Recent Developments of the Classification of Topological Insulators and Superconductors

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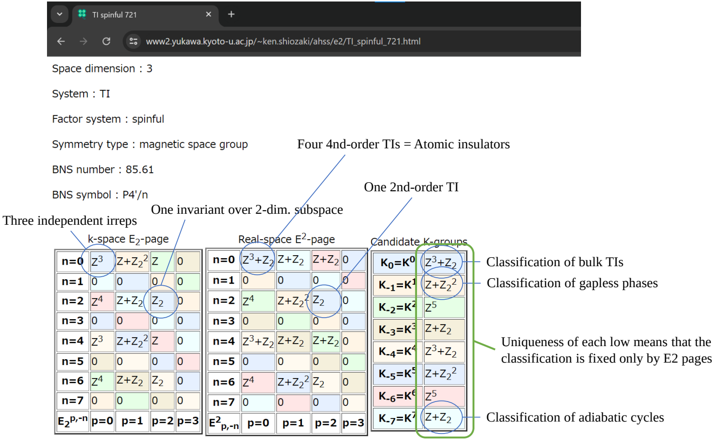
2024/6/26, @Zurich

Based on [1] KS, Masatoshi Sato, Kiyonori Gomi, 1802.06694; [2] KS, Charles Zhaoxi Xiong, Kiyonori Gomi, 1810.00801; [3] KS, Seishiro Ono, 2304.01827.

Take home message

- We determined about 60% of the classification of crystalline free fermion topological phases [KS=Ono, 2304.01827].
- The results are summarized in the webpage (link). (The link is found from [KS=Ono, 2304.01827].)
- Rich information!



Space dimension : 3

System : TI

Factor system : spinful

Symmetry type : magnetic space group

BNS number : 85.61

BNS symbol : P4'/n

k-space E_2-page

n=0	Z^3	$Z+Z_2^2$	Z	0
n=1	0	0	0	0
n=2	Z^4	$Z+Z_2$	Z_2	0
n=3	0	0	0	0
n=4	Z^3	$Z+Z_2^2$	Z	0
n=5	0	0	0	0
n=6	Z^4	$Z+Z_2$	Z_2	0
n=7	0	0	0	0
$E_2^{p,-n}$	p=0	p=1	p=2	p=3

Real-space E^2-page

n=0	Z^3+Z_2	$Z+Z_2$	$Z+Z_2$	0
n=1	0	0	0	0
n=2	Z^4	$Z+Z_2^2$	Z_2	0
n=3	0	0	0	0
n=4	Z^3+Z_2	$Z+Z_2$	$Z+Z_2$	0
n=5	0	0	0	0
n=6	Z^4	$Z+Z_2^2$	Z_2	0
n=7	0	0	0	0
$E^2_{p,-n}$	p=0	p=1	p=2	p=3

Candidate K-groups

$K_0=K^0$	Z^3+Z_2
$K_{-1}=K^1$	$Z+Z_2^2$
$K_{-2}=K^2$	Z^5
$K_{-3}=K^3$	$Z+Z_2$
$K_{-4}=K^4$	Z^3+Z_2
$K_{-5}=K^5$	$Z+Z_2^2$
$K_{-6}=K^6$	Z^5
$K_{-7}=K^7$	$Z+Z_2$

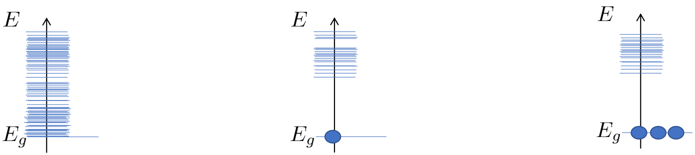
Outline

- Math behind invertible phases: generalized (co)homology theory [2] KS=Xiong=Gomi, 1810.00801
- Real-space Atiyah-Hirzebruch spectral sequence for crystalline invertible phases [2] KS=Xiong=Gomi, 1810.00801
- Case of free fermions with transnational symmetry: k-space Atiyah-Hirzebruch spectral sequence [1] KS=Sato=Gomi, 1802.06694
- Classification tables [3] KS=Ono, 2304.01827

- Compared to the k-space classification of topological insulators/superconductors, the real-space classification may be less familiar.
- I mainly talk about real-space construction.
- I can not talk about the details. This slide is super brief.

Invertible phases

... Gapped phases without ground state degeneracy



Gapless phases
Critical systems, SSB phases of continuous symmetry, ...

Invertible phases
Topological insulators/superconductors, Haldane chain, integer quantum Hall states, ...

Long-range entangled phases
Fractional quantum Hall states, superconductors with dynamical gauge field, (SSB of discrete symmetry, fraction phases), ...

Gapped phases = topological phases

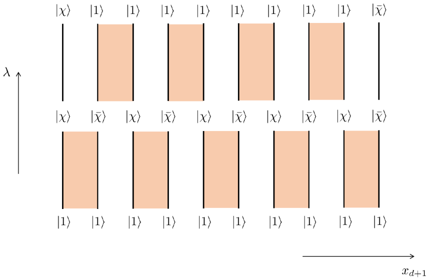
Two properties of invertible phases which are closely related:

(i) Invertibility $\to$ Ω-spectrum structure

(ii) Bulk-boundary correspondence $\to$ boundary map of homological theories

(i) Invertibility and Kitaev's Ω-spectrum conjecture [Kitaev '11, '13, '15 (videos), Gaiotto=Johnson-Freyd, 1712.07950]

- Given a d-D invertible state $|\chi\rangle$, there exists its "inverse" $|\bar{\chi}\rangle$ such that $|\chi\rangle \otimes |\bar{\chi}\rangle \sim |1\rangle \otimes |1\rangle$.
- Using this, given a d-D invertible state $|\chi\rangle$, an adiabaic cycle (Thouless pump) in $(d+1)$-D is canonically constructed:



- This suggests the homotopy equivalence

$$F_d \sim \Omega F_{d+1} := \{|\chi_{d+1}(t)\rangle : [0,1] \to F_{d+1} \,|\, |\chi_{d+1}(0)\rangle = |\chi_{d+1}(1)\rangle = |1_{d+1}\rangle\},$$

where $F_d =$ the "space of d-D invertible states".

(i) Invertibility and Kitaev's Ω-spectrum conjecture (cont.) [Kitaev '11, '13, '15 (videos), Gaiotto=Johnson-Freyd, 1712.07950]

- The sequence of spaces $\{F_d\}_{d\in\mathbb{Z}}$ with the relations $F_d \sim \Omega F_{d+1}$ is called the Ω-spectrum.
- As a matter of mathematical fact, an Ω-spectrum $\{F_d\}_{d\in\mathbb{Z}}$ defines a generalized cohomology and homology theories:

$$h^d(X,Y) := [X/Y, F_d]$$

: Family of d-D invertible states over the parameter space (X,Y),

$$h_n(X,Y) := \mathrm{colim}_{k\to\infty}[S^{n+k}, (X/Y)\wedge F_k]$$

: "Degree-n" invertible phases over the real space (X,Y).

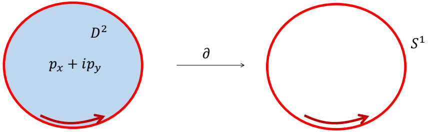
(The meaning of "degree-n" is explained soon later.)

- Lesson. Physics with invertibility/adiabatic pump should be described by a generalized (co)homology theory.

(ii) Bulk-boundary correspondence is the boundary map of homology

- If bulk is nontrivial, the boundary exhibits a gapless/SSB/LRE phase protected by anomalous symmetry.
- Ex: Haldane chain protected by either TRS or $\mathbb{Z}_2 \times \mathbb{Z}_2$ onsite symm.

- The bulk-boundary correspondence reminds us the boundary map of homology theory



$$h_n(D^2, \partial D^2) \xrightarrow{\partial} \to h_{n-1}(\partial D^2)$$

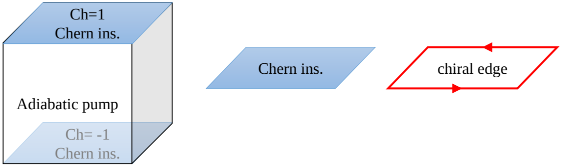
What is the degree n?

