Partial point group operation and Symmetry protected topological phases

Ken Shiozaki YITP, Kyoto university
with
Hassan Shapourian University of Chicago
→ MIT-Harvard joint posdoc
Shinsei Ryu University of Chicago

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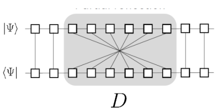
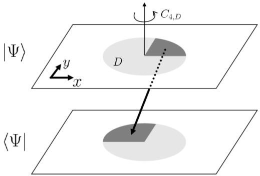
- Refs:
  KS, Hassan Shapourian, Shinsei Ryu, arXiv:1609.05970.
  Hassan Shapourian, KS, Shinsei Ryu, arXiv:1607.03896.
  KS, Shinsei Ryu, arXiv:1607.06504.

Outline

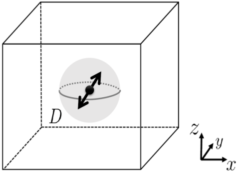
- Introduction
  - Rényi entropy
    — for the purpose of introducing the partial symmetry transformation
  - Partial point group transformation and symmetry protected phases
- Partial point group transformation
  - Partial rotation
  - Partial inversion



Partial reflection

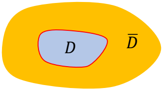


Partial rotation



Partial inversion

Rényi entropy



- Def:

$$S_{R,N} = \frac{1}{1-N} \ln \mathrm{tr}\,[\rho_D^N], \qquad \rho_D = \mathrm{tr}_{\bar{D}}[|\psi\rangle\langle\psi|],$$
$$S_R = \lim_{N\to 1} S_{R,N}.$$

- $\mathrm{tr}\,[\rho_D^N]$ can be written as the expectation value of the *partial* replica permutation operator T_D for the replica ground state $|\Psi\rangle = |\psi\rangle \otimes \cdots \otimes |\psi\rangle$,

$$\mathrm{tr}\,[\rho_D^N] = \langle \Psi | T_D | \Psi \rangle .$$

- For fermions,

$$T_D(f_1^\dagger(x), \ldots, f_N^\dagger(x))\,T_D^{-1} = \begin{cases} (f_1^\dagger(x), \ldots, f_N^\dagger(x))M_T & (x \in D), \\ (f_1^\dagger(x), \ldots, f_N^\dagger(x)) & (x \notin D), \end{cases}$$

$$M_T = \begin{pmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & \cdots & & \\ & & & 0 & -1 \\ 1 & & & & 0 \end{pmatrix}.$$

- Introduce the fermion basis $\tilde{f}_1^\dagger, \ldots \tilde{f}_N^\dagger$ diagonalizing M_T as

$$\tilde{f}_k = \frac{1}{\sqrt{N}}(f_1^\dagger + \omega_k f_2^\dagger + \omega_k^2 f_3^\dagger + \cdots + \omega_k^{N-1} f_N^\dagger),$$

$$\omega_k = e^{\frac{2\pi i(k-1/2)}{N}}, \qquad k = 1, \ldots, N,$$

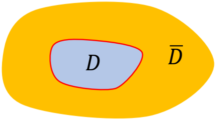
$$T_D \tilde{f}_k^\dagger(x) T_D^{-1} = \begin{cases} -\omega_k \tilde{f}_k^\dagger(x) & (x \in D), \\ \tilde{f}_k^\dagger(x) & (x \notin D). \end{cases}$$

- When $|\psi\rangle$ preserves the $U(1)$ symmetry, $\mathrm{tr}\,[\rho_D^N]$ is further recast as the product of the ground state expectation value of the partial $U(1)$ transformation as

$$\mathrm{tr}\,[\rho_D^N] = \prod_{\ell=-\frac{N-1}{2},-\frac{N-1}{2}+1,\ldots,\frac{N-1}{2}} \langle\psi| U_{\frac{2\pi\ell}{N}}\Big|_D |\psi\rangle,$$

  where $U_\theta|_D$ is the partial $U(1)$ transformation.
- Partial onsite transformation.

Bulk-boundary correspondence for gapped phases



- For a gapped ground state (= a short-range entangled (SRE) state) $|\psi\rangle$, the reduced density matrix $\rho_D = \mathrm{tr}_{\bar{D}}[|\psi\rangle\langle\psi|]$ is well-approximated by the *physical* boundary excitation H_{bdy} living on ∂D that emerges when we cut the system at D.

$$\rho_D \sim \frac{e^{-\beta H_{\mathrm{bdy}}}}{\mathrm{tr}\left[e^{-\beta H_{\mathrm{bdy}}}\right]}, \qquad \beta \sim \xi \sim \frac{1}{m}.$$

