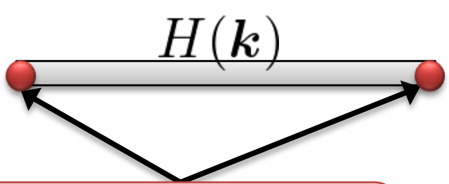
Topological insulators and superconductors

Bulk ・・・ insulator
Boundary ・・・ topologically protected gapless states
localized at boundary

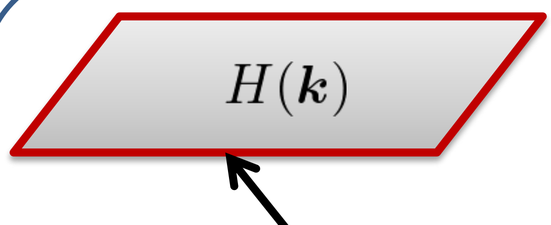
1-dimension



zero energy bound states

(Ex) Jackiw-Rebbi soliton,
SSH model,
spinless p+ip SC,
・・・

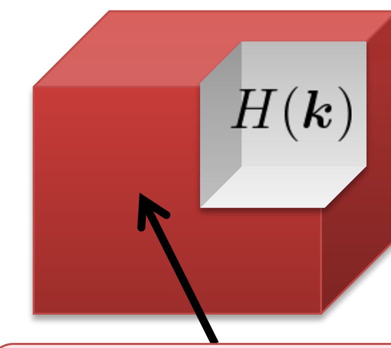
2-dimension



gapless edge states

(Ex) Quantum Hall effect,
Quantum spin Hall effect,
・・・

3-dimension



gapless surface states

(Ex) Z_2 topological insulator,
・・・

A toy model for the integer quantum Hall state

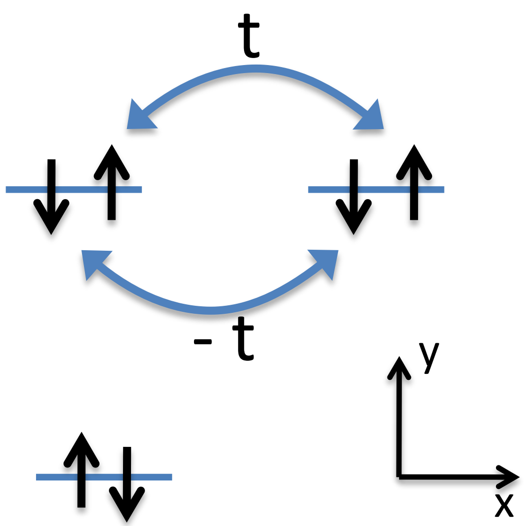
$$H = \sum_n \left[c_n^\dagger \frac{\sigma_3 - i\sigma_1}{2} c_{n+\hat{x}} + c_n^\dagger \frac{\sigma_3 - i\sigma_2}{2} c_{n+\hat{y}} + h.c. \right] + m \sum_n c_n^\dagger \sigma_3 c_n$$

diagonalize

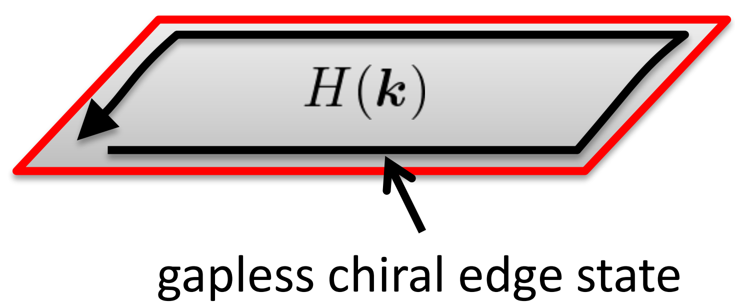
n : site

σ : spin

$$H = \sin k_x \sigma_1 + \sin k_y \sigma_2 + (m + \cos k_x + \cos k_y)\, \sigma_3$$



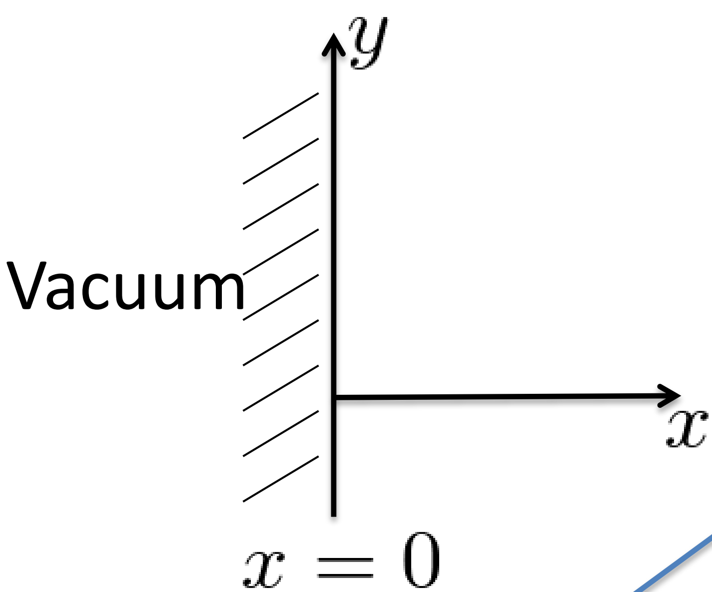
（Ex）Quantum Hall effect



A toy model for the integer quantum Hall state

$$H = \sin k_x \sigma_1 + \sin k_y \sigma_2 + (m + \cos k_x + \cos k_y)\,\sigma_3$$

Finite system



Question

Do the gapless states localized at boundary exist ?

Answer

$-2 < m < 0$,
$0 < m < 2$ → Yes

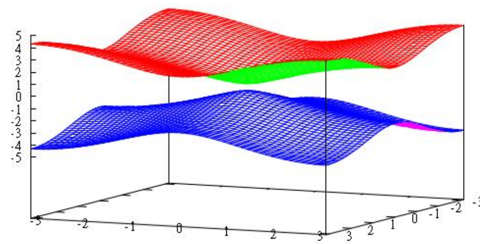
Otherwise → No

What's happen ?

