Higher Berry structure of matrix product states

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with
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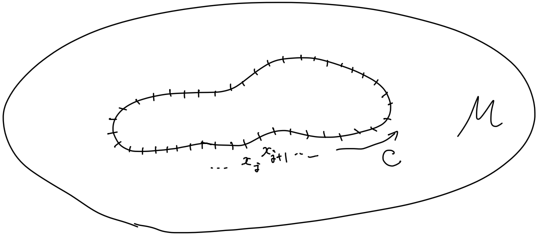
Ref: arXiv:2305.08109.

Related papers:
- Adam Artymowicz, Anton Kapustin, Nikita Sopenko, arXiv:2305.06399.
- Marvin Qi, David T. Stephen, Xueda Wen, Daniel Spiegel, Markus J. Pflaum, Agnés Beaudry, Michael Hermele, arXiv:2305.07700.

Berry phase Berry, 84

- The Berry phase is a $U(1)$-valued quantity defined from a one-parameter family of pure states in a 0-dim system, a quantum mechanical system.
- Consider a one-parameter family of normalized states $|\psi(x)\rangle \in \mathbb{C}^N, \langle\psi(x)|\psi(x)\rangle = 1$, a circle $x \in C \cong S^1$.
- We introduce a triangulation $|C| = \{[x_j, x_{j+1}]\}_{j=1}^n$ of C.
- Because two nearest-neighbor states $|\psi(x_j)\rangle$ and $|\psi(x_{j+1})\rangle$ are "close" to each other, implying $\langle\psi(x_j)|\psi(x_{j+1})\rangle \neq 0$, the $U(1)$ phase of the inner product $\langle\psi(x_j)|\psi(x_{j+1})\rangle$ is well-defined.



Cont.

- For $|C|$, one can define a $U(1)$-valued quantity

$$e^{i\gamma(|C|)} := \prod_{j=1}^{n} \frac{\langle \psi(x_j) | \psi(x_{j+1}) \rangle}{|\langle \psi(x_j) | \psi(x_{j+1}) \rangle|} \in U(1).$$

- $e^{i\gamma(|C|)}$ is manifestly invariant under gauge transformations $|\psi(x_j)\rangle \mapsto |\psi(x_j)\rangle\, e^{i\xi_j}$.
- The Berry phase is defined as the limit

$$e^{i\gamma(C)} = \lim_{|x_j - x_{j+1}| \to 0} e^{i\gamma(|C|)}.$$

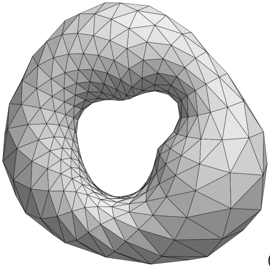
- In numerical calculations, the lattice formula $e^{i\gamma(|C|)}$ is usually sufficient.

Berry "flux"

- Similarly, there is a lattice definition of the Berry curvature (field strength).
- For a 2-simplex $\Delta^2 = (012)$ in the parameter space $\mathcal{M}$, we define the "Berry flux" $e^{iF(\Delta^2)}$ as the Berry phase of the boundary of Δ^2.

$$e^{iF(\Delta^2)} := e^{i\gamma(\partial\Delta^2)}.$$

- If the 2-simplex Δ^2 is small enough, the flux piercing Δ^2 is small $e^{iF(\Delta^2)} \sim 1$.
- Thus, $F(\Delta^2)$ can be considered as a $\mathbb{R}$-valued quantity $F(\Delta^2) \in \mathbb{R}$.



(figure from wikipedia)

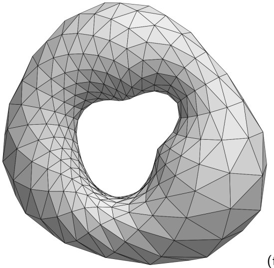
Chern number

- For a 2-submanifold $\Sigma \subset \mathcal{M}$ and its fixed triangulation $|\Sigma|$, a $\mathbb{Z}$-valued quantity (the first Chern number in the limit of $\mathrm{Area}(\Delta^2) \to 0$), is defined.

$$\nu(|\Sigma|) = \frac{1}{2\pi} \sum_{\Delta^2 \in |\Sigma|} F(\Delta^2) \in \mathbb{Z}.$$

- The quantization $\nu(|\Sigma|) \in \mathbb{Z}$ is manifest:

$$e^{2\pi i \nu(|\Sigma|)} = \prod_{\Delta^2 \in |\Sigma|} e^{i\gamma(\partial\Delta^2)} \equiv 1.$$