  Here, $\xi \sim \frac{1}{m}$ is the correlation length of the bulk.
- The partial $U(1)$ transformation $U_\theta|_D$ behaves as the $U(1)$ transformation $U_{\mathrm{bdy},\theta}$ for the boundary excitation H_{bdy}.
- Therefore, with the assumption of the bulk-boundary correspondence, the ground state expectation value $\langle\psi|U_\theta|_D|\psi\rangle$ of the partial $U(1)$ transformation is written as the expectation value of the $U(1)$ transformation for the boundary system.

$$\langle\psi|U_\theta|_D|\psi\rangle \sim \frac{\mathrm{tr}\left[U_{\mathrm{bdy},\theta}\, e^{-\beta H_{\mathrm{bdy}}}\right]}{\mathrm{tr}\left[e^{-\beta H_{\mathrm{bdy}}}\right]}.$$

Ex1: (2+1)D Chern insulator

- $D = D^2$: a 2D disc. $|\partial D| = 2\pi L$.
- Bulk:

$$H = \sum_k f_k^\dagger [k_x \sigma_x + k_y \sigma_y + (m - \epsilon k^2)\sigma_z] f_k.$$

- Boundary:

$$H_{\mathrm{bdy}} = \frac{2\pi}{L} \sum_{n \in \mathbb{Z}+\frac{1}{2}} n : \gamma_n^\dagger \gamma_n : - \frac{1}{24}.$$

- Partial $U(1)$ transformation

$$\langle \psi | U_\theta |_D | \psi \rangle \sim \frac{\mathrm{tr}\,[e^{-i\theta Q_{\mathrm{bdy}}} e^{-\xi H_{\mathrm{bdy}}}]}{\mathrm{tr}\,[e^{-\xi H_{\mathrm{bdy}}}]}, \qquad Q_{\mathrm{bdy}} = \sum_{n \in \mathbb{Z}+\frac{1}{2}} : \gamma_m^\dagger \gamma_m : .$$

- By using the S transformation, it is approximated by the vacuum contribution

$$\langle \psi | U_\theta |_D | \psi \rangle \sim \exp\left[-\frac{2\pi\xi}{L} \frac{1}{2} (\frac{\theta}{2\pi})^2 \right], \qquad -\pi < \theta < \pi.$$

- Rényi entropy

$$S_{R,N} = \frac{N+1}{24N} \times \frac{2\pi L}{\xi} \cdots \xrightarrow[N\to 1]{} S_R = \frac{1}{12} \times \frac{2\pi L}{\xi} + \cdots .$$

Ex2: (3+1)D topological insulator

- $D = D^3$: a 3D disc. $|\partial D| = S^2$ with the radius R.
- Bulk:

$$H = \sum_{\boldsymbol{k}} f_k^\dagger [\boldsymbol{k}\cdot\boldsymbol{\sigma}\tau_x + (m - \epsilon k^2)\tau_z] f_{\boldsymbol{k}}.$$

- Boundary:

$$H_{\rm bdy} = \frac{1}{R}\int d\Omega (\gamma_\uparrow^\dagger, \gamma_\downarrow^\dagger)\begin{pmatrix} 0 & -i\partial_\theta - \frac{1}{\sin\theta}\partial_\phi - \frac{i\cot\theta}{2} \\ -i\partial_\theta + \frac{1}{\sin\theta}\partial_\phi - \frac{i\cot\theta}{2} & 0 \end{pmatrix}\begin{pmatrix}\gamma_\uparrow \\ \gamma_\downarrow\end{pmatrix}$$

- Partial $U(1)$ transformation

$$\langle\psi|U_\theta|_D|\psi\rangle \sim \frac{{\rm tr}\,[e^{-i\theta Q_{\rm bdy}} e^{-\xi H_{\rm bdy}}]}{{\rm tr}\,[e^{-\xi H_{\rm bdy}}]}, \qquad Q_{\rm bdy} = \int d\Omega : \gamma_\uparrow^\dagger\gamma_\uparrow + \gamma_\downarrow^\dagger\gamma_\downarrow : .$$

- Since the bdy excitations is free, one can compute the partial $U(1)$ transformation analyticalally.

$$\langle\psi|U_\theta|_D|\psi\rangle = \exp\left[-\frac{R^2}{\xi^2}\left\{\frac{{\rm Li}_3(-e^{-i\theta}) + {\rm Li}_3(-e^{i\theta})}{2} + \frac{3}{4}\zeta(3)\right\} - \ln\left|\cos\frac{\theta}{2}\right| + \cdots\right]$$