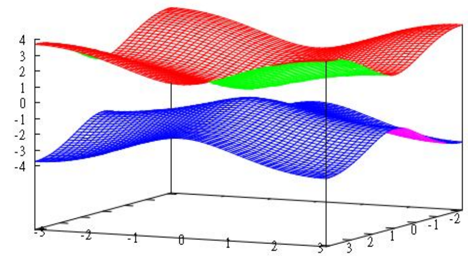
Energy dispersion near $m \sim -2$

$$H = \sin k_x \sigma_1 + \sin k_y \sigma_2 + (m + \cos k_x + \cos k_y)\,\sigma_3$$

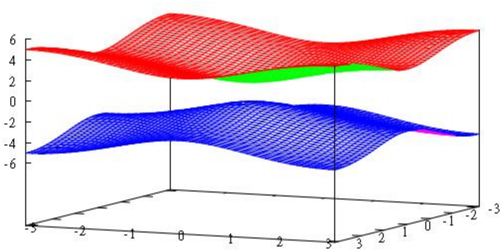
$$E(k_x, k_y) = \pm\sqrt{\sin^2 k_x + \sin^2 k_y + (m + \cos k_x + \cos k_y)^2}$$



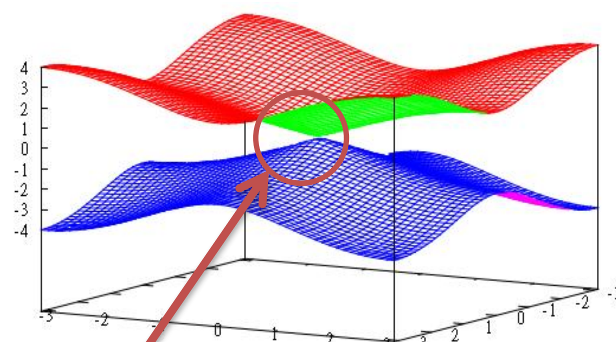
$m = -2.3$



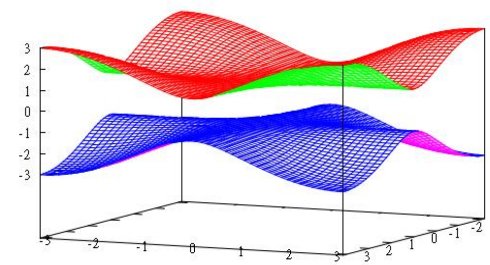
$m = -1.7$



$m = -3$



$m = -2$



$m = -1$

Energy gap is closed at $\boldsymbol{k} = \boldsymbol{0}$

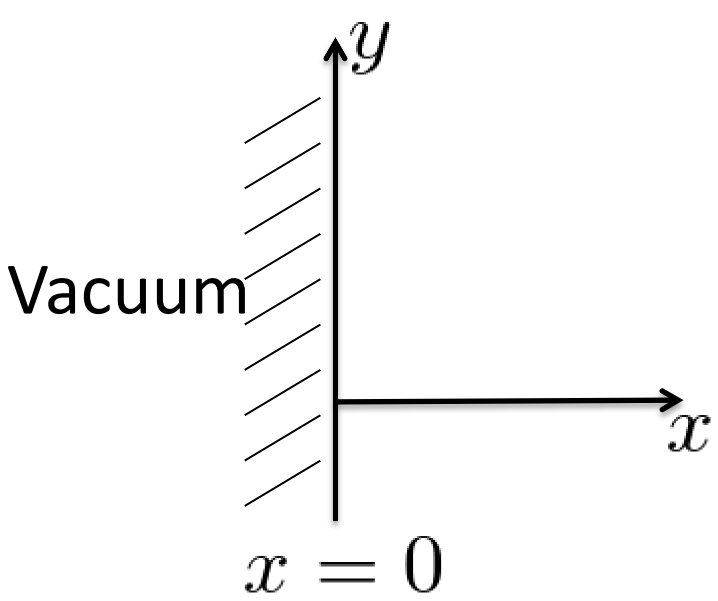
The effective model for the Γ point (*k=0*)

$$H = \sin k_x \sigma_1 + \sin k_y \sigma_2 + (m + \cos k_x + \cos k_y)\,\sigma_3$$

$$\Downarrow\ \boldsymbol{k} \sim \mathbf{0}$$

$$H = k_x \sigma_1 + k_y \sigma_2 + \left(m + 2 - k^2\right)\sigma_3$$

Finite system



$$H = -i\partial_x \sigma_1 + k_y \sigma_2 + \left(m + 2 - k_y^2 + \partial_x^2\right)\sigma_3$$

$$\Downarrow\ k_y = 0$$ (for only time-saving)

$$H|_{k_y=0} = -i\partial_x \sigma_1 + \left(m + 2 + \partial_x^2\right)\sigma_3$$

$$= \begin{pmatrix} m + 2 + \partial_x^2 & -i\partial_x \\ -i\partial_x & -(m + 2 + \partial_x^2) \end{pmatrix}$$

$$\Downarrow\ \phi \propto e^{\lambda x}$$

$$H|_{k_y=0} = \begin{pmatrix} m + 2 + \lambda^2 & -i\lambda \\ -i\lambda & -(m + 2 + \lambda^2) \end{pmatrix}$$

An analytic solution for the gapless mode

$$H|_{k_y=0} = \begin{pmatrix} m+2+\lambda^2 & -i\lambda \\ -i\lambda & -(m+2+\lambda^2) \end{pmatrix}$$

The condition for the existence of zero energy states $\det\left(H|_{k_y=0}\right)=0$

$$m+2+\lambda^2 = +\lambda\ ,\ -\lambda$$

$$\lambda_1 = \frac{1}{2}+\sqrt{\frac{1}{4}-(m+2)} \quad \phi_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \lambda_3 = -\frac{1}{2}+\sqrt{\frac{1}{4}-(m+2)} \quad \phi_3 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda_2 = \frac{1}{2}-\sqrt{\frac{1}{4}-(m+2)} \quad \phi_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \lambda_4 = -\frac{1}{2}-\sqrt{\frac{1}{4}-(m+2)} \quad \phi_4 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The condition for Localizing at $x \sim 0$ $\mathrm{Re}(\lambda) < 0$

(A) $m+2<0 \Rightarrow \lambda_2\ ,\ \lambda_4$

(B) $m+2>0 \Rightarrow \lambda_3\ ,\ \lambda_4$

An analytic solution for the gapless mode

(A)

$$m+2<0$$

λ_2 , λ_4

$$\phi(x)=\alpha\phi_2 e^{\lambda_2 x}+\beta\phi_4 e^{\lambda_4 x}$$
$$=\alpha\begin{pmatrix}1\\-i\end{pmatrix}e^{\lambda_2 x}+\beta\begin{pmatrix}1\\i\end{pmatrix}e^{\lambda_4 x}$$

(B)

$$m+2>0$$

λ_3 , λ_4

$$\phi(x)=\alpha\phi_3 e^{\lambda_3 x}+\beta\phi_4 e^{\lambda_4 x}$$
$$=\alpha\begin{pmatrix}1\\i\end{pmatrix}e^{\lambda_3 x}+\beta\begin{pmatrix}1\\i\end{pmatrix}e^{\lambda_4 x}$$

$$\alpha+\beta=0$$

Zero energy bound state

B.C.