(figure from wikipedia)

To 1-dim invertible states in quantum spin systems

- We want to discuss generalizing the story above to invertible ($\cong$ short-range entangled $\cong$ unique gapped) states in 1-dim quantum spin systems.
- The essential difference is that a 1-dim system is inherently infinite-dimensional:

$$\mathcal{H}_{\text{total}} = \bigotimes_{x \in \mathbb{Z}} \mathcal{H}_x, \quad \mathcal{H}_x \cong \mathbb{C}^d.$$

- One can try the same definition of the Berry phage in 0-dim systems to 1-dim systems but with finite system size with length L.
- However, the inner product between two different invertible states decays exponentially

$$\langle \psi | \psi' \rangle \sim e^{-\alpha L}.$$

- So the $U(1)$ phase is of $\langle \psi | \psi' \rangle$ might be numerically ill-defined...
- This suggests we must reconstruct the "higher Berry phase" in a way suitable for 1-dim infinite systems.

Kapustin=Spodyneiko

- Kapustin and Spodyneiko proposed "higher Berry curvature" in d-dim systems.

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Higher-dimensional generalizations of Berry curvature

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- In the form of correlation functions of the local Hamiltonians h_x and its external derivatives dh_x, they constructed a $\Omega^{(d+2)}(\mathcal{M})$-valued quantity ($(d+2)$- differential form), a generalization of the Berry curvature in d-dim.
- I am not going into detail, but let me explain the relationship with the Ω-spectrum structure of the space of invertible states.

Ω-spectrum proposal [Kitaev, 11, 13, 15, ... in videos]

- A proposal on the topological structure behind the space of invertible states.
- Let F_d be the "space of d-dim invertible states"(which has not been rigorously defined yet).
- Proposal: The sequence of the spaces $\{F_d\}_{d\in\mathbb{Z}}$ forms an Ω-spectrum of the generalized cohomology theory. Namely, there is a homotopy equivalence

$$\Omega F_{d+1} \sim F_d,$$

where

$$\Omega F_{d+1} = \left\{ S^1 \to F_{d+1} \,\middle|\, |\Psi(0)\rangle = |\Psi(2\pi)\rangle = |1\rangle \right\}$$

is the based loop space of F_{d+1}. ($|1\rangle$ is a trivial tensor product state.)
- Physically, ΩF_{d+1} is the "space of adiabatic cycles in $(d+1)$-dim".
- Implication: The adiabatic cycles in $(d+1)$-dim are classified by the invertible phases in d-dim.

$$\pi_0(\Omega F_{d+1}) = \pi_0(F_d).$$

This is consistent with physics: Thouless pump, Floquet cycle,...
- For free fermions, $\pi_0(\Omega^k F_{d+k}) = \pi_0(F_d)$ is well-established [Teo-Kane '10].

What is the space of invertible states in 1-dim quantum spin systems?

puzzle

- For 0-dim,

$$E_0 \sim \mathbb{C}P^{N-1}(= \mathbb{C}^N/\mathbb{C}^\times) \hookrightarrow \mathbb{C}P^\infty = BU(1) \sim K(\mathbb{Z}, 2),$$

  the classifying space of $U(1)$. The embedding is given by

$$\psi = (\psi_1, \dots \psi_N) \mapsto (\psi_1, \dots, \psi_N, 0).$$

- If the Ω-spectrum structure is true,

$$E_0 \sim \Omega E_1 \quad \Rightarrow \quad E_1 \sim K(\mathbb{Z}, 3)$$

  is consistent.
- What is $K(\mathbb{Z}, 3)$?
  $\rightarrow$ the classifying space of infinite projective unitary group $BPU(\infty) \sim K(\mathbb{Z}, 3)$.
- Where is $PU(\infty)$-bundle structure in 1-dim invertible states?
- A family of matrix product states (MPSs) is a $PU(D)$-bundle.
  Haegeman=Mariën=Osborne=Verstraete 14
- MPS should be a good starting point.

Matrix product state (MPS) Vidal, Perez-Garcia=Verstraete=Wolf=Cirac

- Total Hilbert space:

$$\mathcal{H}^{(L)}_{\text{total}} = \bigotimes_{x=1}^{L} \mathcal{H}_x, \quad \mathcal{H}_x = \mathbb{C}^N.$$

- A pure state $|\psi\rangle \in \mathcal{H}^{(L)}_{\text{total}}$ is specified by the wave function

$$|\psi\rangle = \sum_{i_1,\dots,i_L} \psi(i_1,\dots,i_L) \, |i_1 \cdots i_L\rangle \, .$$

- The MPS representation is a way to write the wave function $\psi(i_1,\dots,i_L)$ in the following form:

$$\psi(i_1,\dots,i_L) = \operatorname{tr}[A^{[1]i_1} \cdots A^{[L]i_L}].$$

  Here, $\{A^{[x]i_x}\}_{x=1}^{N}$ is set of $D_j \times D_{j+1}$ matrices.
- An MPS representation is given by successive singular value decompositions from left to right.