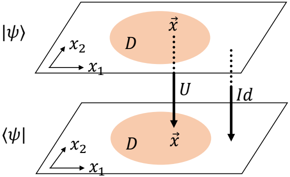
- Rényi entropy

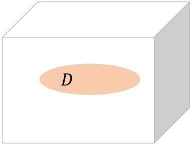
$$S_{R,N} = \frac{9\zeta(3)}{4}\frac{1+N+N^2}{3N^2}\frac{R^2}{\xi^2} - \frac{\ln 2}{3} + \cdots \xrightarrow[N\to 1]{} S_R = \frac{9\zeta(3)}{4}\frac{R^2}{\xi^2} - \frac{\ln 2}{3} + \cdots .$$

Partial onsite symmetry transformation

- Partial onsite symmetry transformation

$\langle\psi|U|_D|\psi\rangle =$





- Symmetry defect surface D with the boundary ∂D.
- There is no interpretation as a topological manifold with a background field. The boundary ∂D is a kind of a singularity.
- Said differently, the expectation value $\langle\psi|U|_D|\psi\rangle$ may depend on the "boundary condition" on the boundary ∂D.

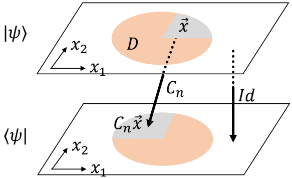
Partial point group transformation [KS-Shapourian-Ryu]

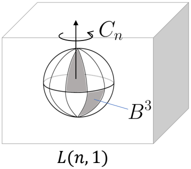
- Similarly, in the presence of point group symmetry G (reflection, rotation, inversion, ...), we may consider the partial point group transformations

$$\langle\psi|g|_D|\psi\rangle\,, \qquad g\in G.$$

- In particular, we fucus on a point group operation that freely acts on the space manifold except for the point group center.
- For instance, n-fold rotation symmetry in (2+1)D.

$$\langle\psi|C_n|_D|\psi\rangle =$$

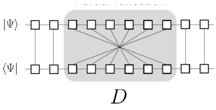




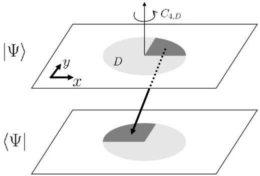
- There is the topological interpretation: the partial point group transformation makes a sort of a cross-cap in the spacetime manifold.
- For the partial n-fold rotation in (2+1)D, the resulting manifold is the (2+1)D manifold with a cross-cap to make the lens space $L(n,1)$.

Exs

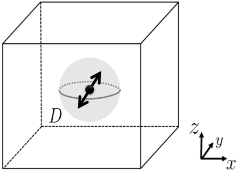
- The partial reflection $x \mapsto -x$ for (1+1)D
  $\Rightarrow$ the real projective plane RP^2.
- The partial n-fold rotation for (2+1)D
  $\Rightarrow$ the lens space $L(n, 1)$.
- The partial inversion $(x, y, z) \mapsto (-x, -y, -z)$ for (3+1)D
  $\Rightarrow$ the 4D real projective space RP^4.



Partial reflection



Partial rotation



Partial inversion

Our claim

(1) For point group symmetry g which acts on the d-dim. space manifold freely except for the point group center, the expectation value of the partial point group transformation $g|_D$ for a g-symmetric short-range entangled (SRE) state $|\psi\rangle$ takes a form as

$$\langle\psi|g|_D|\psi\rangle = \exp\left[i\theta + \gamma - \alpha\frac{|\partial D|}{\xi^{d-1}} + \cdots\right].$$

Here, θ, γ are scale-independent constants, α is a complex constant, and ξ is the correlation length of bulk.

(2) The scale-independent $U(1)$ phase $e^{i\theta}$ is indeed quantized. I.e. $e^{i\theta}$ does not change under the continuous deformation of $|\psi\rangle$ with keeping the short range correlation and the g symmetry.

- A comment:
  Since the partial point group transformation $g|_D$ is not a symmetry of the system, we have the loss of the amplitude proportional to the number of dof living in the boundary ∂D.

Where we were from

- We encountered the partial point group transformation as the *order parameter* $\langle\psi|\mathcal{O}_{\mathrm{SPT}}|\psi\rangle$ of symmetry protected topological (SPT) phases with point group symmetry. [KS-Shapourian-Ryu]
- Why?
- SPT phases are believed to be described by invertible TQFTs $(\mathrm{dim}\mathcal{H}_{M_d} = 1)$.
- A point group symmetry operation becomes onsite or orientation-reversing symmetry. (Ex: C_4 rotation $\to \mathbb{Z}_4$ onsite)
- For onsite symmetry, (the torsion part of) SPT phases are classified by (the torsion part of) the cobordism group. Precisely, an SPT phase can be viewed as a homomorphism

$$\Omega^{\mathrm{str}}_{d+1}(BG) \to U(1), \qquad Z : M \to Z(M).$$

- Therefore, an SPT phase is detected by the path-integral over the generator manifolds of the cobordism group $\Omega^{\mathrm{str}}_{D}(BG)$.
- In some cases, a generator manifold M_{gen} is given by a kind of cross-cap so that it can be "simulated" by the expectation value $\langle\psi|g|_D|\psi\rangle$ of the partial point group transformation of a point group operator g.