For example $\phi|_{x=0}=\begin{pmatrix}0\\0\end{pmatrix}$

Message・・・
the existence of the gapless modes is determined by the Hamiltonian's parameters

An analytic solution for the gapless mode

For $k_y \neq 0$ $\quad H = -i\partial_x \sigma_1 + k_y \sigma_2 + \left(m + 2 - k_y^2 + \partial_x^2\right)\sigma_3$

The chiral edge mode $m + 2 > 0$

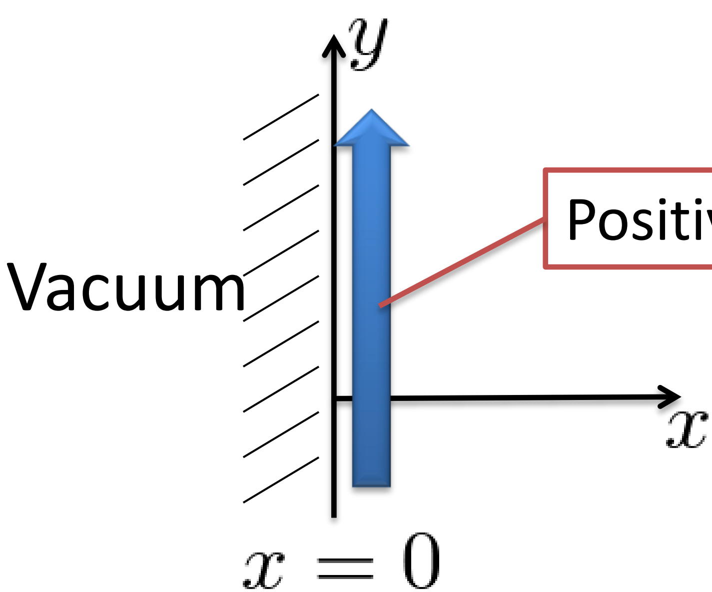
$$\phi_{k_y}^{\{0\}}(x) \sim \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{x}{2}} \sinh\left(\sqrt{\frac{1}{4} - \left(m + 2 - k_y^2\right)}\; x\right)$$

$$\left(-\sqrt{m+2} < k_y < \sqrt{m+2} \right)$$

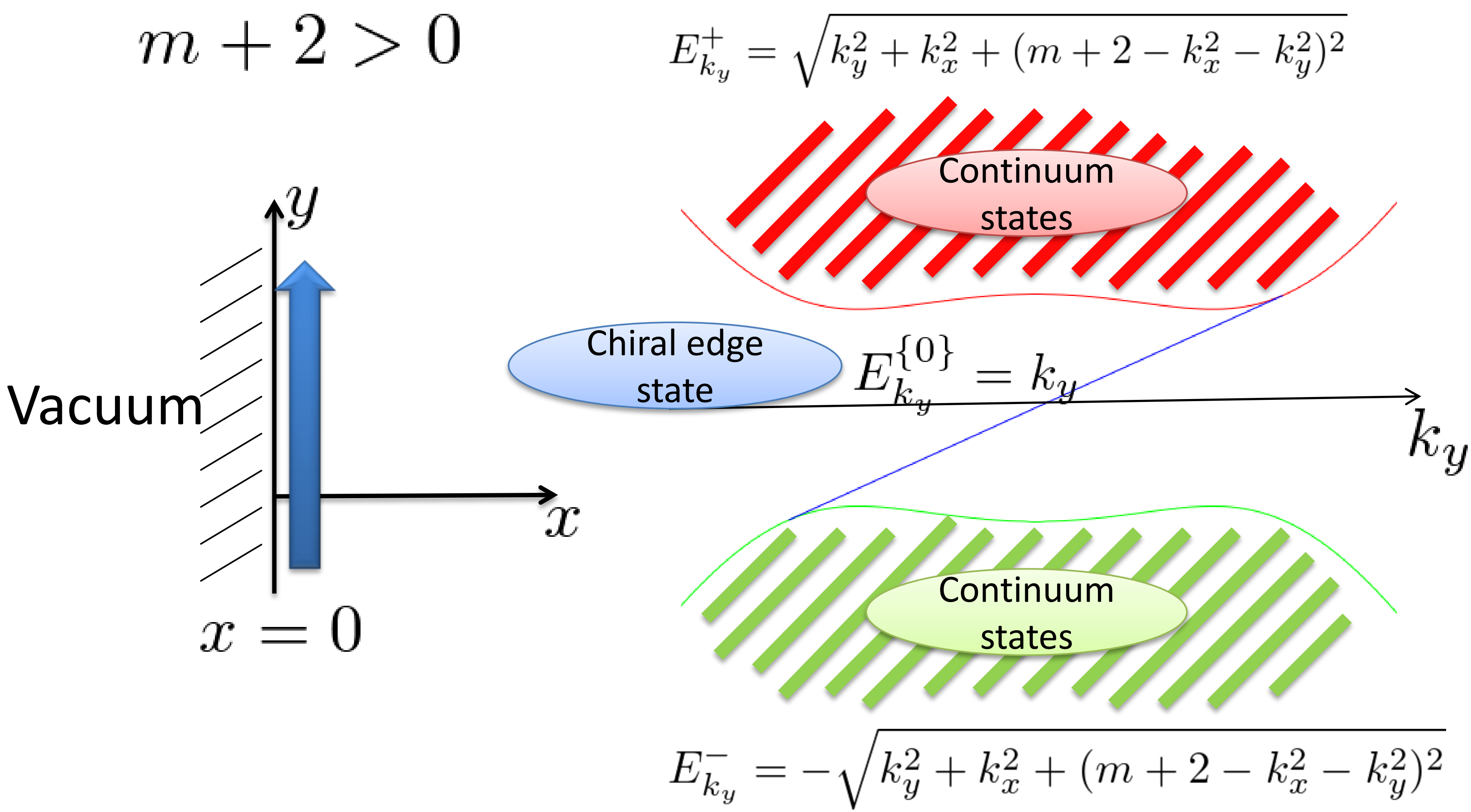
$$H\phi_{k_y}^{\{0\}}(x) = k_y \sigma_2\, \phi_{k_y}^{\{0\}}(x) = k_y\, \phi_{k_y}^{\{0\}}(x)$$

$$\sigma_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = + \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Positive chirality