- $h_{n=0}(X, Y)$: classification of invertible phase over (X, Y), since

$$h_0(D^n, \partial D^n) \underset{\text{Poincare}}{=} h_{-n}(\mathrm{pt}) = h^n(\mathrm{pt}) = [\mathrm{pt}, F_n] = \pi_0(F_n).$$

- The meaning of the generic n-th homology is obtained by considering what is the physical phenomenon living in the boundary of the n-th homology.

$$\xrightarrow{\partial} \underset{\text{Adiabatic pump}}{n=1} \xrightarrow{\partial} \underset{\text{Invertible phase}}{n=0} \xrightarrow{\partial} \underset{\text{Anomaly}}{n=-1} \xrightarrow{\partial}$$



What is the degree n? (cont.)

$\vdots$

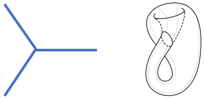
- $h_1(X,Y) :=$ the classification of adiabatic pumps over the real space X which may create an invertible state on $Y \subset X$.
- $h_0(X,Y) :=$ the classification of invertible phases over the real space X which may have anomaly on $Y \subset X$.
- $h_{-1}(X,Y) :=$ the classification of anomalies over the real space X which may have a "source/sink" of an anomalous excitation on $Y \subset X$.

$\vdots$

How useful is it?

- There are practical benefits to studying invertible phases:

- The real space X can be an arbitrary real space and need not be a manifold (manifold is locally Euclidean). For example, we can ask what is the classification of invertible phases over a trijunction,the Klein bottle, etc.



- The generalized homology description is based on real spaces on which invertible phases defined, meaning that it is straightforward to implement spatial symmetry (point and space group symmetry). $\Rightarrow$ Equivariant homology $h_n^G(X, Y)$.
- In particular, the Atiyah-Hirzebruch Spectral Sequence (AHSS) gives us a systematic way to thinking the interplay of crystalline symmetry and physics of invertible phases.

Real-space AHSS: Literature

- Free fermion topological phases are classified by K-homology over real spaces [Kitaev, 0901.2686]:

Periodic table for topological insulators and superconductors

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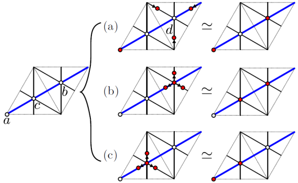
the same component — a contradiction.

A gapped local system on a compact metric space L (say, a manifold with or without boundary) is characterized by a K-homology class $\xi \in K_q^{\mathbb{R}}(L)$, where q is defined (mod 8). K-homology (see e.g. [30]) and the related noncommutative geometry [31] are advanced

- Generalized (co)homology/Ω-spectrum proposal [Kitaev '11,'13 (videos)].
  $\to$ Even for many-body invertible phases, they can be classified by a generalized homology over real spaces.
- Two similar strategies appeared in the study of crystalline invertible phases:
  - Dimensional reduction— to classify invertible phases with crystalline symmetry [Song=Huang=Fu=Hermele 1604.08151],
  - Lattice homotopy— to classify LSM-type theorems [Po=Watanabe=Jian=Zaletel 1703.06882].

Real-space AHSS: Literature (cont.)

- These two procedures are summarized as *trivializing something nontrivial living in low-dimensional real spaces by something nontrivial living in higher-dimensional real spaces*: [Figure from Huang=Song=Huang=Hermele, 1705.09243]



- Reminds us the Atiyah-Hirzebruch spectral sequence (AHSS) of the generalized homology theory.
- We reconstruct these studies in terms of the AHSS of the generalized homology theory KS=Xiong=Gomi 1810.00801.
- Our contribution:
  - The complete mathematical structure behind the real-space approach.
  - Importance of higher differentials and group extension— they can be interpreted and computed using physics knowledge.
- AHSS paper at the same time: Song=Fang=Qi 1810.11013, Song=Huang=Qi=Fang=Hermele 1810.02330, Else=Thorngren 1810.10539.
- Math paper: Freed=Hopkins 1901.06419.

(Homological) spectral sequence in general

- A spectral sequence starts from the E^1-page, which is something commutable.

$$E^1 = \{E^1_{p,-n}\}_{p,n}$$

- Compute the rth differential $(n = 1, 2, \dots,)$

$$d^r_{p,-n} : E^r_{p,-n} \to E^r_{p-r,-n+r-1}, \qquad d^r \circ d^r = 0.$$

- The next page is defined as the homology of d^r,

$$E^{r+1}_{p,-n} = \mathrm{Ker}\ d^r_{p,-n} / \mathrm{Im}\ d^r_{p+r,-n-r+1}.$$

- When this iteration converges at some E^q-page:

$$E^1 \Rightarrow E^2 \Rightarrow \cdots E^q = E^{q+1} = \cdots =: E^\infty,$$

the E^∞-page $\{E^\infty_{p,-n-p}\}_p$ "approximates" the homology group $h_{-n}(X, Y)$.

Real-space AHSS (1) : Cell decomposition

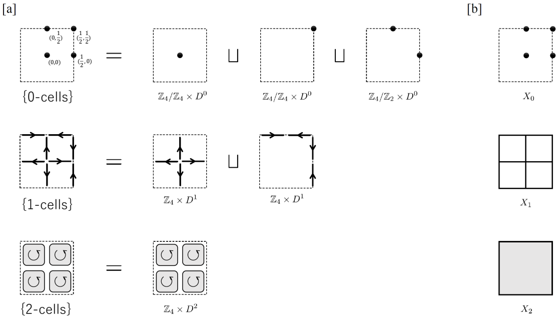
- The starting point of the AHSS is to give a filtration of the space X,

$$X_0 \subset X_1 \subset \cdots \subset X.$$

- A useful filtration is the symmetric cell-decomposition

$$X = \{\text{0-cells}\} \cup \{\text{1-cells}\} \cup \{\text{2-cells}\} \cup \cdots, \quad X_0 = \{\text{0-cells}\}, \quad X_p = X_{p-1} \cup \{p\text{-cells}\}.$$

- Ex: 2d real space with C_4 rotation and $\mathbb{Z}^2$ translation symmetry:

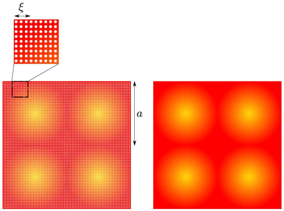


Topological Crystalline Liquid [Thorngren=Else 1612.00846]

- 1-cell, 2-cell??
- Please keep in your mind the following picture:

$$a \text{ (the scale of lattice translation)} \quad \gg \quad \xi \text{ (the scale of microscopic degrees of freedom)}.$$

- $\Rightarrow$ $(p \geq 1)$-D invertible states over p-cells make sense.
- This is not the situation of cond-mat, however, it is useful and may works for topological classification because the effective theory would be "topological". (Figure from [Thorngren=Else 1612.00846])



(a) Smooth state (b) Topological limit

FIG. 1. (a) In a smooth state, the lattice spacing and the correlation length ξ are much less than the unit cell size a and the radius of spatial variation. (b) The topological response of a crystalline topological liquid is captured by a spatially-dependent TQFT that captures the spatial dependence within each unit cell but "forgets" about the lattice.

Real-space AHSS (2) : E^1-page

- We want to compute the homology group $h_0^G(X)$, the classification of invertible phases over the real space X with crystalline symmetry G.
- The E^1-page is defined by

$$E^1_{p,-n} := h^G_{p-n}(X_p, X_{p-1}),$$

the $(p-n)$-th homology over X_p relative to X_{p-1}.