$$Z(M_{\mathrm{gen}}) \sim \langle\psi|g|_D|\psi\rangle \quad \text{for the } U(1) \text{ phase part.}$$

(Ex: $\Omega^{\mathrm{Pin}_+}_4(pt) = \mathbb{Z}_{16}$ generated by $RP^4 \to$ the partial inversion)
- Since $\mathrm{Hom}(\mathrm{Tor}\ \Omega^{\mathrm{str}}_{d+1}(BG), U(1))$ is a torsion, $e^{i\theta}$ should be quantized.

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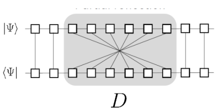
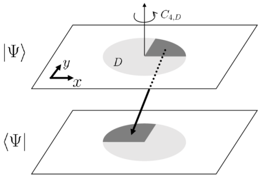
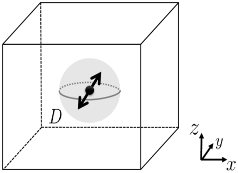


Partial reflection

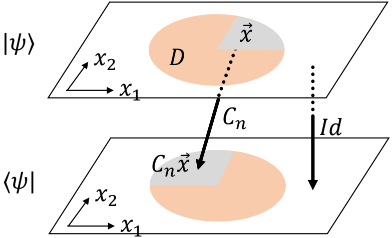


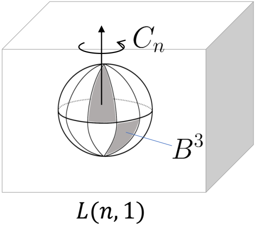
Partial rotation



Partial inversion

Partial rotations for (2+1)D



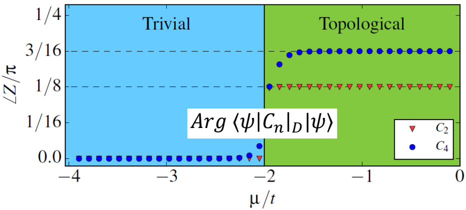


$$\langle\psi|C_n|_D|\psi\rangle = \exp\left[i\theta + \gamma - \alpha\frac{|\partial D|}{\xi} + \cdots\right].$$

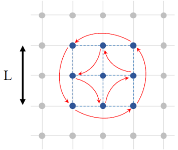
Ex: (2+1)D $(p_x - ip_y)$-superconductor

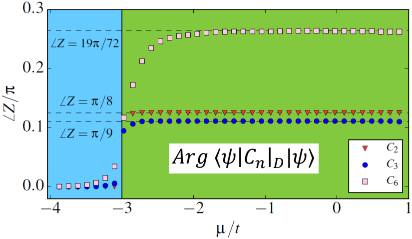
- Numerical calculation for a lattice model

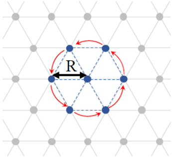
$$H = \underbrace{-t\sum_{\langle i,j\rangle}(f_i^\dagger f_j + h.c.)}_{\text{hopping}} \underbrace{-\mu\sum_i f_i^\dagger f_i}_{\text{chemical}} + \underbrace{\Delta\sum_{\langle i,j\rangle}(e^{-i\theta_{i,j}} f_i^\dagger f_j^\dagger + h.c.)}_{(p_x - ip_y)\text{-gap function}}.$$



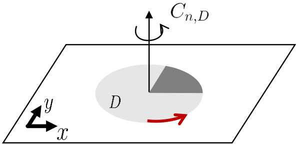
t: hopping
μ : chemical potential
$\Delta = t$







Edge CFT calculation (cf. [Tu-Zhang-Qi, 12] "momentum polarization")



- The bulk-boundary correspondence: the reduced density matrix over D of the gapped ground state $|\psi\rangle$ is approximated by an edge CFT.

$$\rho_D = \mathrm{tr}_{\bar{D}}(|\psi\rangle\langle\psi|) \sim \frac{e^{-\xi H_{\mathrm{edge}}}}{Z},$$

  where $\xi \sim \frac{1}{m} \ll |\partial D|$ is the correlation length of bulk.
- Then, the partial C_n rotation is same as the $\frac{2\pi}{n}$ translation on the edge CFT.

$$\langle\psi|C_n|_D|\psi\rangle \sim \frac{\mathrm{tr}\left[e^{-i:P:\frac{2\pi L}{n}} e^{-\xi H_{\mathrm{edge}}}\right]}{Z}$$

- This is a high temperature partition function.