The energy spectrum of the finite system

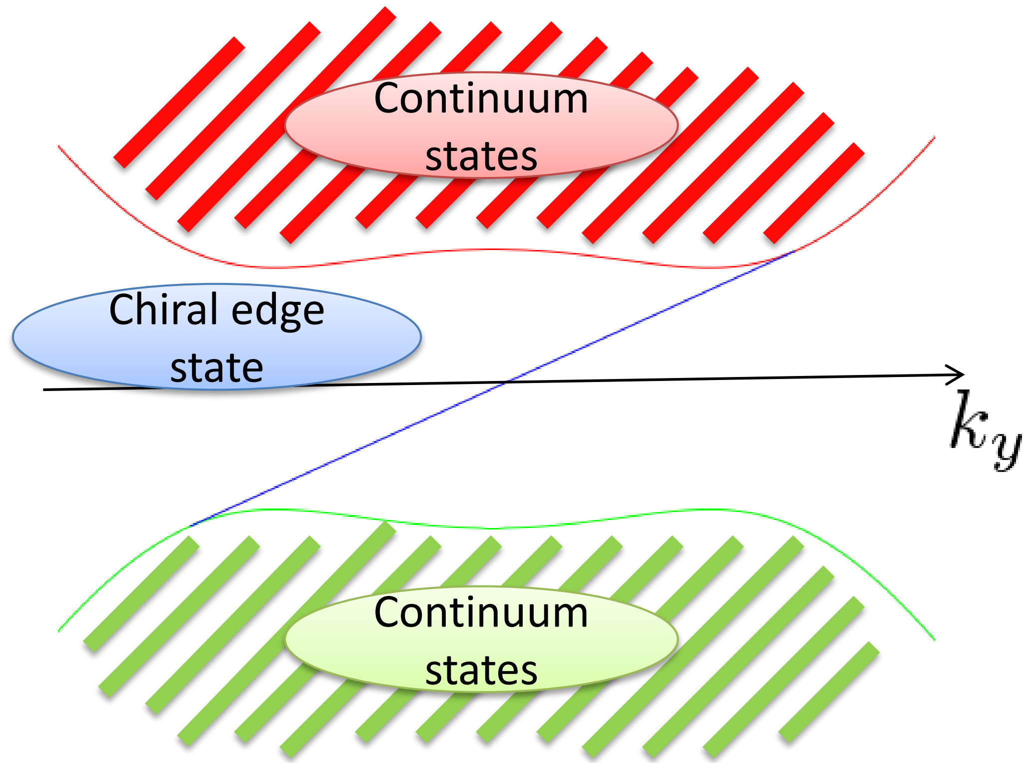


The meaning of “Topologically protected”

- This gapless state is stable against the deformation of the Hamiltonian unless the energy gap is closed.

- Independent of the Boundary conditions.

- Insensitive to disorder, impossible to localize.



Whether do there exist the gapless states is determined by the bulk behavior.

Topological classification of bulk insulators

Consider 2-dimensinal insulating systems

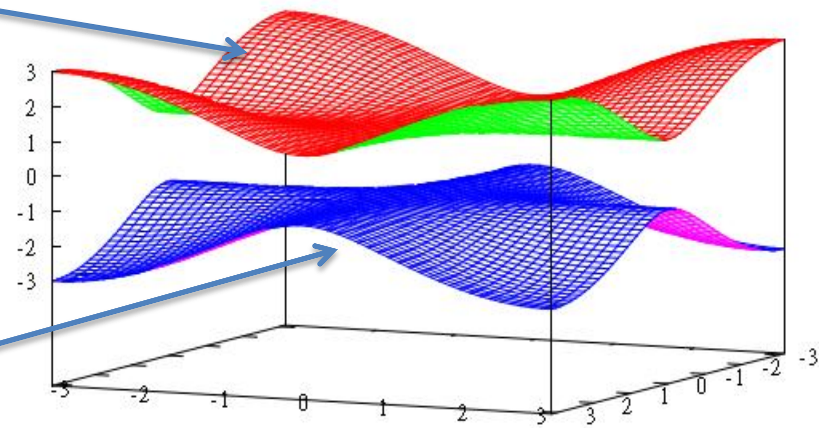
$$H(k_x, k_y) = \varepsilon(\boldsymbol{k}) + \boldsymbol{R}(\boldsymbol{k}) \cdot \boldsymbol{\sigma} \qquad R = \sqrt{\boldsymbol{R}^2}$$

Conduction band

$$u_+(\boldsymbol{k}) = \frac{1}{\sqrt{2R(R+R_3)}} \begin{pmatrix} R + R_3 \\ R_1 + iR_2 \end{pmatrix}$$

Valence band

$$u_-(\boldsymbol{k}) = \frac{1}{\sqrt{2R(R-R_3)}} \begin{pmatrix} R - R_3 \\ -R_1 - iR_2 \end{pmatrix}$$



Ground state $=$ BZ $\xrightarrow[\text{Map}]{u_-(\boldsymbol{k})}$ Spinor space

The global structure of spinor space

Parameterize uniquely

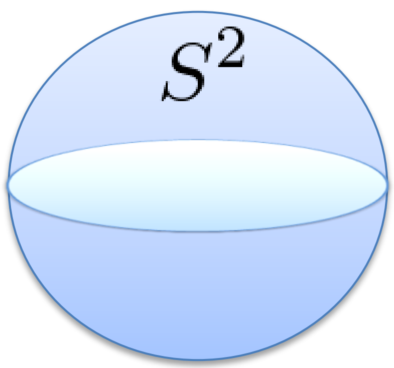
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