- It can be rewritten as

$$E^1_{p,-n} \cong \prod_{j \in \{p\text{-cells}\}} h_{p-n}^{G_{D^p_j}}(D^p_j, \partial D^p_j) \cong \prod_{j \in \{p\text{-cells}\}} \tilde{h}_{p-n}^{G_{D^p_j}}(D^p_j / \partial D^p_j (\cong S^p))$$

$$\cong \prod_{j \in \{p\text{-cells}\}} h_{-n}^{G_{D^p_j}}(pt) \qquad \text{(suspension iso.)}.$$

- Thus, $E^1_{p,-q}$ is collection of ablian groups of n-D invertible phases with the little group $G_{D^p_j}$, which fixes the p-cell D^p_j.

Real-space AHSS (3.1): The first differential d^1

- The E^1-page hosts the "local information" of invertible phases.
- We should properly glue the local information together, which is partly done by the first differential $d^1_{p,-q} : E^1_{p,-q} \to E^1_{p-1,-q}$.
- Math def:

$$d^1_{p,-n} : E^1_{p,-n} = h^G_{p-n}(X_p, X_{p-1}) \xrightarrow{\partial \text{ (bulk−bdy)}} h^G_{p-n-1}(X_{p-1}) \xrightarrow{\text{inclusion}} h^G_{p-n-1}(X_{p-1}, X_{p-2}) = E^1_{p-1,-n}$$

- This is essentially the bulk-boundary correspondence: how invertible states over p-cells show anomalies over adjacent $(p-1)$-cells. — easy to compute.
- Adiabatic deformation of 0D states [Huang=Song=Huang=Hermele 1705.09243] $\Rightarrow d^1_{1,0}$. Lattice homology [Po=Watanabe=Jian=Zaletel 1703.06882] $\Rightarrow d^1_{1,-1}$.

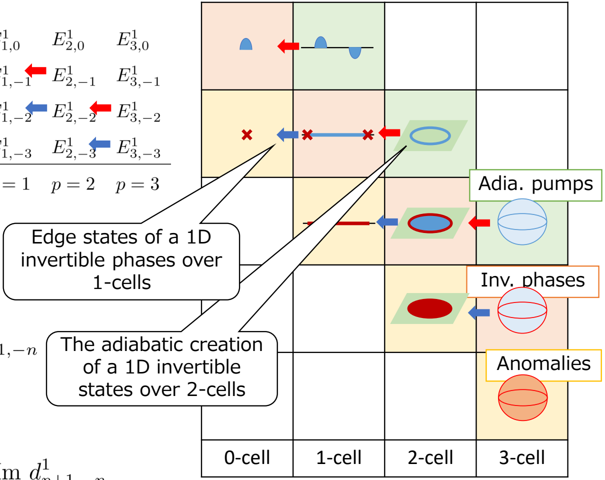
Real-space AHSS (3.1): The first differential d^1 (cont.)

0D Inv	$n=0$	$E^1_{0,0}$	$E^1_{1,0}$	$E^1_{2,0}$	$E^1_{3,0}$
1D Inv	$n=1$	$E^1_{0,-1}$	$E^1_{1,-1}$	$E^1_{2,-1}$	$E^1_{3,-1}$
2D Inv	$n=2$	$E^1_{0,-2}$	$E^1_{1,-2}$	$E^1_{2,-2}$	$E^1_{3,-2}$
3D Inv	$n=3$	$E^1_{0,-3}$	$E^1_{1,-3}$	$E^1_{2,-3}$	$E^1_{3,-3}$
	$E^1_{p,-n}$	$p=0$	$p=1$	$p=2$	$p=3$

$$d^1_{p,-n} : E^1_{p,-n} \to E^1_{p-1,-n}$$

Ker/Im

$$E^2_{p,-n} := \mathrm{Ker}\ d^1_{p,-n} / \mathrm{Im}\ d^1_{p+1,-n}$$

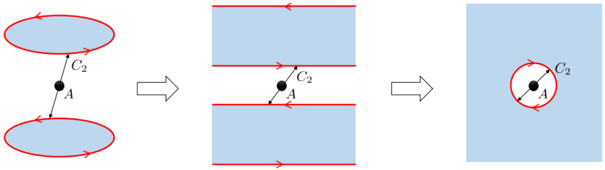


AHSS (3.2): The second differential d^2

- This is not the end of the story.
- The E^2-page is still the "local information" in the sense that they are glued together only on the 1-skeleton X_1, not the full space X.
- We should further compute the anomaly-free condition between p- and $(p-2)$-cells, which is the second differential

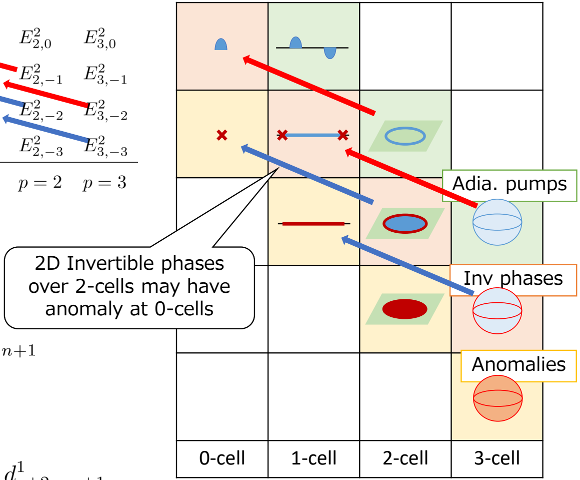
$$d^2_{p,-n} : E^2_{p,-n} \to E^2_{p-2,-n+1}.$$

- An example of nontrivial second differential $d^2_{2,-2}$ is 2D even parity superconductor: On a single $p_x + ip_y$ superconductor, the even parity condition $\hat{C}_2^2 = \mathrm{Id}$ enforces a vortex defect at the inversion center with Majorana zero, meaning that the system can not be gapped.



Real-space AHSS (3.2): The second differential d^2 (cont.)

0D Inv	$n=0$	$E^2_{0,0}$	$E^2_{1,0}$	$E^2_{2,0}$	$E^2_{3,0}$
1D Inv	$n=1$	$E^2_{0,-1}$	$E^2_{1,-1}$	$E^2_{2,-1}$	$E^2_{3,-1}$
2D Inv	$n=2$	$E^2_{0,-2}$	$E^2_{1,-2}$	$E^2_{2,-2}$	$E^2_{3,-2}$
3D Inv	$n=3$	$E^2_{0,-3}$	$E^2_{1,-3}$	$E^2_{2,-3}$	$E^2_{3,-3}$
	$E^2_{p,-n}$	$p=0$	$p=1$	$p=2$	$p=3$



$$d^2_{p,-n} : E^2_{p,-n} \to E^2_{p-2,-n+1}$$

⇓ Ker/Im

$$E^3_{p,-n} := \mathrm{Ker}\ d^1_{p,-n} / \mathrm{Im}\ d^1_{p+2,-n+1}$$

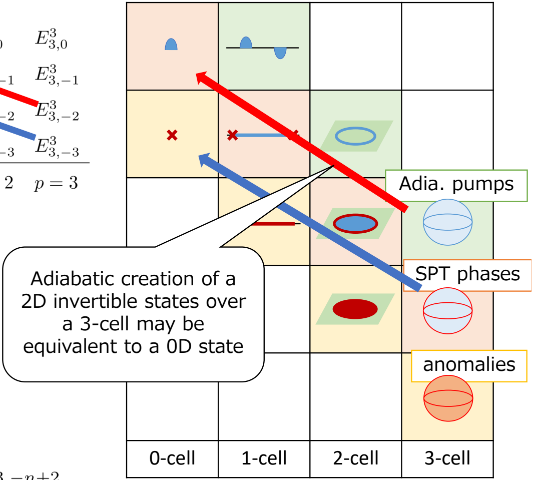
Real-space AHSS (3.3): The third differential d^3