- For right-moving (chiral) CFT, $P = H_{\mathrm{edge}}$.

$$\frac{\mathrm{tr}\,\left[e^{-i{:}P{:}\frac{2\pi L}{n}}e^{-\xi H_{\mathrm{edge}}}\right]}{Z} = \frac{e^{-\frac{2\pi i c}{24n}}}{Z}\sum_{a\in \mathrm{reps}}\chi_a(\frac{i\xi}{L}-\frac{1}{n}).$$

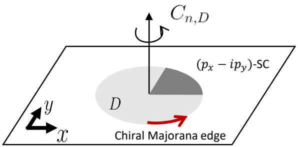
- Applying the $(ST^{-n}S)$ modular transformation, it can be written as a low-temperature partition function and is approximated by the vacuum.

$$\frac{i\xi}{L}-\frac{1}{n}\xrightarrow{S}-\frac{1}{\frac{i\xi}{L}-\frac{1}{n}}\xrightarrow{T^{-n}}\frac{\frac{in\xi}{L}}{\frac{i\xi}{L}-\frac{1}{n}}\xrightarrow{S}\frac{iL}{n^2\xi}+\frac{1}{n},$$

$$\begin{aligned}\mathrm{tr}\,\left[e^{-i{:}P{:}\frac{2\pi L}{n}}e^{-\xi H_{\mathrm{edge}}}\right] &= e^{-\frac{2\pi i c}{24n}}\sum_a\sum_b(ST^{-n}S)_{ab}\chi_b(\frac{iL^2}{n^2\xi}+\frac{1}{n}) \\ &\sim e^{-\frac{2\pi i c}{24n}}\sum_a\sum_b(ST^{-n}S)_{ab}e^{(\frac{2\pi i}{n}-\frac{2\pi L}{n^2\xi})(h_b-\frac{c}{24})}.\end{aligned}$$

- Note that $ST^{-n}S$ the modular transformation to make the lens space $L(n,1)$ in the surgery of two solid tori.

Ex: (2+1)D $(p_x - ip_y)$-superconductor



- Bulk Hamiltonian

$$H = \sum_{\boldsymbol{k}=(k_x,k_y)} \left[f_{\boldsymbol{k}}^\dagger \left(\frac{k^2}{2m} - \mu\right) f_{\boldsymbol{k}} + \frac{\Delta}{2}(k_x - ik_y) f_{\boldsymbol{k}}^\dagger f_{-\boldsymbol{k}}^\dagger + h.c. \right].$$

- Rotation symmetry

$$C_\theta f_{r,\phi}^\dagger C_\theta^{-1} = e^{-i\theta/2} f_{r,\phi+\theta}^\dagger, \qquad C_{2\pi} = (-1)^F.$$

- The edge Majorana excitation on ∂D is given by the Jackiw-Rebi trick.
- Instead of dealing with the open space D directly, we consider the spatially-varying mass $\mu(r)$ so that $\mu(r)$ represents the phase boundary between the topological ($\mu(r) < 0$ for $r < L$) phase and the trivial ($\mu(r) > 0$ for $r > L$) phase.

$$\gamma\left(\frac{L\phi}{2\pi}\right) \sim \left[e^{\frac{i\phi}{2} + \frac{\pi i}{4}} f_{r,\phi} + e^{-\frac{i\phi}{2} - \frac{\pi i}{4}} f_{r,\phi}^\dagger \right] e^{-\int^r \frac{\mu(r')}{\Delta} dr'},$$

$$\underbrace{\gamma(\ell)^\dagger = \gamma(\ell)}_{\text{real condition}}, \qquad \underbrace{\gamma(\ell + 2\pi L) = -\gamma(\ell)}_{\text{APBC (NS sector)}}.$$

- Plugging this into the bulk Hamiltonian, we have the edge Hamiltonian

$$H_{\mathrm{NS}} = \frac{2\pi\Delta}{L}\left(\sum_{n\in\mathbb{Z}+\frac{1}{2},n>0} n\gamma_{-n}\gamma_n - \frac{1}{48}\right).$$

- The CFT data:

$$c = \frac{1}{2}, \qquad (h_1, h_\psi, h_\sigma) = (0, \frac{1}{2}, \frac{1}{16}),$$

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \qquad T = e^{-\frac{\pi i}{24}}\begin{pmatrix} 1 & & \\ & -1 & \\ & & e^{\frac{\pi i}{8}} \end{pmatrix},$$

Virasolo rep. $= [1] \oplus [\psi]$ (NS sector).