relative phase

$u \geq 0$ fix

$$0 \leq \theta \leq \pi$$

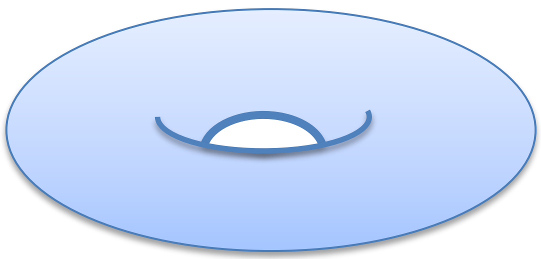
$$0 \leq \phi \leq 2\pi$$



$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

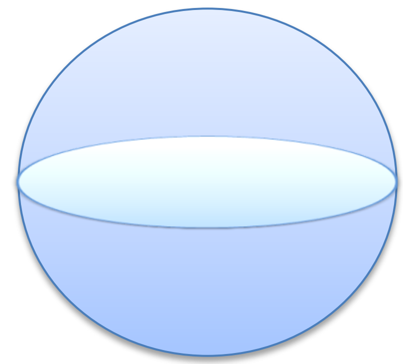
Ground state topology

Brillouin zone = torus



$u_-(\boldsymbol{k})$

spinor space = sphere



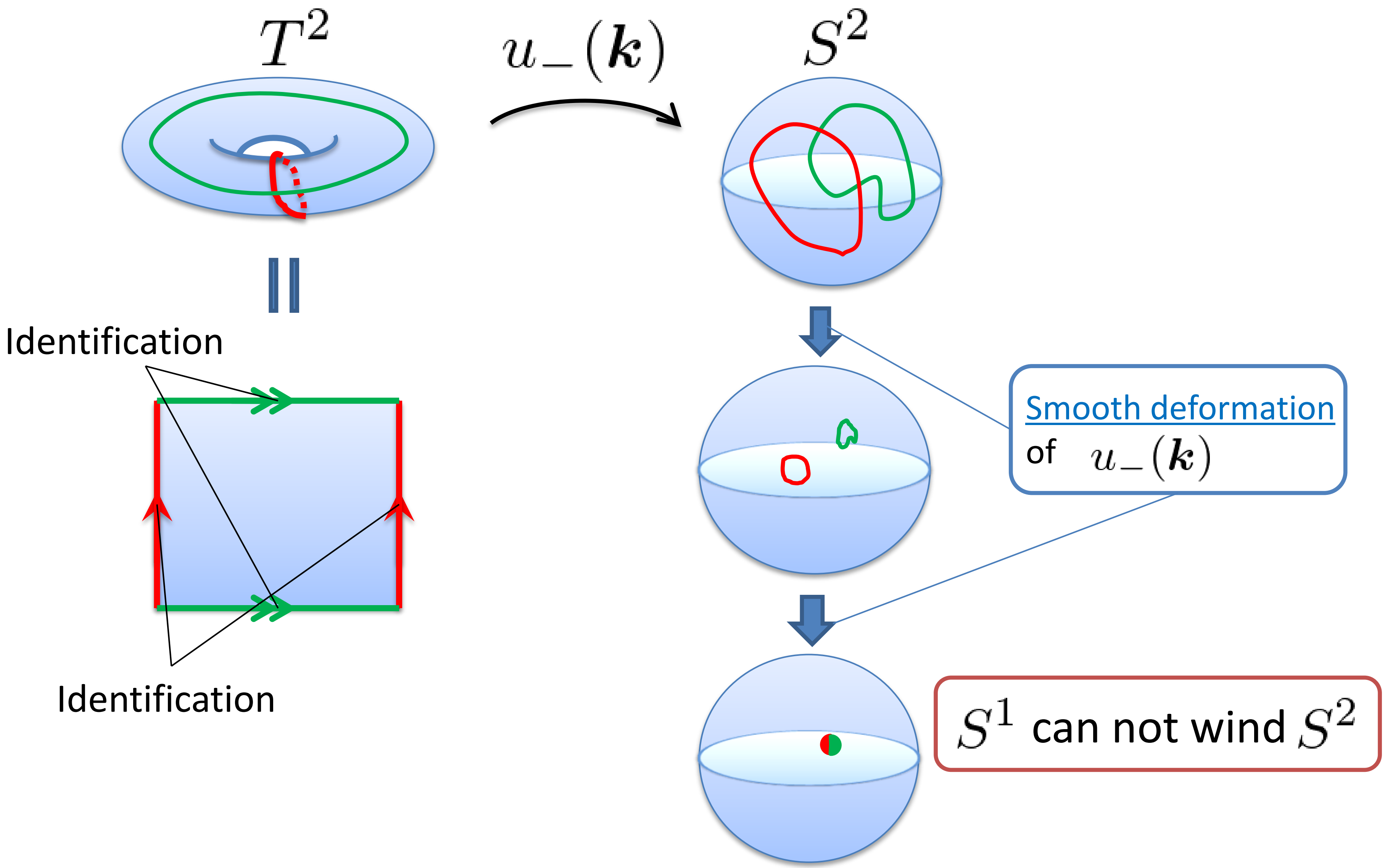
$$T^2 \longrightarrow S^2$$

$$\Psi \qquad\qquad \Psi$$

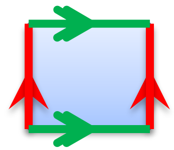
$$(k_x, k_y) \longmapsto u_-(\boldsymbol{k}) = \begin{pmatrix} \cos\frac{\theta(\boldsymbol{k})}{2} \\ e^{i\phi(\boldsymbol{k})}\sin\frac{\theta(\boldsymbol{k})}{2} \end{pmatrix}$$

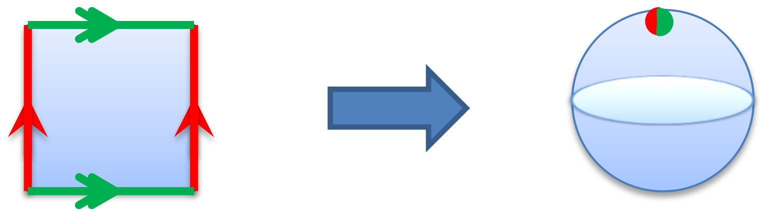
Topology ・・・ The study of geometrical properties that are insensitive to **smooth deformations**