0D Inv	$n=0$	$E^3_{0,0}$	$E^3_{1,0}$	$E^3_{2,0}$	$E^3_{3,0}$
1D Inv	$n=1$	$E^3_{0,-1}$	$E^3_{1,-1}$	$E^3_{2,-1}$	$E^3_{3,-1}$
2D Inv	$n=2$	$E^3_{0,-2}$	$E^3_{1,-2}$	$E^3_{2,-2}$	$E^3_{3,-2}$
3D Inv	$n=3$	$E^3_{0,-3}$	$E^3_{1,-3}$	$E^3_{2,-3}$	$E^3_{3,-3}$
	$E^3_{p,-n}$	$p=0$	$p=1$	$p=2$	$p=3$

$$d^3_{p,-n} : E^3_{p,-n} \to E^3_{p-3,-n+2}$$

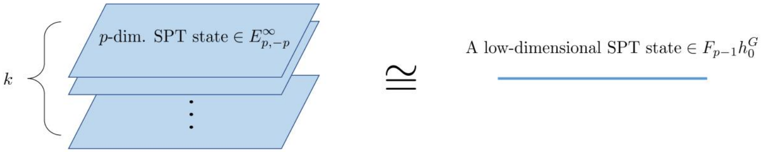
Ker/Im

$$E^4_{p,-n} := \text{Ker } d^1_{p,-n} / \text{Im } d^1_{p+3,-n+2} = E^\infty_{p,-n}$$



Real-space AHSS (4): Group extension

- E^{∞}-page itself does not provide the group structure of invertible phases.
- This is because the stacking higher dimensional invertible phases may be a lower-dimensional invertible phase. $\Rightarrow$ nontrivial group extension.



- Ex. Trivial extension:

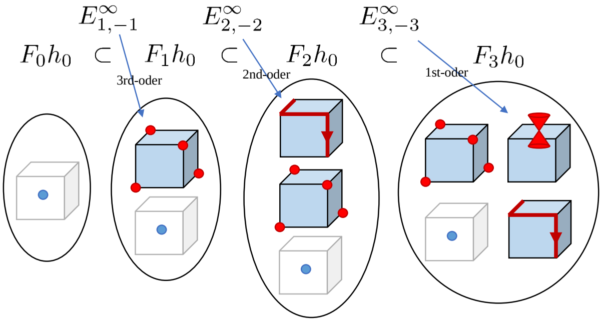
$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 + \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0.$$

Ex. Nontrivial extension:

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$

Real-space AHSS (4): Group extension (cont.)

- Introduce
$$F_p h_0 := \mathrm{Im}\,[h_0^G(X_p) \to h_0^G(X)], \qquad p = 0, 1, \dots,$$
the classification of invertible phases at most p-dimensions.
- E^∞-page is the "ratio" between adjacent layers of filtration: $F_p h_0 / F_{p-1} h_0 \cong E^\infty_{p,-p}$.



- Remark. $E^\infty_{p,-p}$ is the classification of $(d-p+1)$th-order invertible phases [Huang=Song=Huang=Hermele 1705.09243, Trifunovic=Brouwer 1805.02598].

LSM type theorems

- Im $d^r_{r,-r}$ = SPT-LSM theorm [Lu 1705.04691, Yang=Jiang=Vishwanath=Ran 1705.05421]:
  In DOF consisting only of Im $d^r_{r,-r} \subset E^1_{0,-1}$, if we get an invertible state, this state must be a nontrivial invertible state.

- $E^{\infty}_{0,-1}$ = LSM theorem:
  DOF consisting only of $E^{\infty}_{0,-1} \subset E^1_{0,-1}$ can not have an invertible state.

Free fermions

- A subclass of invertible phases — invertible phases of fermion bi-linear Hamiltonian

$$\hat{H}=\sum_{xy,ij} c_{xi}^{\dagger}\mathcal{H}_{xi,yj}c_{yj}.$$

- Discussed to be the same as the classification of Dirac mass $M(\boldsymbol{r})$ over the d-D Euclidean space $\mathbb{R}^d$ with crystalline G symmetry.

$$\mathcal{H}=\sum_{\mu=1}^{d} -i\partial_\mu\gamma_\mu+M(\boldsymbol{r}).$$

$\Rightarrow$ K-homology $K_0^G(\mathbb{R}^d)$.

- The real-space AHSS applies to free fermions (K-homology) as well.
- Current status:
    - The first differential $d^1_{p,-n}$ is easy to compute $\Rightarrow$ the list of E^2-pages for 1651 magnetic space groups [KS=Ono 2304.01827].
    - There is an algorithm to compute the second differential $d^2_{2,-2}$ for TRS superconductors with conventional pairing symmetry [Ono=KS=Watanabe 2206.02489].
    - No algorithm for higher differentials $d^{r\geq 2}_{p,-n}$ and solving the group extension.

Free fermions with lattice translational invariance — Band theory

- With lattice translation symmetry, the free fermion Hamiltonian $\mathcal{H}$ is characterized by a parameter family of Hermite matrices $\mathcal{H}_{\boldsymbol{k}}$:

$$\mathcal{H}_{x,y} = \mathcal{H}_{x-y} \xrightarrow{\text{Fourier tr.}} \mathcal{H}_{\boldsymbol{k}}.$$

- The classification of invertible phases of free fermions is the same as the classification of Hamiltonians $\mathcal{H}_{\boldsymbol{k}}$ over the Brillouin zone (BZ) torus T^d
$\Rightarrow$ K-cohomology theory $K^{-n}(T^d)$ [Freed=Moore 1208.5055].

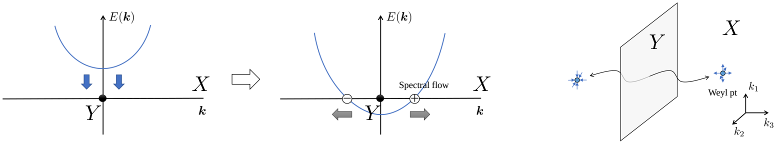
- K-cohomology theory $K^{-n}(X)$: the classification of "functions" over the space X. (contravariant functor)
- K-homology theory $K_n(X)$: the classification of "textures" over the space X. (covariant functor)

Connected homomorphism of K-cohomology

- The degree-0 K-group $K^0(X)$ is represented by a pair of gapped Hamiltonian $(H_{\boldsymbol{k}}, H'_{\boldsymbol{k}})$ over X [1].
- We want to understand the connected homomorphism

$$d : K^0(Y) \to K^1(X, Y).$$

- This is naturally understood as the creation of gapless points associated with the topological transition over $Y \subset X$.



- This also implies that degree-1 K-group $K^1(X)$ is the classification of gapless Hamiltonians.

[1] Precisely, the K-group $K^0(X)$ is represented by a triple (E, H, H'), where E is a vector bundle over X and H, H' are flat and gapped Hamiltonians acting on E [Karoubi's book].

Physical meaning of degree n of K-cohomology group $K^{-n}(X)$

- $K^{-1}(X)$: One-parameter family of gapped Hamiltonian over X.

$$\mathcal{H}_{\boldsymbol{k},\theta} = \boldsymbol{k} \cdot \boldsymbol{\gamma} + \theta\Gamma.$$

$\cong$ the gauge transformation over X.

- $K^{0}(X)$: Gapped Hamiltonian over X.

$$\mathcal{H}_{\boldsymbol{k}} = \boldsymbol{k} \cdot \boldsymbol{\gamma} + M.$$

- $K^{1}(X)$: Gapless Hamiltonian over X.

$$\mathcal{H}_{\boldsymbol{k}} = \boldsymbol{k} \cdot \boldsymbol{\gamma}.$$

- $K^{2}(X)$: Singular Hamiltonian over X.

$$\mathcal{H}_{\boldsymbol{k}} = \mathrm{Im}\ \log \left[k_d + i \sum_{\mu=1}^{d-1} k_\mu \gamma_\mu \right].$$

k-space AHSS: d_1

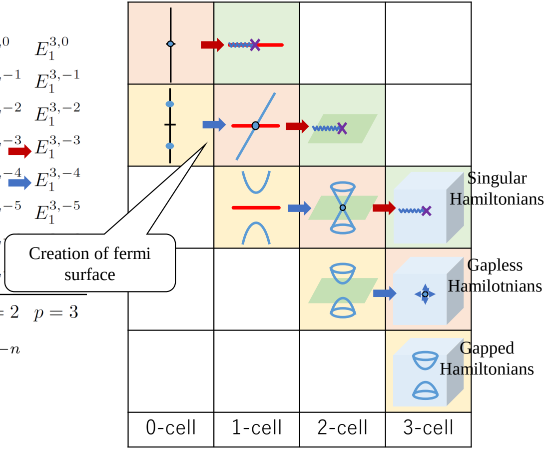
- The mathematical structure is dual to the real-space AHSS. All the arrows are reversed.

