Wess=Zumino項とChern=Simons項の関係

塩崎　謙

March 3, 2025

Abstract

懸垂ΣM上のハミルトニアンを構成することにより，WZ項とCS項は値を変えずにマップされることを見る．

1　定義

1.1　WZ項

Mを向き付けのある閉じた偶数$2n$次元多様体として，滑らかなマップ

$$g : M_{2n} \to G, \quad G = U(n) \text{ or } SO(n), \tag{1.1}$$

に対して，$(2n+1)$次元への拡張

$$\tilde{g} : X \to G, \quad \partial X = M, \quad \tilde{g}|_M = g, \tag{1.2}$$

を考える．拡張の存在は仮定する． WZ項を

$$\mathrm{WZ}_{2n}[g] = 2\pi \frac{n!}{(2n+1)!(2\pi i)^{n+1}} \int_X \mathrm{tr}\,[(\tilde{g} d\tilde{g}^{-1})^{2n+1}] \tag{1.3}$$

$$= 2\pi \frac{-n!}{(2n+1)!(2\pi i)^{n+1}} \int_X \mathrm{tr}\,[(\tilde{g}^{-1} d\tilde{g})^{2n+1}] \tag{1.4}$$

として定める．係数は，$G = U(n)$の場合に

$$\mathrm{WZ}[g] \in \mathbb{R}/2\pi\mathbb{Z} \tag{1.5}$$

に値を取るように定めている．例えば，

$$\mathrm{WZ}_2[g] = \frac{1}{12\pi} \int_X \mathrm{tr}\,[(\tilde{g}^{-1} d\tilde{g})^3]. \tag{1.6}$$

1.2　CS項

Mを向き付けのある閉じた奇数$2n-1$次元多様体として，滑らかなマップ

$$P : M_{2n} \to B, \quad B = U(n+m)/U(n) \times U(m) \text{ or } O(n+m)/O(n) \times O(m), \quad P^2 = P \text{ (projection)}, \tag{1.7}$$

に対して，$2n$次元への拡張

$$\tilde{P} : X \to B, \quad \partial X = M, \quad \tilde{P}|_M = P, \quad \tilde{P}^2 = \tilde{P} \tag{1.8}$$

を考える．拡張の存在は仮定する． CS項を

$$\mathrm{CS}_{2n-1}[P] = 2\pi \frac{1}{n!}(\frac{i}{2\pi})^n \int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n}] \tag{1.9}$$

として定める．係数は，$B = U(n+m)/U(n) \times U(m)$の場合に

$$\mathrm{CS}_{2n-1}[P] \in \mathbb{R}/2\pi\mathbb{Z} \tag{1.10}$$

に値を取るように定めている．例えば，

$$\mathrm{CS}_3[P] = -\frac{1}{4\pi}\int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^4]. \tag{1.11}$$

2 WZ項→CS項

滑らかなマップ$g : M_{2n} \to G$と拡張$\tilde{g} : X \to G$に対して，懸垂ΣX上の平坦ハミルトニアンを

$$\tilde{H} = \cos\theta\sigma_z + \sin\theta\tilde{\Gamma}, \quad \tilde{\Gamma} = \begin{pmatrix} & \tilde{g} \\ \tilde{g}^\dagger & \end{pmatrix}, \quad \theta \in [0,\pi], \tag{2.1}$$

として定める．

$$\tilde{H} = \sigma_z(\cos\theta + \sin\theta\sigma_z\tilde{\Gamma}) = \sigma_z e^{\theta\sigma_z\tilde{\Gamma}} = e^{-\theta\sigma_z\tilde{\Gamma}}\sigma_z \tag{2.2}$$

と書くことができる．固有値-1への射影が

$$\tilde{P} = \frac{1-\tilde{H}}{2} \tag{2.3}$$

として得られる． ΣM_{2n}上のCS項は

$$\mathrm{CS}_{2n+1}[P] = 2\pi\frac{1}{(n+1)!}(\frac{i}{2\pi})^{n+1}\int_{\Sigma X}\mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n+2}] \tag{2.4}$$

$$= 2\pi\frac{1}{(n+1)!}(\frac{i}{2\pi})^{n+1}\int_{\Sigma X}\frac{1}{2^{2n+3}}\mathrm{tr}\,[(1-\tilde{H})(d\tilde{H})^{2n+2}]. \tag{2.5}$$

さて

$$d\tilde{H} = \underbrace{(-\sin\theta\sigma_z d\theta + \cos\theta\tilde{\Gamma}d\theta)}_{X} + \underbrace{\sin\theta d\tilde{\Gamma}}_{Y} =: X + Y, \tag{2.6}$$