The classification of the Map $T^2 \to S^2$



The classification of the Map $T^2 \to S^2$

We can identify  with  without loss of generality



S^2

The classification of the Map $T^2 \to S^2$

||

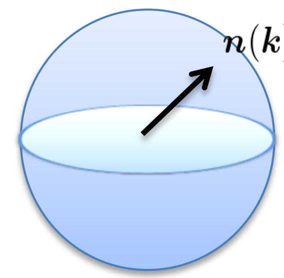
The classification of the Map $S^2 \to S^2$

||

Integer $\mathbb{Z}$ which counts the number of times S^2 covers S^2

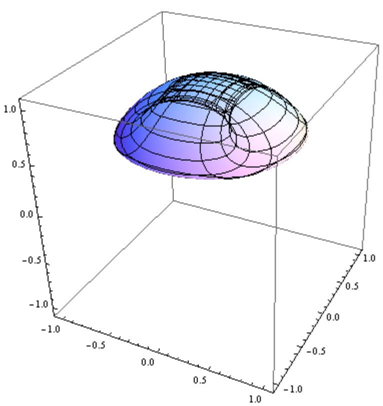
Example

$$u_-(\boldsymbol{k}) = \frac{1}{\sqrt{2R(R-R_3)}} \begin{pmatrix} R - R_3 \\ -R_1 - iR_2 \end{pmatrix} \sim \begin{pmatrix} \cos\frac{\theta(\boldsymbol{k})}{2} \\ e^{i\phi(\boldsymbol{k})} \sin\frac{\theta(\boldsymbol{k})}{2} \end{pmatrix}$$



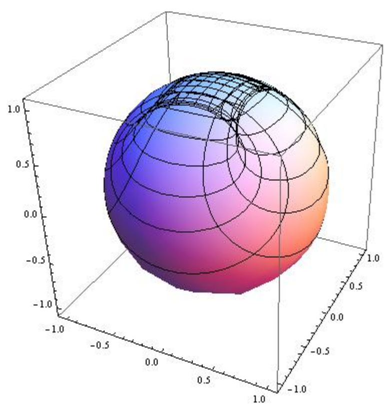
$$\boldsymbol{n}(\boldsymbol{k}) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\phi \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\phi \\ \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \end{pmatrix} = -\frac{\boldsymbol{R}(\boldsymbol{k})}{R(\boldsymbol{k})}$$

$$H = \sin k_x \sigma_1 + \sin k_y \sigma_2 + (m + \cos k_x + \cos k_y)\,\sigma_3$$



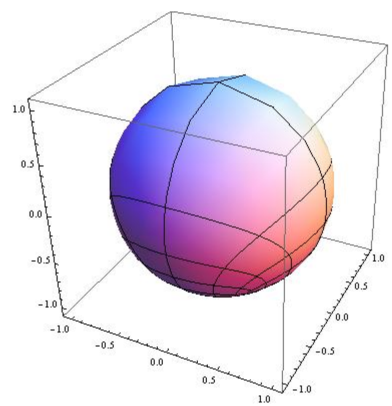
m=-2.2

Z = 0



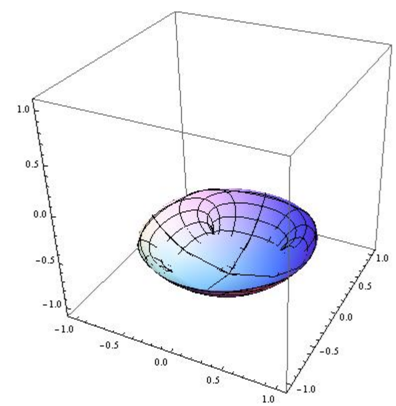
m=-1.8

Z = -1



m=1.8

Z = 1



m=2.2

Z = 0

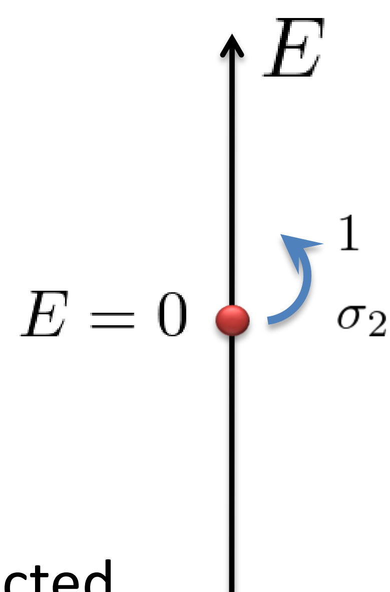
How about 1-dimensional systems ?

$$H = -i\partial_x \sigma_1 + \left(m + 2 + \partial_x^2\right) \sigma_3$$
$$= \begin{pmatrix} m + 2 + \partial_x^2 & -i\partial_x \\ -i\partial_x & -(m + 2 + \partial_x^2) \end{pmatrix}$$

$$\phi^{\{0\}}(x) \sim \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{x}{2}} \sinh\left(\sqrt{\frac{1}{4} - (m+2)}\ x\right)$$
$$H\phi^{\{0\}}(x) = 0$$

There exist a zero energy state,
but not stable along perturbation

$$\mu\ \phi^{\{0\}}(x) = \mu\ \phi^{\{0\}}(x)$$
$$\Delta\sigma_2\ \phi^{\{0\}}(x) = \Delta\ \phi^{\{0\}}(x)$$

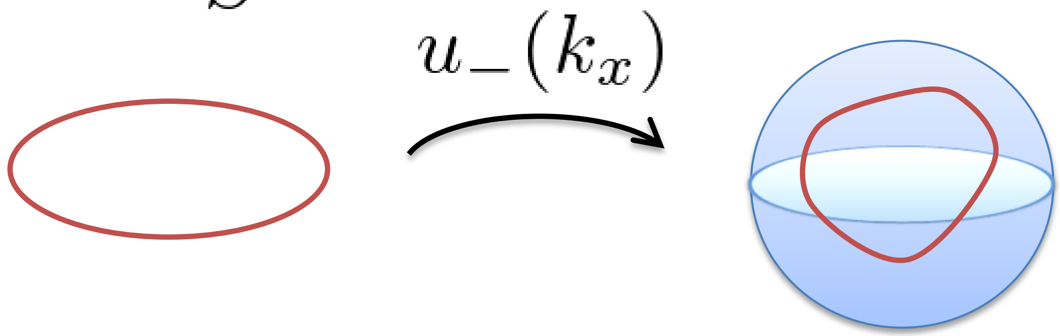


This zero mode is not topologically protected

How about 1-dimensional systems ?

This is consistent in the topological classification ground state

Brillouin zone = S^1



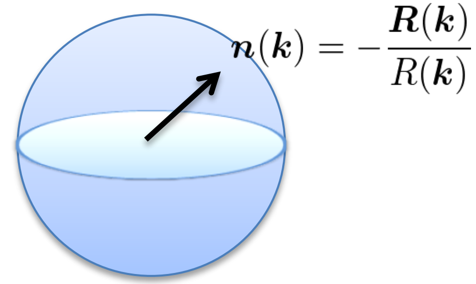
$$S^1 \longrightarrow S^2$$

S^1 can not wind S^2 ⇒ Trivial

How about 1-dimensional systems ?