Translational invariant MPS Perez-Garcia=Verstraete=Wolf=Cirac

- Let $\hat{T}_r$ be the translation operator

$$\hat{T}_r \left| i_1 \cdots i_L \right\rangle = \left| i_2 \cdots i_L i_1 \right\rangle .$$

- Fact. Translation invariant state $\hat{T}_r \left|\psi\right\rangle = \left|\psi\right\rangle$ has a translational invariant MPS representation

$$\left|\psi\right\rangle = \left|\{A^i\}_i\right\rangle_L := \sum_{i_1,\ldots,i_L} \mathrm{tr}\left[A^{i_1} \cdots A^{i_L}\right] \left|i_1 \cdots i_L\right\rangle .$$

- In the rest of this talk, I focus only on translational invariant states and MPSs.
- (D-MPS) A D-MPS is the set of $D \times D$ matrices

$$\{A^i\}_{i=1}^N, \quad A^i \in \mathrm{Mat}_{D\times D}(\mathbb{C}).$$

D is called the bond dimension.

Injective MPS Perez-Garcia=Verstraete=Wolf=Cirac

- D-MPS may be a cat state, a superposition of macroscopically different states.
- To remove such MPSs, we impose "injectivity" on MPS:
- (Transfer matrix) We introduce the transfer matrix $T \in \mathrm{End}(\mathrm{Mat}_{D\times D}(\mathbb{C}))$ by

$$T(X) := \sum_{i=1}^{N} A^i X A^{i\dagger}.$$

- (Injectivity) D-MPS is injective iff
  (i) The largest eigenvalue λ_0 of T in magnitude $|\lambda|$ is unique, and its eigenspace is non-degenerate.
  (ii) The corresponding eigenvector X is positive $X > 0$.
- T is completely positive $\to \lambda_0 > 0$ (positive real number).
- If D-MPS is injective, the correlation function decays exponentially.

$$\langle \hat{O}_x \hat{O}'_y \rangle - \langle \hat{O}_x \rangle \langle \hat{O}'_y \rangle \sim e^{-|x-y|/\xi}, \quad \xi = -1/\log(|\lambda_1|/|\lambda_0|),$$

where λ_1 is the second largest eigenvalue in magnitude.
$\to$ "invertible" state.

Canonical form Perez-Garcia=Verstraete=Wolf=Cirac

- (Canonical form) A D-MPS is in the canonical form when

$$\sum_{i=1}^{N} A^i A^{i\dagger} = 1_D \quad \text{and} \quad \sum_{i=1}^{N} A^{i\dagger} \Lambda^2 A^i = \Lambda^2$$

  hold, where Λ is a positive-definite diagonal matrix whose entries are the Schmidt eigenvalues and satisfy $\mathrm{tr}\,[\Lambda^2] = 1$.
- This is a normalization condition for MPSs.
- Remark. Injective D-MPS can be in the canonical form:

$$\sum_{i=1}^{N} A^i X A^{i\dagger} = \lambda_0 X \quad \Rightarrow \quad \sum_{i=1}^{N} \left(\frac{1}{\sqrt{\lambda_0}} X^{-1/2} A^i X^{1/2}\right)\left(\frac{1}{\sqrt{\lambda_0}} X^{-1/2} A^i X^{1/2}\right)^\dagger = 1_D,$$

$$\sum_{i=1}^{N} A^{i\dagger} Y_0 A^i = Y_0, Y_0 = U^\dagger \Lambda^2 U \quad \Rightarrow \quad \sum_{i=1}^{N} (U A^{i\dagger} U^\dagger) \Lambda^2 (U^\dagger A^i U)^\dagger = \Lambda^2.$$

- In the rest of my talk, I assume D-MPSs are injective and in the canonical form.