- For the edge Majorana, the partial rotation is indeed the translation

$$C_\theta \gamma(\ell) C_\theta^{-1} = \gamma(\ell + \frac{\theta L}{2\pi})$$

- The partial C_n rotation is given as

$$\langle\psi|C_n|_D|\psi\rangle \sim \begin{cases} \exp\left[-\frac{(n^2+2)\pi i}{24n} - (1 - \frac{1}{n^2})\frac{1}{48}\frac{2\pi L}{\xi} + \cdots\right] & (n \text{ even}), \\ \exp\left[-\frac{(n^2-1)\pi i}{24n} - \ln\sqrt{2} - (1 + \frac{1}{n^2})\frac{1}{48}\frac{2\pi L}{\xi} + \cdots\right] & (n \text{ odd}). \end{cases}$$

- For some *n*s:

$$\langle\psi|C_2|_D|\psi\rangle \sim \exp\left[-\frac{\pi i}{8} - \frac{3}{4}\cdot\frac{1}{48}\cdot\frac{2\pi L}{\xi} + \cdots\right],$$

$$\langle\psi|C_3|_D|\psi\rangle \sim \exp\left[-\frac{\pi i}{9} - \ln\sqrt{2} - \frac{11}{9}\cdot\frac{1}{48}\cdot\frac{2\pi L}{\xi} + \cdots\right],$$

$$\langle\psi|C_4|_D|\psi\rangle \sim \exp\left[-\frac{3\pi i}{16} - \frac{15}{16}\cdot\frac{1}{48}\cdot\frac{2\pi L}{\xi} + \cdots\right],$$

$$\langle\psi|C_5|_D|\psi\rangle \sim \exp\left[-\frac{\pi i}{5} - \ln\sqrt{2} - \frac{27}{25}\cdot\frac{1}{48}\cdot\frac{2\pi L}{\xi} + \cdots\right],$$

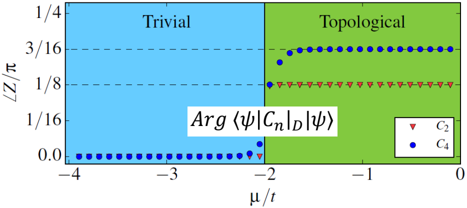
$$\langle\psi|C_6|_D|\psi\rangle \sim \exp\left[-\frac{19\pi i}{72} - \frac{35}{36}\cdot\frac{1}{48}\cdot\frac{2\pi L}{\xi} + \cdots\right].$$

- There $U(1)$ phases exactly match with the numerical calculation.

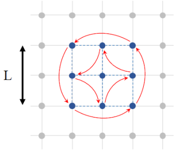
Ex: (2+1)D $(p_x - ip_y)$-superconductor

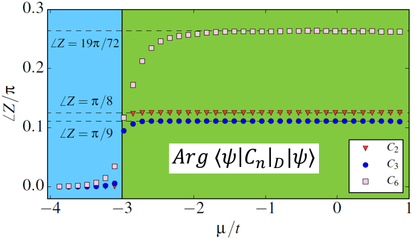
- Numerical calculation for a lattice model

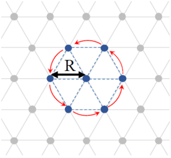
$$H = \underbrace{-t \sum_{\langle i,j \rangle} (f_i^\dagger f_j + h.c.)}_{\text{hopping}} \underbrace{- \mu \sum_i f_i^\dagger f_i}_{\text{chemical}} + \underbrace{\Delta \sum_{\langle i,j \rangle} (e^{-i\theta_{i,j}} f_i^\dagger f_j^\dagger + h.c.)}_{(p_x - ip_y)\text{-gap function}} .$$



t: hopping
μ : chemical potential
$\Delta = t$

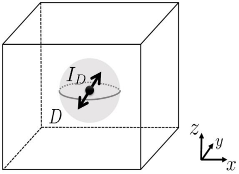






Partial inversion for (3+1)d

(3+1)D bulk

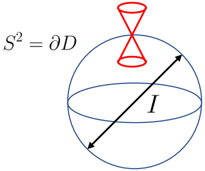


Partial inversion

$$\langle\psi|I|_D|\psi\rangle = |\langle\psi|I|_D|\psi\rangle|\, e^{i\theta}$$

$=$

(2+1)D surface



Antipodal map

$$\mathrm{tr}\left[\mathrm{I}\ \mathrm{e}^{-\xi\, H_{surf}}\right] = \mathrm{tr}\left[\mathrm{I}\ \mathrm{e}^{-\xi\, H_{surf}}\right] e^{i\theta}$$

$$\langle\psi|I|_D|\psi\rangle = \exp\left[i\theta + \gamma - \alpha\frac{|\partial D|}{\xi^2} + \cdots\right].$$

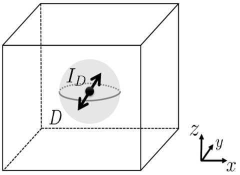
Ex: (3+1)D odd parity superconductors with inversion symmetry