If there are a certain appropriate symmetry, the zero mode is topologically protected.

$$H(k_x) = \varepsilon(k_x) + \boldsymbol{R}(k_x) \cdot \boldsymbol{\sigma}$$



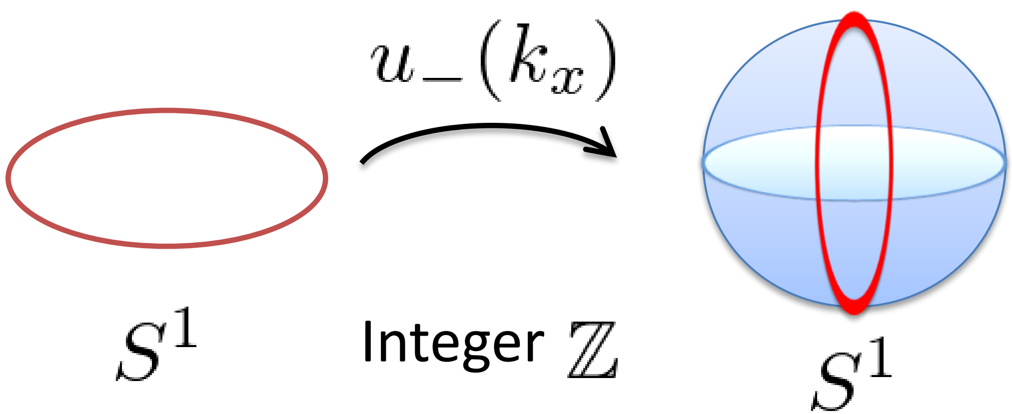
(Ex) Chiral symmetry

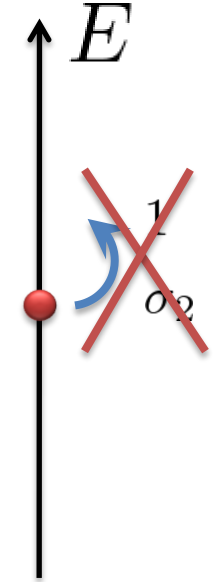
$$\sigma_2 H(x) \sigma_2 = -H(x)$$

$$\sigma_2 H(k_x) \sigma_2 = -H(k_x)$$

$$\varepsilon(k_x) = R_2(k_x) = 0$$

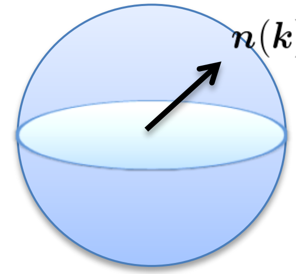
$$n_2(k_x) = 0$$





How about 1-dimensional systems ?

$$H(k_x) = \varepsilon(k_x) + \boldsymbol{R}(k_x)\cdot\boldsymbol{\sigma}$$



$$\boldsymbol{n}(\boldsymbol{k}) = -\frac{\boldsymbol{R}(\boldsymbol{k})}{R(\boldsymbol{k})}$$

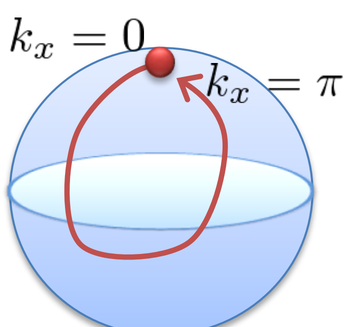
(Ex) Particle-Hall symmetry (i)

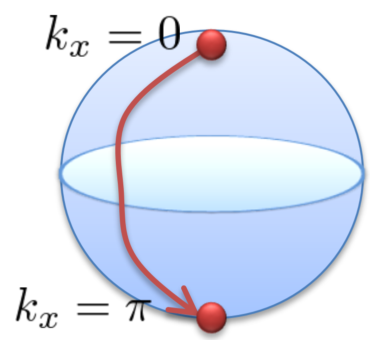
$$\sigma_1 H^*(x)\sigma_1 = -H(x)$$

$$\sigma_1 H^*(k_x)\sigma_1 = -H(-k_x)$$

$$\begin{pmatrix} R_1(k_x) \\ R_2(k_x) \\ R_3(k_x) \end{pmatrix} = \begin{pmatrix} -R_1(-k_x) \\ -R_2(-k_x) \\ R_3(-k_x) \end{pmatrix}$$

$$\boldsymbol{n}(0), \boldsymbol{n}(\pi) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

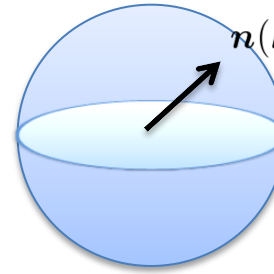




Trivial ⇔ $\mathbb{Z}_2$ ⇔ Non-trivial

How about 1-dimensional systems ?

$$H(k_x) = \varepsilon(k_x) + \boldsymbol{R}(k_x) \cdot \boldsymbol{\sigma}$$



$$\boldsymbol{n}(\boldsymbol{k}) = -\frac{\boldsymbol{R}(\boldsymbol{k})}{R(\boldsymbol{k})}$$

(Ex) Particle-Hall symmetry (ii)

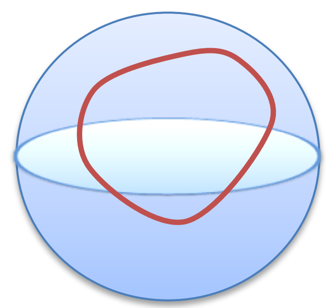
$$\sigma_2 H^*(x) \sigma_2 = -H(x)$$

$$\Downarrow$$

$$\sigma_2 H^*(k_x) \sigma_2 = -H(-k_x)$$

$$\begin{pmatrix} R_1(k_x) \\ R_2(k_x) \\ R_3(k_x) \end{pmatrix} = \begin{pmatrix} R_1(-k_x) \\ R_2(-k_x) \\ R_3(-k_x) \end{pmatrix}$$