Gauge structure of MPS Perez-Garcia=Verstraete=Wolf=Cirac

- Theorem. (Fundamental theorem of MPS.) For two D-MPSs $\{A_0^i\}_{i=1}^N, \{A_1^i\}_{i=1}^N$ with the matrices of the Schmidt eigenvalues Λ_0 and Λ_1, if $^\exists L > cD^4$ $(c = O(1))$ and $^\exists e^{i\alpha} \in U(1)$ s.t. $|\{A_0^i\}_i\rangle_L = e^{i\alpha}\,|\{A_1^i\}_i\rangle_L$, then $^{\exists 1}e^{i\theta_{01}} \in U(1)$ and $^\exists V_{01} \in U(D)$ s.t.

$$A_0^i = e^{i\theta_{01}} V_{01} A_1^i V_{01}^\dagger, \quad i = 1, \dots N, \quad \text{and} \quad V_{01}\Lambda_1 = \Lambda_0 V_{01}.$$

  Here, V_{01} is unique up to $U(1)$ phases.
- The matrix V_{01} is unique as an element of the projective unitary group

$$PU(D) = U(D)/(V_{01} \sim e^{i\beta} V_{01}).$$

- Thus, a family of D-MPS over a parameter space $\mathcal{M}$ is a $U(1) \times PU(D)$ bundle over $\mathcal{M}$.
- Remark. $U(1)$ part is from 0-dim nature because we are considering translational invariant MPSs.

Charastristic class of $PU(D)$ bundle Grothendieck 66

- Let's consider a D-MPS bundle over $\mathcal{M}$.
- Let $\{U_\alpha\}_\alpha$ be a good cover of $\mathcal{M}$. [1]
- On two patch intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have transition functions $(e^{i\theta_{\alpha\beta}(x)}, V_{\alpha\beta}(x))$ by

$$A^i_\beta(x) = e^{i\theta_{\alpha\beta}(x)} V^\dagger_{\alpha\beta}(x) A^i_\alpha(x) V_{\alpha\beta}(x).$$

- Over $x \in U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, consider two deformations ($x \in U_{\alpha\beta\gamma}$ is omitted)

$$A^i_\gamma = e^{i\theta_{\alpha\gamma}} V^\dagger_{\alpha\gamma} A^i_\alpha V_{\alpha\gamma}, \qquad A^i_\gamma = e^{i\theta_{\beta\gamma}} V^\dagger_{\beta\gamma} A^i_\beta V_{\beta\gamma} = e^{i\theta_{\beta\gamma}} e^{i\theta_{\alpha\beta}} V^\dagger_{\beta\gamma} V^\dagger_{\alpha\beta} A^i_\alpha V_{\alpha\beta} V_{\beta\gamma},$$

and using the uniqueness of $V_{\alpha\gamma}$, we have the 1-cocycle condition for $\{V_{\alpha\beta}\}_{\alpha\beta}$:

$$V_{\alpha\beta}(x) V_{\beta\gamma}(x) = e^{i\phi_{\alpha\beta\gamma}(x)} V_{\alpha\gamma}(x), \quad {}^\exists e^{i\phi_{\alpha\beta\gamma}(x)} \in U(1), \quad x \in U_{\alpha\beta\gamma}.$$

- The formula Ohyama=Ryu, 2304.05356.:

$$e^{i\phi_{\alpha\beta\gamma}(x)} = \mathrm{tr}\,[\Lambda^2_\alpha(x) V_{\alpha\beta}(x) V_{\beta\gamma}(x) V_{\gamma\alpha}(x)].$$

[1] All sets and all non-empty intersections of finitely-many sets are contractible.

cont.

- There is a consistency condition on $e^{i\phi_{\alpha\beta\gamma}(x)}$. $V_{\alpha\beta}(V_{\beta\gamma}V_{\gamma\delta}) = (V_{\alpha\beta}V_{\beta\gamma})V_{\gamma\delta}$ gives us

$$e^{i\phi_{\alpha\beta\delta}(x)}e^{i\phi_{\beta\gamma\delta}(x)} = e^{i\phi_{\alpha\beta\gamma}(x)}e^{i\phi_{\alpha\gamma\delta}(x)}, \quad x \in U_{\alpha\beta\gamma\delta},$$

  meaning that $e^{i\phi_{\alpha\beta\gamma}(x)}$ is a 2-cocycle in $\check{Z}^2(\mathcal{M}, \underline{U(1)})$.