$n=0$	$E_1^{0,0}$	$E_1^{1,0}$	$E_1^{2,0}$	$E_1^{3,0}$
$n=1$	$E_1^{0,-1}$	$E_1^{1,-1}$	$E_1^{2,-1}$	$E_1^{3,-1}$
$n=2$	$E_1^{0,-2}$	$E_1^{1,-2}$	$E_1^{2,-2}$	$E_1^{3,-2}$
$n=3$	$E_1^{0,-3}$	$E_1^{1,-3}$	$E_1^{2,-3}$	$E_1^{3,-3}$
$n=4$	$E_1^{0,-4}$	$E_1^{1,-4}$	$E_1^{2,-4}$	$E_1^{3,-4}$
$n=5$	$E_1^{0,-5}$	$E_1^{1,-5}$	$E_1^{2,-5}$	$E_1^{3,-5}$
$n=6$	$E_1^{0,-6}$	$E_1^{1,-6}$	$E_1^{2,}$ [illegible]	[illegible]
$n=7$	$E_1^{0,-7}$	$E_1^{1,-7}$	$E_1^{2,}$ [illegible]	[illegible]
$E_1^{p,-n}$	$p=0$	$p=1$	$p=2$	$p=3$

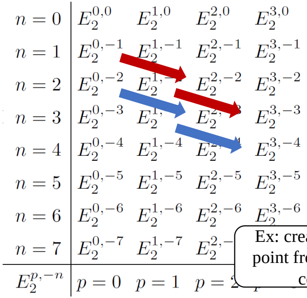
$$d_1 : E_1^{p,-n} \to E_1^{p+1,-n}$$

⇩ Ker/Im

$$E_2^{p,-n} := \text{Ker } d_1^{p,-n} / \text{Im } d_1^{p-1,-n}$$



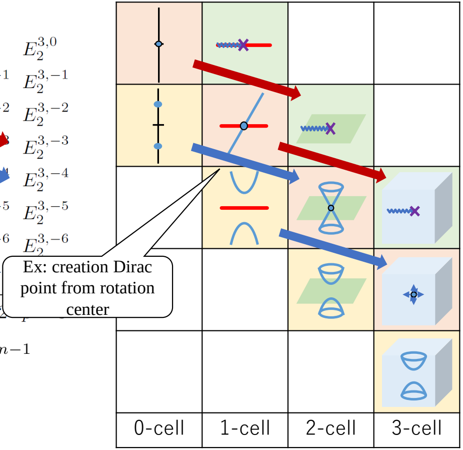
k-space AHSS: d_2



$$d_2 : E_2^{p,-n} \to E_2^{p+2,-n-1}$$

Ker/Im

$$E_3^{p,-n} := \mathrm{Ker}\ d_2^{p,-n} / \mathrm{Im}\ d_2^{p-2,-n+1}$$



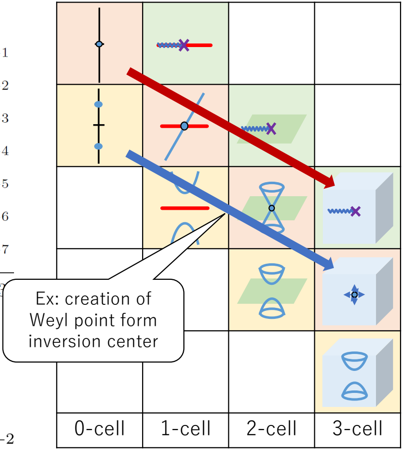
k-space AHSS: d_3

$n=0$	$E_3^{0,0}$	$E_3^{1,0}$	$E_3^{2,0}$	$E_3^{3,0}$
$n=1$	$E_3^{0,-1}$	$E_3^{1,-1}$	$E_3^{2,-1}$	$E_3^{3,-1}$
$n=2$	$E_3^{0,-2}$	$E_3^{1,-2}$	$E_3^{2,-2}$	$E_3^{3,-2}$
$n=3$	$E_3^{0,-3}$	$E_3^{1,-3}$	$E_3^{2,-3}$	$E_3^{3,-3}$
$n=4$	$E_3^{0,-4}$	$E_3^{1,-4}$	$E_3^{2,-4}$	$E_3^{3,-4}$
$n=5$	$E_3^{0,-5}$	$E_3^{1,-5}$	$E_3^{2,-5}$	$E_3^{3,-5}$
$n=6$	$E_3^{0,-6}$	$E_3^{1,-6}$	$E_3^{2,-6}$	$E_3^{3,-6}$
$n=7$	$E_3^{0,-7}$	$E_3^{1,-7}$	$E_3^{2,-7}$	$E_3^{3,-7}$
$E_3^{p,-n}$	$p=0$	$p=1$	$p=2$	$p=3$

$$d_3 : E_3^{p,-n} \to E_3^{p+3,-n-2}$$

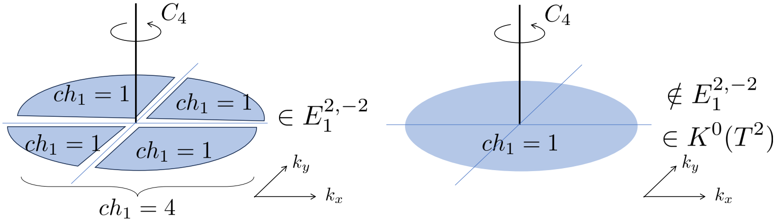
Ker/Im

$$E_4^{p,-n} := \mathrm{Ker}\ d_3^{p,-n} / \mathrm{Im}\ d_3^{p-3,-n+2} = E_\infty^{p,-n}$$



k-space AHSS: Group extension

- E_∞-page is insufficient for the complete classification of the band structure.
- This is because E_1-page misses the higher-dimensional structure of Hamiltonian at the high-symmetric point.
- Ex: C_4-rotation symmetry:



- To get the group structure for E_∞-page, we have to solve the group extension problem: If a stack of band structures is non-trivial at a high-symmetry point, does a non-trivial band structure appear in adjacent high-dimensional cells?

Example of group extension: 1D gapless edge state of class D superconductors

Space dimension : 1

System : SC

Factor system : spinless

Symmetry type : magnetic rod group

OG number : 1.1

BNS symbol : p1

Class : D

Pairing symmetry : A

Character table of A:

1
1

k-space E_2-page

n=0	Z_2^2	Z
n=1	0	0
n=2	Z	Z_2
n=3	0	0
n=4	0	Z
n=5	0	0
n=6	Z	0
n=7	Z_2^2	0
$E_2^{p,-n}$	p=0	p=1

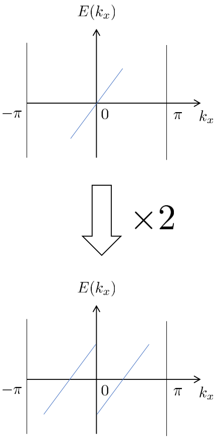
Real-space E^2-page

n=0	Z_2	Z_2
n=1	Z_2	Z_2
n=2	Z	Z
n=3	0	0
n=4	0	0
n=5	0	0
n=6	Z	Z
n=7	0	0
$E^2_{p,-n}$	p=0	p=1

Candidate K-groups

$K_0=K^0$	Z_2^2
$K_{-1}=K^1$	$Z+Z_2$
$K_{-2}=K^2$	Z
$K_{-3}=K^3$	0
$K_{-4}=K^4$	0
$K_{-5}=K^5$	Z
$K_{-6}=K^6$	Z
$K_{-7}=K^7$	Z_2

Nontrivial group extension



$(1,0) \in \mathbb{Z}_2^2 = E_2^{0,1}$

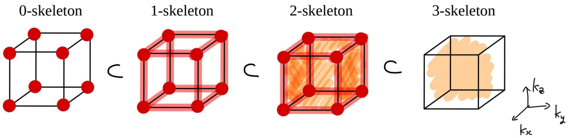
$1 \in \mathbb{Z} = E_2^{1,0}$

k-space AHSS: Current status

- The first differential $d_1^{p,-n}$ is easy to compute
  $\Rightarrow$ the list of E_2-pages for 1651 magnetic space groups [KS=Ono 2304.01827].
- No algorithm for higher differentials $d_{r\geq 2}^{p,-n}$.
- No algorithm for solving the group extension.