- Inversion symmetry

$$I f_j^\dagger(\boldsymbol{x}) I^{-1} = f_i^\dagger(-\boldsymbol{x})\mathcal{I}_{ij}, \qquad I^2 = (-1)^F, \qquad (\boldsymbol{x} = (x,y,z)).$$

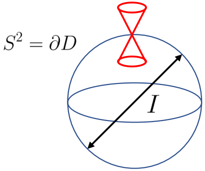
- The classification of SPT phases is given by $\mho^4_{\mathrm{pin}_+}(pt) = \mathrm{Hom}(\Omega_4^{\mathrm{pin}_+}(pt), U(1)) = \mathbb{Z}_{16}$. [Kitaev, Fidkowski-Chen-Vishwanath, You-Xu, Kapustin-Thorngren-Turzillo-Wang, ...]
- The generator manifold of $\Omega_4^{\mathrm{pin}_+}(pt)$ is the $4D$ real projective space RP^4.
- RP^4 is simulated by the ground state expectation value of the partial inversion defined by

$$I|_D f_j^\dagger(\boldsymbol{x})(I|_D)^{-1} = \begin{cases} f_i^\dagger(-\boldsymbol{x})\mathcal{I}_{ij} & (\boldsymbol{x} \in D), \\ f_j^\dagger(\boldsymbol{x}) & (\boldsymbol{x} \notin D). \end{cases}$$



Partial inversion

=



Antipodal map

A generator model

- A generator model of $\mho^4_{\mathrm{pin}_+}(pt) = \mathbb{Z}_{16}$ is given by the topological superconductor for the He-B phase.
- The bulk Hamiltonian is given by

$$H = \sum_{\boldsymbol{k}=(k_x,k_y,k_z)} \Psi^\dagger_{\boldsymbol{k}} \left[(\frac{k^2}{2m} - \mu)\tau_z + \Delta \tau_x \boldsymbol{k} \cdot \sigma \right] \Psi_{\boldsymbol{k}},$$

where $\Psi(\boldsymbol{k}) = (f_{\uparrow,\boldsymbol{k}}, f_{\downarrow,\boldsymbol{k}}, f^\dagger_{\downarrow,-\boldsymbol{k}}, -f^\dagger_{\uparrow,-\boldsymbol{k}})$ is the Nambu fermion.
- Inversion symmetry:

$$I f^\dagger_{\sigma,\boldsymbol{x}} I^{-1} = i f^\dagger_{\sigma,-\boldsymbol{x}}, \qquad I^2 = (-1)^F.$$

- The partial inversion on ∂B^3 = the antipodal map on $\partial B^3 = S^2$.
- The surface excitation $\gamma(\theta,\phi)$ is explicitly written by the bulk complex fermion as in

$$\gamma(\theta,\phi) \sim \Big[- e^{-i\phi/2} \cos\frac{\theta}{2} \{ f^\dagger_\uparrow(r,\theta,\phi) + i f_\downarrow(r,\theta,\phi) \} - e^{i\phi/2} \sin\frac{\theta}{2} \{ f^\dagger_\downarrow(r,\theta,\phi) - i f_\uparrow(r,\theta,\phi) \} \Big] e^{-\int^r \frac{\mu(r')}{\Delta} dr'},$$

where $\mu(r)$ represents the boundary at the radius $r = R$ between the topological $\mu < 0$ and trivial $(\mu > 0)$ regions.

- APBC for the ϕ-direction (like the Schwinger gauge)

$$\gamma(\theta, \phi + 2\pi) = -\gamma(\theta, \phi).$$

- The partial inversion $I_{\rm surf}$ acts on the surface fermions as

$$I_{\rm surf}\gamma^\dagger(\theta, \phi) I_{\rm surf}^{-1} = -i\gamma(\pi - \theta, \phi + \pi).$$

- Plugging $\gamma(\theta, \phi), \gamma^\dagger(\theta, \phi)$ into the bulk Hamiltonian, we have the surface Hamiltonian

$$H_{\rm surf} = \int \sin\theta d\theta d\phi (\gamma(\theta, \phi), -\gamma(\theta, \phi)) \mathcal{H} \begin{pmatrix} \gamma(\theta, \phi) \\ -\gamma^\dagger(\theta, \phi) \end{pmatrix},$$

$$\mathcal{H} = \frac{\Delta}{R} \begin{pmatrix} 0 & -i\partial_\theta - \frac{1}{\sin\theta}\partial_\phi - \frac{i\cot\theta}{2} \\ -i\partial_\theta + \frac{1}{\sin\theta}\partial_\phi - \frac{i\cot\theta}{2} & 0 \end{pmatrix}.$$