- The change of U(1) phase $V_{\alpha\beta}(x) \mapsto V_{\alpha\beta}(x)e^{i\xi_{\alpha\beta}(x)}$ leads the equivalence relation by the 2-coboundary $e^{i\phi} \sim e^{i\phi}\delta e^{i\xi}$. Thus, $[e^{i\phi_{\alpha\beta\gamma}(x)}] \in \check{H}^2(\mathcal{M}, \underline{U(1)})$.
- Take a lift $\mathbb{R}/2\pi\mathbb{Z} \ni \phi_{\alpha\beta\gamma}(x) \to \tilde{\phi}_{\alpha\beta\gamma}(x) \in \mathbb{R}$, we have a $\mathbb{Z}$-valued 3-cocycle

$$c = \frac{1}{2\pi}\delta\tilde{\phi} \in \check{Z}^3(\mathcal{M}, \mathbb{Z}),$$

  and its equivalence class (from choices of lifts) is the characteristic class (Dixmier-Douady class) of $PU(D)$-bundle

$$[c] \in \check{H}^3(\mathcal{M}, \mathbb{Z}).$$

cont.

- However, $[c]$ never represents a free part of $\check{H}^3(\mathcal{M},\mathbb{Z})$.
- This is because we can choose $V_{\alpha\beta}(x)$ to be a $SU(D)$ matrix by using the $U(1)$ phase freedom. In doing so, the 2-cocycle $e^{i\phi_{\alpha\beta\gamma}(x)}$ is quantized to a $\mathbb{Z}_D$-value

$$e^{i\phi_{\alpha\beta\gamma(x)}} \in \mathbb{Z}_D = \{e^{\frac{2\pi i p}{D}} | p = 0, \dots, D-1\},$$

resulting in

$$D \times [c] = 0 \in \check{H}^3(\mathcal{M},\mathbb{Z}).$$

- Therefore, D-MPS bundle can provide only the D-torsion part of $H^3(\mathcal{M},\mathbb{Z})$,

$$\{x \in H^3(\mathcal{M},\mathbb{Z}) | Dx = 0\}.$$

- cf. See Ohyama=Terashima=KS for an explicit construction of MPS when $\mathcal{M} = \mathbb{R}P^2 \times S^1$ and $\mathcal{M} = L(3,1) \times S^1$.

Is constant bond dimension D physically reasonable?

- No.
- The bond dimension D is just a parameter to represent a true gapped state $|\psi\rangle$ with an MPS $|\mathrm{MPS}\rangle$ with an error $\langle\psi|\mathrm{MPS}\rangle = 1-\epsilon$.
- Even in the class of MPSs with finite bond dimensions (like AKLT state), it is easy to make a continuous path from different bond dimensions:

$$|\psi(t)\rangle = (1-t)\,|D\text{−MPS}\rangle + t\,|D'\text{−MPS}\rangle\,.$$

- It is natural to think of a family of MPS over $\mathcal{M}$ whose bond dimension D_x is also a function over $\mathcal{M}$.
- In this talk, the interest is a lattice formulation of the higher Berry phase. We introduce a triangulation $|\mathcal{M}|$ of $\mathcal{M}$.
- Input: a set of D_p-MPSs over the vertices p of $|\mathcal{M}|$.

What is the "measure" of the space of MPSs?

- How to estimate the distance between two MPSs?
- For the purpose of making the higher Berry phase, a measure may be given by the spectrum of the mixed transfer matrix.
- For physically different D_0-MPS $\{A_0^i\}_{i=1}^N$ and D_1-MPS $\{A_1^i\}_{i=1}^N$, we define the mixed transfer matrix $T_{01} \in \mathrm{End}(\mathrm{Mat}_{D_0 \times D_1}(\mathbb{C}))$ by

$$T_{01}(X) := \sum_{i=1}^N A_0^i X A_1^{i\dagger}.$$

- If two MPSs are physically "close" to each other, the spectrum of T_{01} should resemble the spectra of transfer matrices for each.

$$T_{00}(X) = \sum_{i=1}^N A_0^i X A_0^{i\dagger}, \quad T_{11}(X) = \sum_{i=1}^N A_1^i X A_1^{i\dagger}.$$

- In particular, the largest eigenvalue λ_{01} of T_{01} in magnitude $|\lambda|$ is unique, its eigenspace is non-degenerate, $|\lambda_{01}| \sim 1$, and there is a finite gap $|\lambda| < |\lambda_{01}| - |\delta\lambda|$ for $\lambda \neq \lambda_{01}$.