From k-space AHSS to topological invariants

- We do not know the concrete expression for many topological invariants.



- Only the number of irreps at high-symmetry points (0D invariants) and the topological invariants using the whole BZ torus (ex: Chern number, winding number) are known.
- For intermediate situations, no systematic construction of topological invariants defined over 1D and 2D subspace of BZ torus is known.
- Remark. Many ad hoc constructions are known. Wilson line, mirror Chern/winding number, glide $\mathbb{Z}_2/\mathbb{Z}_4$ invariant, etc.

- The k-space AHSS can be used to construct k-space topological invariants systematically.
- We have constructed all the topological invariants defined over the 1-dimensional subspace of BZ torus, for the cases where the Altland-Zirnbauer symmetry class over 1-cell is either AIII, AI, DIII, or CI [Ono=KS 2311.15814]. (website)

Classificaion table: Strategy

- The real-space and k-space AHSSs give the different E^{∞}- and E_{∞}-pages, respectively.
- But converge the same classification group! [2]

$$K_{-n}(\mathbb{R}^d) \cong K^n(T^d).$$

- Even if no formulas of higher-differentials d^r, d_r of AHSS for $r \geq 2$ are known, we can try all possible d^r, d_r, because they are represented by integer-valued matrices

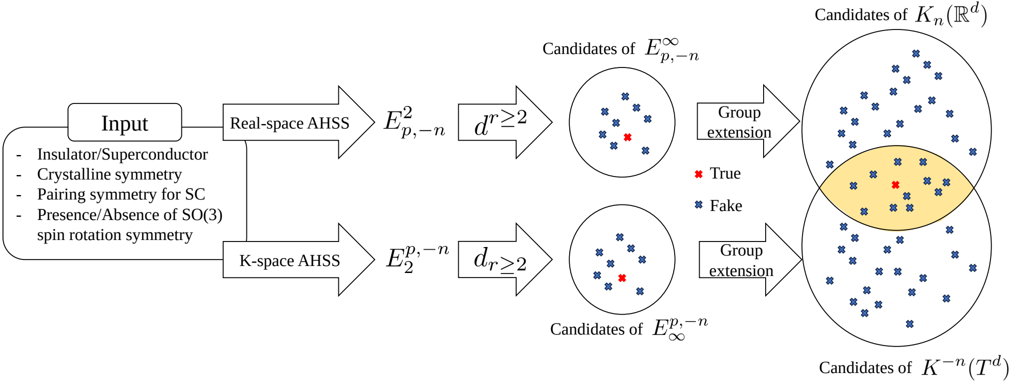
$$\begin{pmatrix} * & * & \cdots \\ * & * & \cdots \\ \vdots & & \ddots \end{pmatrix}.$$

- Therefore, all possible E^{∞}- and E_{∞}-pages can be listed in a brute-force calculation.
- Also, the group extension of $\mathbb{Z}$-modules is characterized by an integer-valued matrix [3], meaning that all the possible group extensions can be listed.
- The true classification group lives in the intersection of possibilities from real-space AHSS and k-space AHSS. $\Rightarrow$ strong constraint on the true classification!

[2] For crystalline symmetry, the isomorphism was proven in [Gomi=Kubota=Thiang 2102.00393].

[3] The linear structure is known as the Baer sum.

Classificaion table: Strategy (cont.)



Summary of results

Symmetry type	Number of symmetry settings	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
3D, magnetic space groups	31050	0.594	0.779	0.886	0.859	0.749	0.663	0.539	0.519
3D, magnetic point groups	2346	0.788	0.869	0.931	0.939	0.941	0.817	0.649	0.682
2D, magnetic layer groups	9264	0.88	0.966	0.975	0.964	0.941	0.914	0.801	0.751
2D, magnetic point groups	2364	0.951	0.978	0.976	0.974	0.984	0.936	0.769	0.795
1D, magnetic rod groups	6488	0.943	0.996	0.999	0.996	0.981	0.975	0.945	0.897
1D, magnetic point groups	1946	0.958	0.997	0.998	0.997	0.995	0.984	0.93	0.904

TABLE VI. The proportion of symmetry classes for which the candidate K-group SK_{-n} is uniquely determined. The first column indicates the spatial dimension and the presence or absence of lattice translational symmetry.

- $n=0$: gapped band structures
- $n=1$: gapless band structures
- $n=7$: adiabatic cycles = gapped Floquet phases = large gauge transformations

Setting:

- Insulators for spinless and spinful electrons.
- Superconductors for spinless and spinful electrons and with and without spin $SO(3)$ symmetry.
- 1651 magnetic space groups.
- For superconductors, all possible pairing symmetry of gap function.
- The results are summarized in the webpage (link).

Meaning of table

TI spinful 721

www2.yukawa.kyoto-u.ac.jp/~ken.shiozaki/ahss/e2/TI_spinful_721.html

Space dimension : 3

System : TI

Factor system : spinful

Symmetry type : magnetic space group

BNS number : 85.61

BNS symbol : P4'/n

Four 4nd-order TIs = Atomic insulators

One 2nd-order TI

One invariant over 2-dim. subspace

Three independent irreps

k-space E_2-page

n=0	Z^3	$Z+Z_2^2$	Z	0
n=1	0	0	0	0
n=2	Z^4	$Z+Z_2$	Z_2	0
n=3	0	0	0	0
n=4	Z^3	$Z+Z_2^2$	Z	0
n=5	0	0	0	0
n=6	Z^4	$Z+Z_2$	Z_2	0
n=7	0	0	0	0
$E_2^{p,-n}$	p=0	p=1	p=2	p=3

Real-space E^2-page

n=0	Z^3+Z_2	$Z+Z_2$	$Z+Z_2$	0
n=1	0	0	0	0
n=2	Z^4	$Z+Z_2^2$	Z_2	0
n=3	0	0	0	0
n=4	Z^3+Z_2	$Z+Z_2$	$Z+Z_2$	0
n=5	0	0	0	0
n=6	Z^4	$Z+Z_2^2$	Z_2	0
n=7	0	0	0	0
$E^2_{p,-n}$	p=0	p=1	p=2	p=3

Candidate K-groups

$K_0=K^0$	Z^3+Z_2
$K_{-1}=K^1$	$Z+Z_2^2$
$K_{-2}=K^2$	Z^5
$K_{-3}=K^3$	$Z+Z_2$
$K_{-4}=K^4$	Z^3+Z_2
$K_{-5}=K^5$	$Z+Z_2^2$
$K_{-6}=K^6$	Z^5
$K_{-7}=K^7$	$Z+Z_2$

Classification of bulk TIs

Classification of gapless phases

Uniqueness of each low means that the classification is fixed only by E2 pages

Classification of adiabatic cycles

Meaning of table (cont.)

