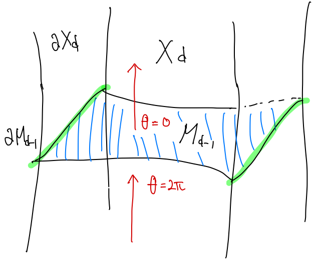
$$U[\theta] = \sum_{\{g_j\}} \prod_{\Delta^d} e^{\frac{i\theta(\Delta^d)}{2\pi} s(\Delta^d) \tilde{\phi}_\nu(g_0,\dots,g_d)} \left|\{g_j\}\right\rangle \left\langle \{g_j\}\right|$$

is G-symmetric.

Texture induced SPT phase

- We also can construct an exactly solvable model of the texture Hamiltonian.
- As for 1D cases, we first introduce a function $\theta : \{\Delta^d\} \to [0, 2\pi]$ which can have jumps $2\pi \to 0$ somewhere, and Let M_{d-1} be the codimension 1 surface on which θ jumps from 2π to 0.
- Introduce the twist operator of the form

$$U_{\text{twist}} = U(M_{d-1})^{-1} U[\theta],$$



$$U[\theta] = \sum_{\{g_{j\in X_d}\}} \prod_{\Delta^d \in X_d} e^{\frac{i\theta(\Delta^d)}{2\pi} s(\Delta^d) \tilde{\phi}_\nu(g_0,\dots,g_d)} \left|\{g_j\}\right\rangle \left\langle\{g_j\}\right|,$$

$$U(M_{d-1}) = \sum_{\{g_{n\in M_{d-1}}\}} \prod_{\Delta^{d-1} \in M_{d-1}} \nu(g_*, g_0, \dots, g_{d-1})^{s(\Delta^{d-1})} \left|\{g_n\}\right\rangle \left\langle\{g_n\}\right|.$$

- We can show that the twist Hamiltonian defined by $U_{\mathrm{twist}} H_0 U_{\mathrm{twist}}^{-1}$ traps the SPT phase one dimension lower, by explicitly computing how G symmetry acts on the ground state manifold of the system with boundary.
- The ground state manifold $|\Psi(\{g_{n\in\partial X_d}\})\rangle$ is explicitly written as

$$
\begin{aligned}
&|\Psi(\{g_{n\in\partial X_d}\})\rangle \\
&= \sum_{\{g_{j\in \mathring{X}_d}\}} \prod_{\Delta^{d-1}\in M_{d-1}} \nu(g_*, g_0, \ldots, g_{d-1})^{-s(\Delta^{d-1})} \prod_{\Delta^d \in X_d} e^{\frac{i\theta(\Delta^d)}{2\pi} s(\Delta^d)\tilde{\phi}_\nu(g_0,\ldots,g_d)} \left|\{g_j\},\{g_n\}\right\rangle ,
\end{aligned}
$$

- from which we can explicitly compute how the G symmetry acts on the boundary states $|\Psi(\{g_{n\in\partial X_d}\})\rangle$. In doing so, it turns out that the nontrivial G action is only on the boundary ∂M_{d-1} of M_{d-1}. We have the following form

$$\hat{g}\,|\Psi(\{g_{n\in\partial X_d}\})\rangle = \mathcal{N}_{\partial M_{d-1}}(g)\mathcal{S}_{\partial X_d}(g)\,|\Psi(\{g_n\}\rangle \tag{1}$$

where $\mathcal{N}_{\partial M_{d-1}}$ and $\mathcal{S}_{\partial X_d}$ are local unitaries acting on ∂M_{d-1} and ∂X_d, respectively, as in

$$\mathcal{S}_{\partial X_d}(g)\,|\Psi(\{g_n\}\rangle = |\Psi(\{gg_n\}\rangle\,, \tag{2}$$

$$\mathcal{N}_{\partial M_{d-1}}(g)\,|\Psi(\{g_n\}\rangle = \prod_{\Delta^{d-2}\in\partial M_{d-1}} \nu(g_*, gg_*, g_0, \ldots, g_{d-2})^{|\Delta^{d-2}|}\,|\Psi(\{g_n\}\rangle\,. \tag{3}$$

- The local unitary $\mathcal{N}_{\partial M_{d-1}}(g)\mathcal{S}_{\partial X_d}(g)$ (restricted to ∂M_{d-1}) is known as an anomalous symmetry action of the boundary of $(d-1)$D SPT phase with $\nu \in Z^d(G, U(1)_s)$. (For example, see Else–Nayak)
- Thus, we conclude that the texture Hamiltonian H_{texture} traps the $(d-1)$D SPT phase on the codimension 1 surface M_{d-1}.

Summary

- For adiabatic cycles of quantum spin systems, the topological invariant is the $U(1)$ phase winding numbers

$$n = \frac{1}{2\pi i}\oint d\log\omega_\theta$$

  of the 2π-periodic $(d+1)$-cocycle $\omega_\theta \in Z^{d+1}(G, U(1))$. The equivalence class $[n]$ takes a value in the group cohomology $H^{d+1}(G, \mathbb{Z})$.

- By tracing the Bockstein homomorphism $H^d(G, U(1)) \cong H^{d+1}(G, \mathbb{Z})$, we can construct a local unitary of adiabatic cycles, which is the same one by Roy–Harper.

- With the group cohomology model, we have checked the desired properties of the adiabatic cycles: we showed that the local unitary pumps the SPT phase on the boundary Roy–Harper, and the texture Hamiltonian traps the SPT phase in one dimension lower.

Matrix product state

- The Matrix Product State (MPS) is quite useful tool to describe unique gapped ground states in spin chains.
- For simplicity, we assume translational symmetry.
- A matrix product state defined by a collection of matrices $\{A_m\}$ is written as

$$|\psi(\{A_m\})\rangle = \sum_{\{m_j\}} \mathrm{Tr}[\cdots A_{m_j} A_{m_{j+1}} \cdots] \, |\cdots m_j m_{j+1} \cdots\rangle \, .$$

Here, $A_m = [A_m]_{\alpha\beta}$ are $D \times D$ matrices with $m = 1, \ldots, \mathrm{dim}\mathcal{H}_j$ the indices of the local Hilbert space $\mathcal{H}_j$, and $\alpha\beta$ stands for the bond Hilbert space. D measures the entanglement between two sites.

Injective MPS and uniqueness

- Unique gapped ground states are described by injective MPSs. (Short-range correlation, and no cat states.)
- The only property of injective MPS we use is the following.

Lemma

Two injective MPSs $|\Psi(\{A_m\})\rangle$ and $|\Psi(\{\tilde{A}_m\})\rangle$ represent the same state iff there exists $e^{i\chi} \in U(1)$ and $W \in U(D)$ such that

$$\tilde{A}_m = e^{i\chi} W^\dagger A_m W.$$

Here, $e^{i\chi}$ is unique and W is unique up to a $U(1)$ phase.

- This is a kind of gauge choice of an MPS. Physical consequences should be independent of this gauge choice.

The Rice-Mele model

- A nontrivial Thouless pump is good illustrated by the Rice-Mele model.
- Free fermion model with nearest neighbor hopping with the staggered amplitude $t+\delta$ and $t-\delta$, and the staggered potential Δ.

$$H = \sum_j \left(\frac{t}{2} + (-1)^j \frac{\delta}{2} \right) (a_j^\dagger a_j + h.c.) + \Delta (-1)^j a_j^\dagger a_j .$$