"Overlap matrix" V_{01}

- Let V_{01} be the eigenvector with $\lambda = \lambda_{01}$ of the mixed transfer matrix T_{01}, namely,

$$\sum_{i=1}^{N} A_0^i V_{01} A_1^{i\dagger} = \lambda_{01} V_{01}, \quad V_{01} \in \mathrm{Mat}_{D_0 \times D_1}(\mathbb{C}).$$

- We can fix $V_{10} = V_{01}^\dagger$.
- The matrix V_{01} plays the role of the inner product $\langle \psi(x_j) | \psi(x_{j+1}) \rangle$ in the discrete Berry phase formula in 0-dim.
- cf. V_{01} is nothing but the transition function when $\{A_0^i\}_i$ and $\{A_1^i\}_i$ represent the same physical state.
- There are two types of gauge transformation of V_{01}:
  (i) $V_{01} \to W_0 V_{01} W_1^\dagger$.
  This comes from the gauge transformations of MPSs $A_n^i \to e^{i\theta_n} W_n^\dagger A_n^i W_n$, $n = 0, 1$.
  (ii) $V_{01} \to z V_{01}$ with $z \in \mathbb{C}^\times$.
  This is because the eigenvalue equation $T_{01}(V_{01}) = \lambda_{01} V_{01}$ does not fix the overall $\mathbb{C}$ number of V_{01}.

2-cocycle and weighting by Schmidt eigenvalues

- Let's construct a gauge invariant quantity from the set of V_{01}s!
- Before we do so, we return to the cases of physically equivalent three MPSs. For physically equivalent three D-MPSs $\{A_0^i\}_i, \{A_1^i\}_i, \{A_2^i\}_i$ with the matrices of Schmidt eigenvalues $\Lambda_0, \Lambda_1, \Lambda_2$, the 2-cocycle $e^{i\phi_{012}}$ is given by

$$e^{i\phi_{012}} = \mathrm{tr}\,[\Lambda_0^2 V_{01} V_{12} V_{20}].$$

- This is not symmetric in the labels $0, 1, 2$. A more symmetric expression that is suitable for physically different MPSs is

$$e^{i\phi_{012}} = \mathrm{tr}\,[\Lambda_0^{\frac{2}{3}} V_{01} \Lambda_1^{\frac{2}{3}} V_{12} \Lambda_2^{\frac{2}{3}} V_{20}].$$

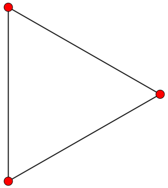
- We employ this formula for MPSs, which are physically different but close to each other.

Higher Berry connection

- For a 2-simplex $\Delta^2 = (012)$ of the discretized parameter space $|\mathcal{M}|$, we define a $U(1)$-valued quantity

$$e^{i\phi_{012}} := \frac{\mathrm{tr}\,[\Lambda_0^{\frac{2}{3}} V_{01} \Lambda_1^{\frac{2}{3}} V_{12} \Lambda_2^{\frac{2}{3}} V_{20}]}{\left|\mathrm{tr}\,[\Lambda_0^{\frac{2}{3}} V_{01} \Lambda_1^{\frac{2}{3}} V_{12} \Lambda_2^{\frac{2}{3}} V_{20}]\right|} \in U(1).$$

$\rightarrow$ invariant under the 1st gauge (i) $V_{01} \rightarrow W_0 V_{01} W_1^\dagger$.

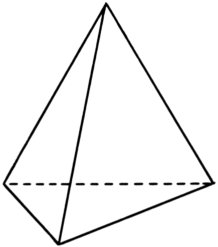


Higher Berry curvature (flux)

- For a 3-simplex $\Delta^3 = (0123)$ of $|\mathcal{M}|$, we define the "higher Berry flux"

$$e^{iF(\Delta^3)} := e^{i\phi_{123}} e^{-i\phi_{023}} e^{i\phi_{013}} e^{i\phi_{012}} \in U(1).$$

  $\rightarrow$ invariant under the 2nd gauge (ii) $V_{01} \rightarrow zV_{01}$ with $z \in \mathbb{C}^\times$.



- If the triangulation $|\mathcal{M}|$ is small enough then $e^{iF(\Delta^3)} \sim 1$ so that we can think $F(\Delta^3)$ as an $\mathbb{R}$-valued quantity.

Topological invariant (Dixmier-Douady class)

- The sum over all 3-simplexes of $|\mathcal{M}|$ is manifestly quantized

$$\nu(|X|) = \frac{1}{2\pi} \sum_{\Delta^3 \in |X|} F(\Delta^3) \in \mathbb{Z},$$

  since

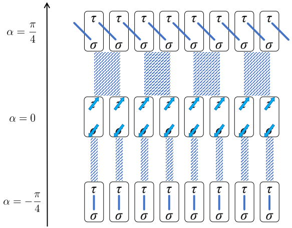
$$e^{2\pi i \nu(|X|)} = \prod_{\Delta^3 \in |X|} e^{2\pi i F(\Delta^3)} = \prod_{\Delta^3} e^{i\phi_{123}} e^{-i\phi_{023}} e^{i\phi_{013}} e^{i\phi_{012}} = 1.$$

- We expect that $\nu(|X|)$ characterizes the free part of the 3rd cohomology group $H^3(\mathcal{M}, \mathbb{Z})$, the topological nature of the parameter family of invertible 1-dim spin systems.