- We no longer have a simple algebraic way to implement the S transformation to approximate the partition function for (2+1)D CFTs. However, this is a free theory, everything is computable. (cf. [Cardy 91, *Operator content and modular properties of higher-dimensional conformal field theories*])

- By using the monopole harmonics, it is straightforward to diagonalize the surface Hamiltonian as in

$$H_{\rm surf} = \frac{\Delta}{R} \sum_{n\in\mathbb{N}} \sum_{m=-(n-1/2),-(n-1/2)+1,\dots,n-1/2} n\chi^\dagger_{n,m}\chi_{n,m},$$

and we show that the partial inversion acts on eigenstates as

$$I_{\rm surf}\chi^\dagger_{n,m}I^{-1}_{\rm surf} = i(-1)^n\chi^\dagger_{n,m}, \quad (n\in\mathbb{N}).$$

- We arrive at the analytic expression of the partial inversion, the expectation value of the antipodal map $I_{\rm surf}$.

$$\langle\psi|I|_D|\psi\rangle \sim \frac{\mathrm{tr}\left[I_{\rm surf}e^{-\frac{\xi}{\Delta}H_{\rm surf}}\right]}{\mathrm{tr}\left[e^{-\frac{\xi}{\Delta}H_{\rm surf}}\right]} = \frac{\prod_{n=1}^\infty(1+i(-q)^n)^{2n}}{\prod_{n=1}^\infty(1+q^n)^{2n}}, \qquad q=e^{-\xi/R}.$$

- Using the Cahen-Mellin integral ($\sim$ the S transformation [Cardy 91]), we have

$$\langle\psi|I|_D|\psi\rangle \sim \frac{\mathrm{tr}\left[I_{\rm surf}e^{-\frac{\xi}{\Delta}H_{\rm surf}}\right]}{\mathrm{tr}\left[e^{-\frac{\xi}{\Delta}H_{\rm surf}}\right]} = \exp\left[-\frac{\pi i}{8}+\frac{\ln 2}{12}-\frac{21}{16}\zeta(3)\frac{R^2}{\xi^2}+\cdots\right].$$

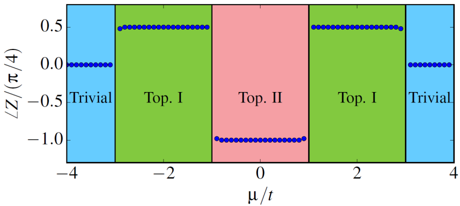
- This matches with the cobordism classification $\mho^4_{\mathrm{pin}_+}(pt)=\mathbb{Z}_{16}$.

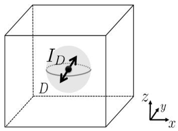
Numerical results

- The lattice model for the HE-B phase on the 3D cubic lattice.

$$H = -t \sum_{\langle i,j \rangle,\sigma} (f^\dagger_{\sigma,i} f_{\sigma,j} + h.c.) - \mu \sum_{\sigma,i} f^\dagger_{\sigma,i} f_{\sigma,j}$$
$$+ \Delta \sum_{\langle i,j \rangle,\sigma,\sigma'} \sum_{\mu=x,y,z} \left\{ (\sigma_\mu i \sigma_y)_{\sigma,\sigma'} f^\dagger_{\sigma,i} f^\dagger_{\sigma',j} + h.c. \right\}.$$

- The topological phase with a single surface Majorana fermion appears for $t < |\mu| < 3t$.
- The numerical result for $t = \Delta$, 12^3 sites for the total system, and 6^3 sites for the partial inversion.



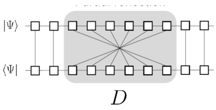


Summary [KS-Shapourian-Ryu, arXiv:1609.05970]

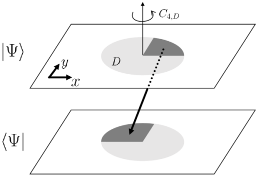
- The partial point group operation may have a topological meaning.
- The expectation value of the partial point group transformation $g|_D$ on a g-symmetric short-range entangled (SRE) state $|\psi\rangle$ takes a form as

$$\langle\psi|g|_D|\psi\rangle = \exp\left[i\theta + \gamma - \alpha\frac{|\partial D|}{\xi^{d-1}} + \cdots\right].$$

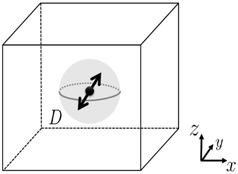
- The $U(1)$ phase $e^{i\theta}$ is quantized, and its value is determined by the SPT phases with point group symmetry to which the SRE state $|\psi\rangle$ belongs.
- cf. The anomaly indicator for the gapped topological ordered surface [Wang-Levin, Tachikawa-Yonekura, Barkeshli et al (16)].



Partial reflection



Partial rotation



Partial inversion