Trivial



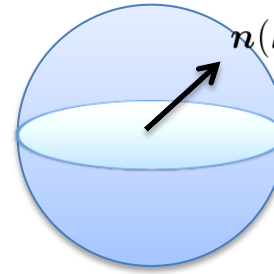
Topological classification depend on the symmetry matrix

$$(\sigma_1 K)(\sigma_1 K) = 1$$

$$(\sigma_2 K)(\sigma_2 K) = -1$$

K ・・・ complex conjugate

How about 1-dimensional systems ?



$$H(k_x) = \varepsilon(k_x) + \boldsymbol{R}(k_x) \cdot \boldsymbol{\sigma}$$

(Ex) Particle-Hall symmetry (i) and Time-reversal symmetry

$$\sigma_1 H^*(x) \sigma_1 = -H(x)$$

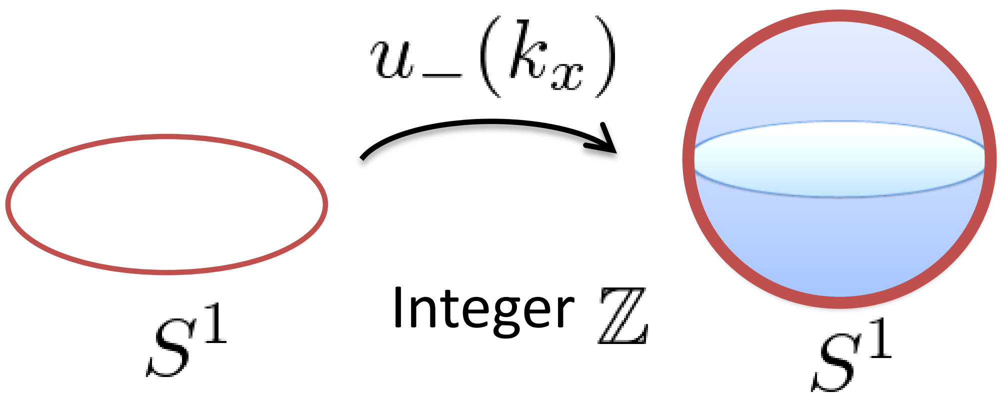
$$H^*(x) = H(x)$$

$$\sigma_1 H^*(k_x) \sigma_1 = -H(-k_x)$$

$$H^*(k_x) = H(-k_x)$$

$$\begin{pmatrix} R_1(k_x) \\ R_2(k_x) \\ R_3(k_x) \end{pmatrix} = \begin{pmatrix} 0 \\ -R_2(-k_x) \\ R_3(-k_x) \end{pmatrix}$$

$$n_1(k_x) = 0$$
$$n_2(-k_x) = -n_2(k_x)$$
$$n_3(-k_x) = n_3(k_x)$$



Topological classification of insulators and superconductors

$H(\boldsymbol{k})$ ··· Insulating Hamiltonian or BdG Hamiltonian
d-dimension Full gapped

1. Determine the local symmetries

- Time reversal symmetry (anti-unitary)
  - Yes → $\Theta^2 = \pm 1$ $\Theta H(\boldsymbol{x})\Theta^{-1} = H(\boldsymbol{x})$
  - No
- Particle hole symmetry (anti-unitary)
  - Yes → $\Xi^2 = \pm 1$ $\Xi H(\boldsymbol{x})\Xi^{-1} = -H(\boldsymbol{x})$
  - No
- Chiral (sublattice) symmetry (unitary)
  - Yes → Π $\Pi H(\boldsymbol{x})\Pi^{-1} = -H(\boldsymbol{x})$
  - No

2. Classify the ground state topology

Topological classification of insulators and superconductors

Schnyder et al.(2008)

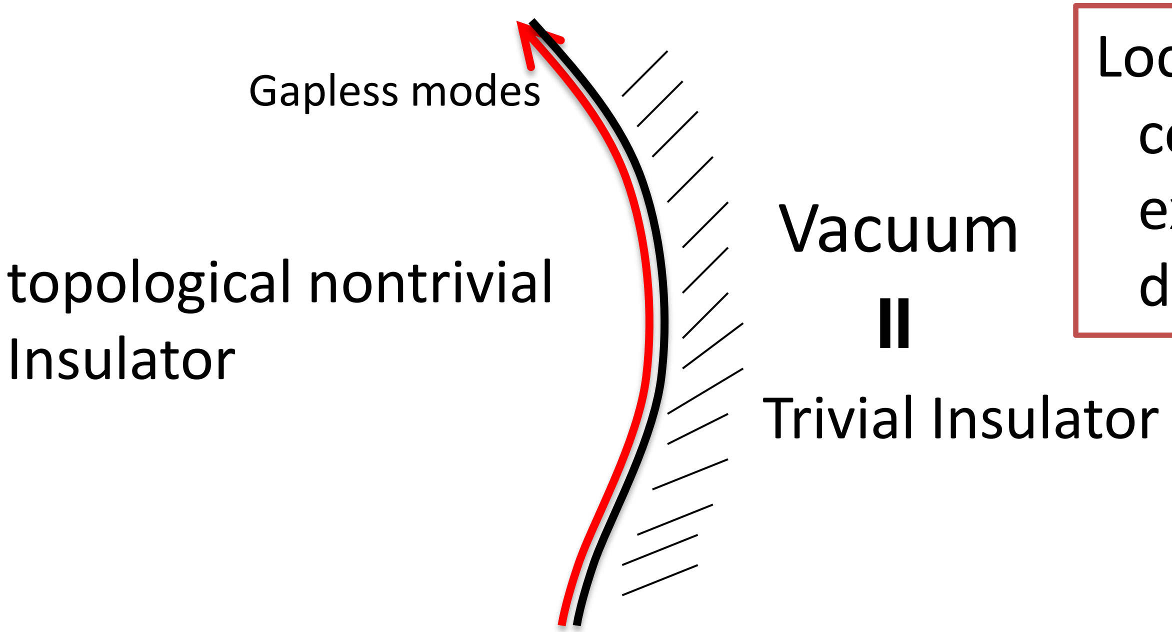
		Symmetry			d							
s	AZ	Θ^2	Ξ^2	Π^2	0	1	2	3	4	5	6	7
0	A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	AI	1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
2	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
3	DIII	−1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
4	AII	−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
5	CII	−1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
6	C	0	−1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
7	CI	1	−1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

∴ Completely classified

Bulk–boundary correspondence

Bulk- boundary correspondence

topological nontrivial ground state $\{\ |u_n(\boldsymbol{k})\rangle\ \}$ ⟺ the existence of topologically protected gapless states localized at boundary



Local symmetry ･･･ consistent with the existence of the disorder