Model (Xueda Wen, et al. + perturbation)

- Two spin 1/2 dof $\boldsymbol{\sigma}_l$ and $\boldsymbol{\tau}_l$ for each site.

$$H_0(\alpha \in [-\frac{\pi}{4}, \frac{\pi}{4}], \boldsymbol{n} \in S^2)$$
$$= \sin(2\alpha) \sum_{l\in\mathbb{Z}} \begin{cases} -\boldsymbol{\sigma}_j \cdot \boldsymbol{\tau}_j & (\alpha \in [-\frac{\pi}{4}, 0]) \\ \boldsymbol{\tau}_l \cdot \boldsymbol{\sigma}_{l+1} & (\alpha \in [0, \frac{\pi}{4}]) \end{cases} + \cos(2\alpha) \sum_{l\in\mathbb{Z}} (-\boldsymbol{n}\cdot\boldsymbol{\sigma}_l + \boldsymbol{n}\cdot\boldsymbol{\tau}_l).$$



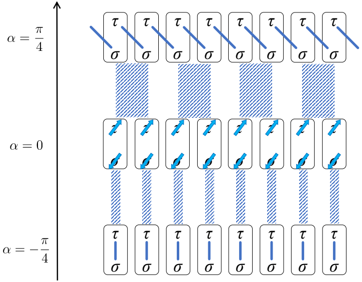
- This is an array of two spin problems, so easy to solve.
- At $\alpha = \pm\frac{\pi}{4}$, $\boldsymbol{n}$ disappears, implying that the parameter space is S^3.
- The bond dimension is $D = 1$ for $\alpha \in [-\frac{\pi}{4}]$ and $D = 2$ for $\alpha \in [0, \frac{\pi}{4}]$.
- This model shows a nontrivial value for the topological invariant introduced by Kapustin=Spodyneiko.

cont.

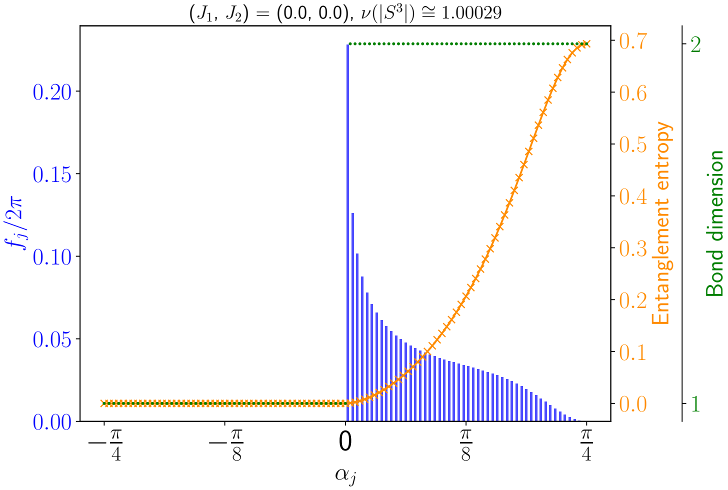
- We add NN and NNN Heisenberg terms to $H_0(\alpha, \boldsymbol{n})$.

$$H(\alpha, \boldsymbol{n}) = H_0(\alpha, \boldsymbol{n}) + J_1 \sum_{l \in \mathbb{Z}} (\boldsymbol{\sigma}_l \cdot \boldsymbol{\tau}_l + \boldsymbol{\tau}_l \cdot \boldsymbol{\sigma}_{l+1}) + J_2 \sum_{l \in \mathbb{Z}} (\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_{l+1} + \boldsymbol{\tau}_l \cdot \boldsymbol{\tau}_{l+1}).$$

- We numerically solve this model by DMRG with TeNPy (Tensor Network Python) package Hauschild=Pollmann.
- In this model, the higher Berry flux is symmetric for the S^2-direction.