と書き，$\tilde{\Gamma}^2 = 1$より

$$XY = YX \tag{2.7}$$

に注意すると，

$$\mathrm{tr}\,[(1-\tilde{H})(d\tilde{H})^{2n+2}] = \mathrm{tr}\,[(1-\tilde{H})(X+Y)^{2n+2}] = (2n+2)\mathrm{tr}\,[(1-\tilde{H})XY^{2n+1}] \tag{2.8}$$

$$= (2n+2)\mathrm{tr}\,\left[\begin{pmatrix}1-\cos\theta & -\sin\theta\tilde{g} \\ -\sin\theta\tilde{g}^\dagger & 1+\cos\theta\end{pmatrix}\begin{pmatrix}-\sin\theta & \cos\theta\tilde{g} \\ \cos\theta\tilde{g}^\dagger & \sin\theta\end{pmatrix}d\theta(\sin\theta)^{2n+1}\begin{pmatrix} & d\tilde{g} \\ d\tilde{g}^\dagger & \end{pmatrix}^{2n+1}\right] \tag{2.9}$$

$$= (2n+2)\mathrm{tr}\,\left[\begin{pmatrix} * & (\cos\theta-1)\tilde{g} \\ (\cos\theta+1)\tilde{g}^\dagger & * \end{pmatrix}d\theta(\sin\theta)^{2n+1}\begin{pmatrix} & (d\tilde{g}d\tilde{g}^\dagger)^n d\tilde{g} \\ (d\tilde{g}^\dagger d\tilde{g})^n d\tilde{g}^\dagger & \end{pmatrix}\right] \tag{2.10}$$

$$= (2n+2)(\sin\theta)^{2n+1}d\theta\mathrm{tr}\,[(\cos\theta-1)\tilde{g}(d\tilde{g}^\dagger d\tilde{g})^n d\tilde{g}^\dagger + (\cos\theta+1)\tilde{g}^\dagger(d\tilde{g}d\tilde{g}^\dagger)^n d\tilde{g}] \tag{2.11}$$

$$= (2n+2)(\sin\theta)^{2n+1}d\theta\mathrm{tr}\,[(\cos\theta-1)(-1)^{n+1}(\tilde{g}^\dagger d\tilde{g})^{2n+1} + (\cos\theta+1)(-1)^n(\tilde{g}^\dagger d\tilde{g})^{2n+1}] \tag{2.12}$$

$$= (-1)^n 2(2n+2)(\sin\theta)^{2n+1}d\theta\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}]. \tag{2.13}$$

よって，

$$\mathrm{CS}_{2n+1}[P] = 2\pi\frac{1}{(n+1)!}(\frac{i}{2\pi})^{n+1}\int_{\Sigma X}\frac{1}{2^{2n+3}}\mathrm{tr}\,[(1-\tilde{H})(d\tilde{H})^{2n+2}] \tag{2.14}$$

$$= 2\pi\frac{1}{(n+1)!}(\frac{i}{2\pi})^{n+1}\int_{\Sigma X}\frac{1}{2^{2n+3}}(-1)^n 2(2n+2)(\sin\theta)^{2n+1}d\theta\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}]. \tag{2.15}$$

$$\int_0^\pi(\sin\theta)^{2n+1}d\theta = \frac{2(2n)!!}{(2n+1)!!} = \frac{2(2^n n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \tag{2.16}$$

より，

$$\mathrm{CS}_{2n+1}[P] = 2\pi\frac{1}{(n+1)!}(\frac{i}{2\pi})^{n+1}\int_X\frac{1}{2^{2n+3}}(-1)^n 2(2n+2)\frac{2^{2n+1}(n!)^2}{(2n+1)!}\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}] \tag{2.17}$$

$$= 2\pi(\frac{i}{2\pi})^{n+1}\int_X(-1)^n\frac{n!}{(2n+1)!}\mathrm{tr}\,[(\tilde{g}^\dagger d\tilde{g})^{2n+1}] \tag{2.18}$$

$$= 2\pi(\frac{i}{2\pi})^{n+1}\int_X(-1)^{n+1}\frac{n!}{(2n+1)!}\mathrm{tr}\,[(\tilde{g}d\tilde{g}^\dagger)^{2n+1}] \tag{2.19}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_X\mathrm{tr}\,[(\tilde{g}d\tilde{g}^\dagger)^{2n+1}] \tag{2.20}$$

$$= \mathrm{WZ}_{2n}[g]. \tag{2.21}$$

よって，懸垂によってWZ項がそのままCS項に値を変えずにマップされる．

3 CS項→WZ項

滑らかなマップ$P : M_{2n-1} \to B$と拡張$\tilde{P} : X \to B$に対して，懸垂ΣX上のユニタリ行列を

$$\tilde{g} = e^{i\theta\tilde{H}} = \cos\theta + i\sin\theta\tilde{H}, \quad \tilde{H} = 1-2\tilde{P}, \quad \theta\in[0,\pi], \tag{3.1}$$