Space dimension : 3

System : SC

Factor system : spinful

Symmetry type : magnetic space group

BNS number : 160.66

BNS symbol : R3m1'

Class : D

Pairing symmetry : A_2

Character table of A_2:

1	m_{100}	3_{001}^{+}	3_{001}^{-}	m_{110}	m_{010}
1	*-1*	*1*	*1*	*-1*	*-1*

k-space E_2-page

n=0	Z_2^6	$Z^2+Z_2^2$	Z_2	0
n=1	Z_2^2	$Z+Z_2^2$	Z	Z_2
n=2	Z	Z_2^3	Z	0
n=3	Z	Z_2	Z_2^2	Z
n=4	0	Z^2	Z_2	0
n=5	0	Z	Z	Z_2
n=6	Z	0	Z	0
n=7	$Z+Z_2^4$	0	0	Z
$\mathbf{E_2^{p,-n}}$	**p=0**	**p=1**	**p=2**	**p=3**

Real-space E^2-page

n=0	Z_2^2	Z_2^3	Z_2	0
n=1	$Z+Z_2$	$Z+Z_2^2$	Z_2^2	Z_2
n=2	Z	Z^2	$Z+Z_2$	Z_2
n=3	0	0	Z	Z
n=4	0	0	0	0
n=5	Z	Z	0	0
n=6	Z	Z^2	Z	0
n=7	Z_2	Z_2	Z	Z
$\mathbf{E^2_{p,-n}}$	**p=0**	**p=1**	**p=2**	**p=3**

Candidate K-groups

$\mathbf{K_0=K^0}$	$Z^3+Z_2^3$	$Z^3+Z_4+Z_2^2$	$Z^3+Z_4^2+Z_2$	$Z^3+Z_2^4$	$Z^3+Z_4+Z_2^3$	$Z^3+Z_2^5$								
$\mathbf{K_{-1}=K^1}$	Z^4	Z^4+Z_2												
$\mathbf{K_{-2}=K^2}$	Z													
$\mathbf{K_{-3}=K^3}$	0													
$\mathbf{K_{-4}=K^4}$	Z^3													
$\mathbf{K_{-5}=K^5}$	Z^4													
$\mathbf{K_{-6}=K^6}$	$Z+Z_2$	$Z+Z_4$	$Z+Z_8$	$Z+Z_2^2$	$Z+Z_4+Z_2$	$Z+Z_2^3$								
$\mathbf{K_{-7}=K^7}$	Z_4^3	$Z_8+Z_4+Z_2$	$Z_8+Z_4^2$	$Z_8^2+Z_2$	$Z_4^2+Z_2^2$	$Z_4^3+Z_2$	$Z_8+Z_2^3$	$Z_8+Z_4+Z_2^2$	$Z_4+Z_2^4$	$Z_4^2+Z_2^3$	$Z_8+Z_2^4$	Z_2^6	$Z_4+Z_2^5$	Z_2^7

Multiple possibilities implies that the classification is NOT fixed by E2 pages

Summary and future works

- The real-space AHSS is applicable for any invertible phases. The underlying mathematical structure is the Ω-spectrum structure of invertible states.
- For free fermions with lattice translational symmetry, k-space AHSS is also available.
- Comparing real-space and k-space E_2-pages gives us a strong constraint for the K-group.
- All calculation details are in KS=Ono, 2304.01827.
  —Cell decomposition, irreducible characters, making the $E1$-page, formulas of d^1, d_1, listing possible d^r, d_r for $r \geq 2$, Baer sum, ...

Open questions:

- Algorithm to compute the higher differentials.
- Algorithm to compute the group extension.
- Algorithm to contract the k-space topological invariants from k-space AHSS [Ono=KS 2311.15814].
- Algorithm to provide the microscopic model Hamiltonians from real-space AHSS.
- Fermionic crystalline equivalence principle? [Debray 2102.02941, Manjunath=Calvera=Barkeshli 2210.02452]

Some useful mathematical facts and physical interpretation

- By design, the classification of n-dim. SPT phases is given by the disconnected parts of F_n,

$$\pi_0(F_n) = [pt, F_n] = h^n(pt).$$

- From the Poincaré duality and the suspension isomorphism,

$$h^n(pt) = h_{-n}(pt) = h_0(D^n, \partial D^n).$$

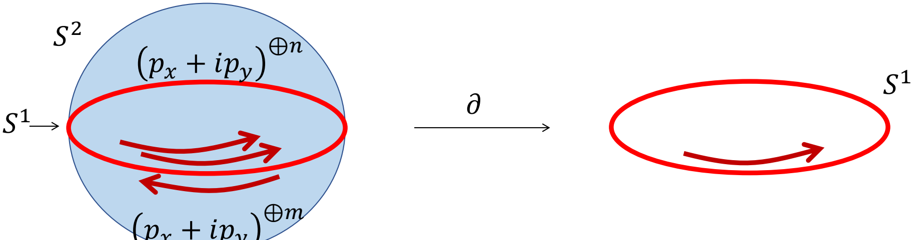
- $h_0(D^n, \partial D^n)$ can be identified with the classification of SPT phases over D^n relative to its boundary ∂D^n.

From SPT phases to a generalized homology theory

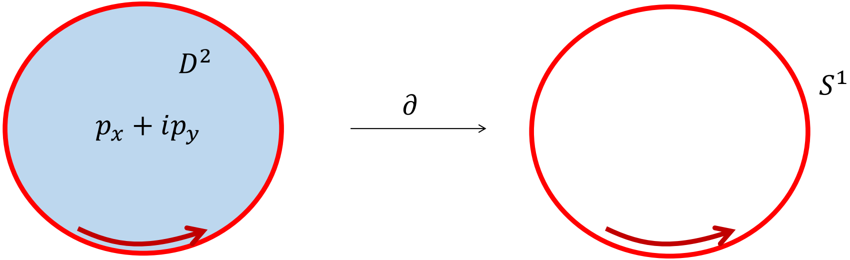
- $h_0(X,Y) :=$ the abelian group of SPT phases over a real space X which may have an quantum anomaly over a real space $Y \subset X$.
- We define the boundary map $\partial : h_0(X,Y) \to h_{-1}(Y)$ as the bulk-boundary correspondence.
- This implies $h_{-1}(Y)$ should be regarded as the abelian group of quantum anomaly over a real space Y.

Ex: Superconductors over $X = S^2$ that may have an anomalous edge state over the equator $Y = S^1$. We have

$$h_0(S^2,S^1) = \mathbb{Z} \times \mathbb{Z}, \qquad h_{-1}(S^1) = \mathbb{Z},$$
$$\partial : h_0(S^2,S^1) \to h_{-1}(S^1), \quad (n,m) \mapsto n - m.$$



- The ordinary bulk-boundary correspondence is the special case of the boundary map ∂ where $X = D^n$ and $Y = \partial D^n$.



"Physical definition" of $h_n(X,Y)$ v.s. the axioms

Let's consider if the above identification of the group $h_n(X,Y)$ with a physical phenomenon related to SPT phases satisfies the axioms.

- ✓ A covariant functor (Because of the real-space picture)
- ✓ (homotopy)
  If $f, f' : X \to X'$ are homotopic, then $f_* = f'_*$.
- ✓ (excision)
  For $A, B \subset X$, the inclusion $A \to A \cup B$ induces an isomorphism $h_n(A, A \cap B) \to h_n(A \cup B, B)$.
- ✓ (additivity)
  $h_n(\sqcup_\lambda X_\lambda, \sqcup_\lambda Y_\lambda) = \sqcup_\lambda h_n(X_\lambda, Y_\lambda)$.
- ✓ (exactness)
  For $Y \subset X$, there is a long exact sequence

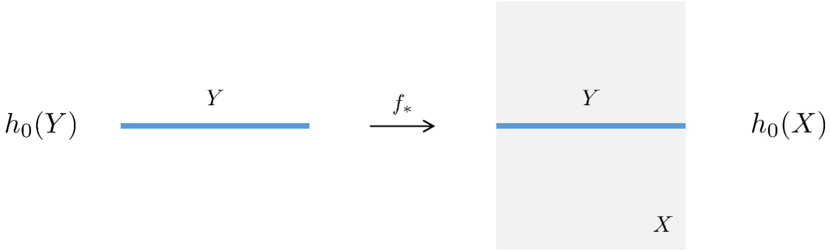
$$\cdots \to h_n(Y) \to h_n(X) \to h_n(X,Y) \xrightarrow{\partial} h_{n-1}(Y) \to \cdots$$

... It looks OK.