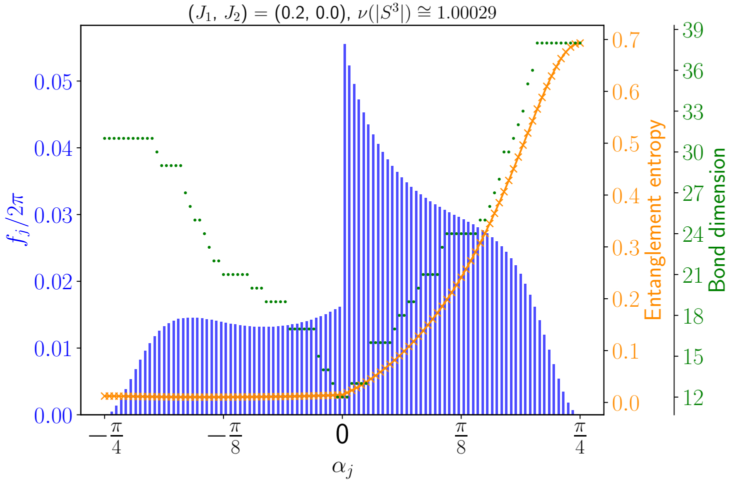


Numerical results



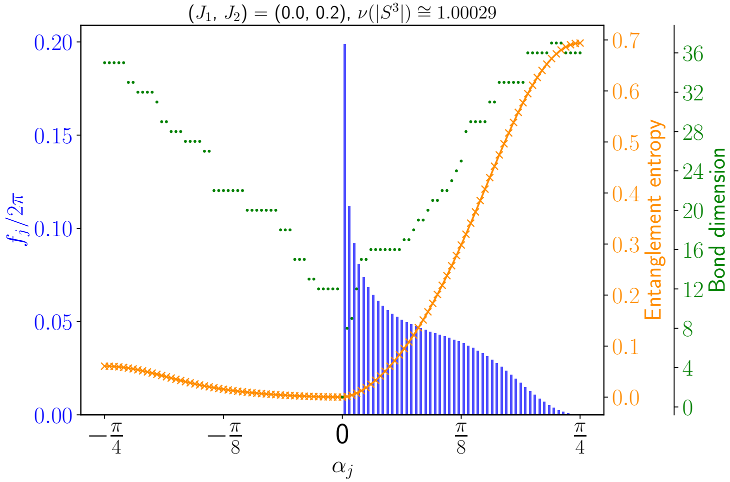
$$f_j = \sum_{S^2} F(\Delta^3), \quad \nu(|S^3|) = \frac{1}{2\pi} \sum_j f_j.$$

cont.



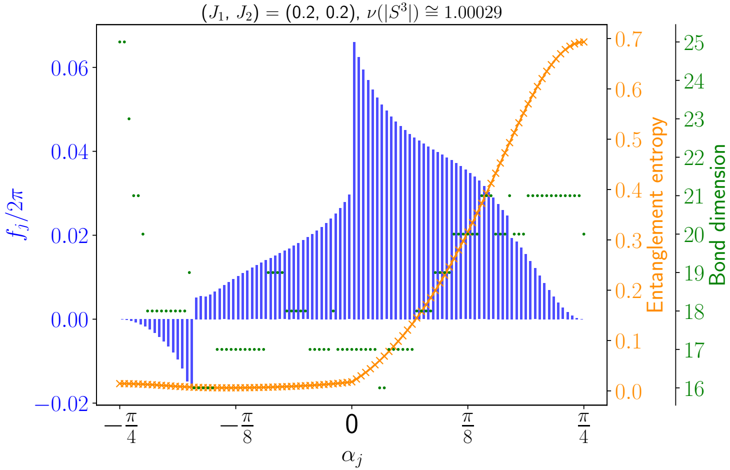
$$f_j = \sum_{S^2} F(\Delta^3), \quad \nu(|S^3|) = \frac{1}{2\pi} \sum_j f_j.$$

cont.



$$f_j = \sum_{S^2} F(\Delta^3), \quad \nu(|S^3|) = \frac{1}{2\pi} \sum_j f_j.$$

cont.



$$f_j = \sum_{S^2} F(\Delta^3), \quad \nu(|S^3|) = \frac{1}{2\pi} \sum_j f_j .$$

Summary

- We studied the geometric structure of the family of MPSs.
- We propose that the eigenvector V_{01} of the mixed transfer matrix T_{01} for physically different but close MPSs resembles the inner product $\langle\psi_0|\psi_1\rangle$ of two states in 0-dim. We constructed the higher Berry phase and the higher Berry curvature of MPSs.
- We demonstrated that by using DMRG the integrated higher Berry curvature over the 3-sphere shows the nontrivial topological invariant $\nu(|\mathcal{M}|) = 1$.

KS, Heinsdorf, Ohyama, 2305.08109.