として定める．ΣM_{2n-1}上のWZ項は

$$\mathrm{WZ}_{2n}[g] = 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_{\Sigma X}\mathrm{tr}\,[(\tilde{g}d\tilde{g}^{-1})^{2n+1}] \tag{3.2}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_{\Sigma X}-\mathrm{tr}\,[i^{2n+1}(2n+1)(\sin\theta)^{2n}d\theta\tilde{H}(d\tilde{H})^{2n} + i^{2n+1}(\sin\theta)^{2n+1}(d\tilde{H})^{2n+1}e^{i\theta\tilde{H}}]. \tag{3.3}$$

App. Aの計算結果を用いた． $(d\tilde{H})^{2n+1}$は次元的な理由で消える．また，

$$\int_0^\pi(\sin\theta)^{2n}d\theta = \frac{\pi(2n-1)!!}{(2n)!!} = \frac{\pi(2n)!}{2^{2n}n!n!} \tag{3.4}$$

に注意すると，

$$\mathrm{WZ}_{2n}[g] = 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_{\Sigma X}-\mathrm{tr}\,[i^{2n+1}(2n+1)(\sin\theta)^{2n}d\theta\tilde{H}(d\tilde{H})^{2n}] \tag{3.5}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_X -i^{2n+1}(2n+1)\frac{\pi(2n)!}{2^{2n}n!n!}\mathrm{tr}\,[\tilde{H}(d\tilde{H})^{2n}] \tag{3.6}$$

$$= 2\pi\frac{n!}{(2n+1)!(2\pi i)^{n+1}}\int_X -i^{2n+1}(2n+1)\frac{\pi(2n)!}{n!n!}\mathrm{tr}\,[(1-2\tilde{P})(d\tilde{P})^{2n}] \tag{3.7}$$

$$= 2\pi\frac{-i^{2n+1}\pi}{n!(2\pi i)^{n+1}}\int_X\mathrm{tr}\,[(1-2\tilde{P})(d\tilde{P})^{2n}]. \tag{3.8}$$

ここで，

$$\int_X \mathrm{tr}\,[(d\tilde{P})^{2n}] = \int_X d\mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n-1}] = \int_{M_{2n-1}} \mathrm{tr}\,[P(dP)^{2n-1}] \tag{3.9}$$

であるが，

$$\mathrm{tr}\,[P(dP)^{2n-1}] = \mathrm{tr}\,[(dP)^{2n-1}(1-P)] = \mathrm{tr}\,[(1-P)(dP)^{2n-1}] \tag{3.10}$$

より

$$\mathrm{tr}\,[P(dP)^{2n-1}] = \frac{1}{2}d\mathrm{tr}\,[P(dP)^{2n-2}] \tag{3.11}$$

となり全微分項であるので落ちる．よって，

$$\mathrm{WZ}_{2n}[g] = 2\pi\frac{2i^{2n+1}\pi}{n!(2\pi i)^{n+1}}\int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n}] \tag{3.12}$$

$$= 2\pi\frac{i^n}{n!(2\pi)^n}\int_X \mathrm{tr}\,[\tilde{P}(d\tilde{P})^{2n}] \tag{3.13}$$

$$= \mathrm{CS}_{2n-1}[P]. \tag{3.14}$$

よって，懸垂により，CS項も値がそのままWZ項にマップされる．

A　$(g^\dagger dg)^n$の計算

よく参照するので，

$$g = e^{i\theta H} = \cos\theta + i\sin\theta H, \quad H:\ \mathrm{Hermite}, \quad H^2 = 1, \quad \partial_\theta H = 0, \tag{A.1}$$

なる場合の

$$(g^{-1}dg)^n \tag{A.2}$$

の計算をしておく．

$$dg = -\sin\theta d\theta + i\cos\theta d\theta H + i\sin\theta dH = e^{i(\theta+\frac{\pi}{2})H} + i\sin\theta dH, \tag{A.3}$$

$$g^{-1}dg = iHd\theta + i\sin\theta e^{-i\theta H}dH. \tag{A.4}$$

ここで，

$$dHH = -HdH \tag{A.5}$$

に注意すると，

$$g^{-1}dg = iHd\theta + i\sin\theta dHe^{i\theta H} \tag{A.6}$$

とも書かれる．

$$X = iHd\theta, \quad Y = i\sin\theta dHe^{i\theta H}, \tag{A.7}$$

と置くと，

$$XY = YX \tag{A.8}$$

に注意．また，$X^{n\geq 2} = 0$に注意すると，

$$(g^{-1}dg)^n = nXY^{n-1} + Y^n \tag{A.9}$$

$$= niHd\theta(i\sin\theta dHe^{i\theta H})^{n-1} + (i\sin\theta dHe^{i\theta H})^n \tag{A.10}$$

$$= \begin{cases} i^n n(\sin\theta)^{n-1}d\theta H(dH)^{n-1} + i^n(\sin\theta)^n(dH)^n e^{i\theta H} & (n\in\ \mathrm{odd}) \\ i^n n(\sin\theta)^{n-1}d\theta H(dH)^{n-1}e^{i\theta H} + i^n(\sin\theta)^n(dH)^n & (n\in\ \mathrm{even}) \end{cases}. \tag{A.11}$$