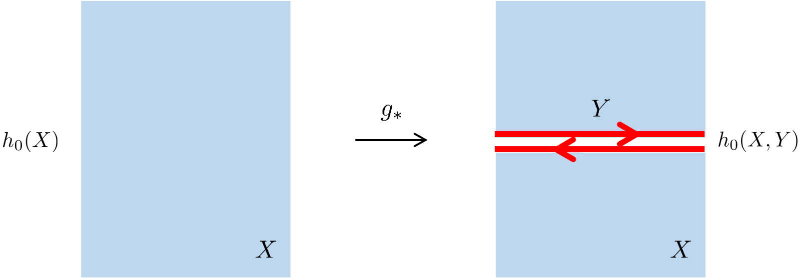
Exactness

$$\cdots \xrightarrow{\partial^1} h_0(Y) \xrightarrow{f_*^0} h_0(X) \xrightarrow{g_*^0} h_0(X,Y) \xrightarrow{\partial^0} h_{-1}(Y) \xrightarrow{f_*^{-1}} \cdots$$

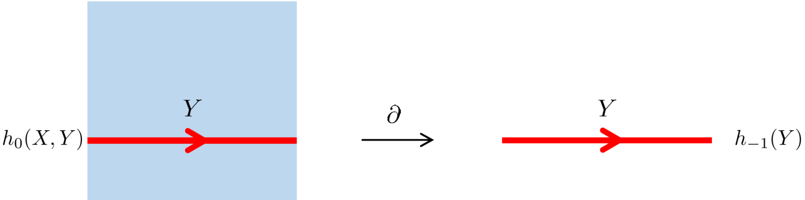
- f_* and g_* are induced homomorphisms of inclusions $f : X \to Y$ and $g : (X, \emptyset) \to (X, Y)$, respectively.
- f_*^0 is regarded as embedding an SPT phase over Y in X.



- g_*^0 is regarded as cutting out Y from X, which leads to anomalous states over Y from an SPT phase over X.



- ∂^0 is the bulk-boundary correspondence.



Real-space AHSS (4): Filtration

- E^{∞}-page itself does not provides the classification of SPT phenomena.
- Introduce the following subgroups of $h_n^G(X,Y)$,

$$F_p h_n := \mathrm{Im}\,[h_n^G(X_p, X_p \cap Y) \to h_n^G(X,Y)], \qquad p = 0,1,\dots.$$

- This has the clear physical meaning. For instance, $F_p h_0$ is the classification of SPT phases over the p-skeleton X_p which persists after being embedded in the whole space X.
- We have a filtration of the homology group

$$0 \subset F_0 h_n \subset F_1 h_n \subset \cdots \subset F_d h_n = h_n^G(X,Y),$$

where d is the space dimension of X.

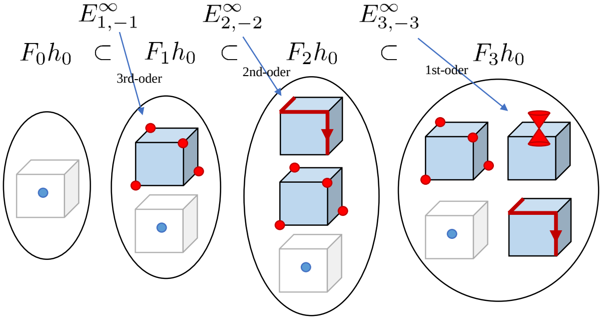
- The following relation connects the E^{∞}-page and the homology group.

$$F_p h_n / F_{p-1} h_n \cong E^{\infty}_{p,n-p}.$$

- The E^{∞}-page has good physical meanings.

Higher-order SPT phases

- $E^{\infty}_{p,-p}$: The classification of $(d-p+1)$th-order SPT phases. (cf. Huang-Song-Huang-Hermele)
- Ex: $3d$ with point group symmetry (without translation symmetry):



- This unifies the terminology of "strong" and "weak" SPT phases and higher-order SPT phases.

Ex: the classification of higher-order TIs with magnetic point group symmetry via the AHSS

[Okuma-Sato-KS, cf. Cornfeld-Chapman, KS]

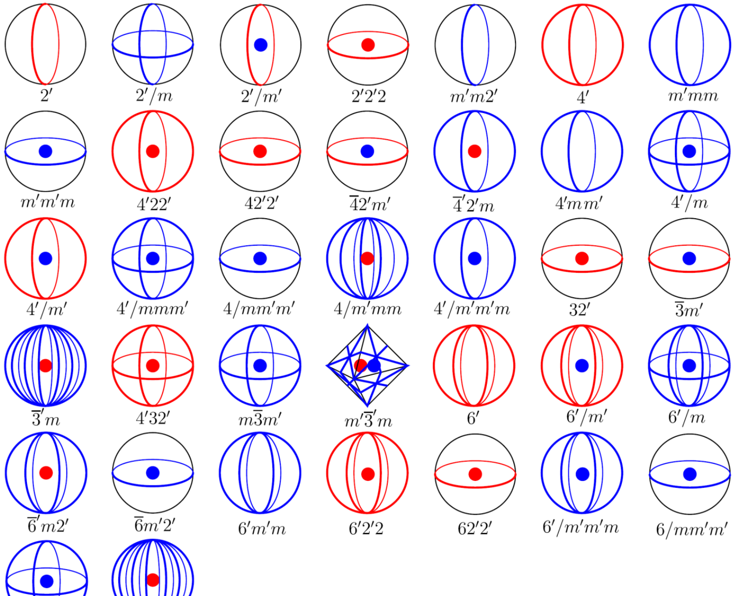
G	$2'$	$2'/m$	$2'/m'$	$2'2'2$	$m'm2'$	$4'$	$m'mm$	$m'm'm$	$4'22'$	$42'2'$	$\overline{4}2'm'$	$\overline{4}'2'm$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$E_{0,0}^{\infty}$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}^2$	$\mathbb{Z}_2$
$E_{2,-2}^{\infty}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

2nd-order TI

G	$4'mm'$	$4'/m$	$4'/m'$	$4'/mmm'$	$4/mm'm'$	$4/m'mm$	$4'/m'm'm$	$32'$	$\overline{3}m'$	$\overline{3}'m$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$E_{0,0}^{\infty}$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}^3$	$\mathbb{Z}_2$
$E_{2,-2}^{\infty}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

G	$4'32'$	$m\overline{3}m'$	$m'\overline{3}'m$	$6'$	$6'/m'$	$6'/m$	$\overline{6}'m2'$	$\overline{6}m'2'$	$6'm'm$	$6'2'2$	$62'2'$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}_2^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^6$
$E_{0,0}^{\infty}$	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$
$E_{2,-2}^{\infty}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

G	$6'/m'm'm$	$6/mm'm'$	$6'/mmm'$	$6/m'mm$	Others†
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	0


$2'$
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$2'/m'$
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$m'm2'$
$4'$
$m'mm$
$m'm'm$
$4'22'$
$42'2'$
$\overline{4}2'm'$
$\overline{4}'2'm$
$4'mm'$
$4'/m$
$4'/m'$
$4'/mmm'$
$4/mm'm'$
$4/m'mm$
$4'/m'm'm$
$32'$
$\overline{3}m'$
$\overline{3}'m$
$4'32'$
$m\overline{3}m'$
$m'\overline{3}'m$
$6'$
$6'/m'$
$6'/m$
$\overline{6}'m2'$
$\overline{6}m'2'$
$6'm'm$
$6'2'2$
$62'2'$
$6'/m'm'm$
$6/mm'm'$

Ex: the classification of higher-order TIs with magnetic point group symmetry via the AHSS

[Okuma-Sato-KS, cf. Cornfeld-Chapman, KS]

G	$2'$	$2'/m$	$2'/m'$	$2'2'2$	$m'm2'$	$4'$	$m'mm$	$m'm'm$	$4'22'$	$42'2'$	$\overline{4}2'm'$	$\overline{4}'2'm$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$E_{0,0}^{\infty}$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}^2$	$\mathbb{Z}_2$
$E_{2,-2}^{\infty}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

2nd-order TI

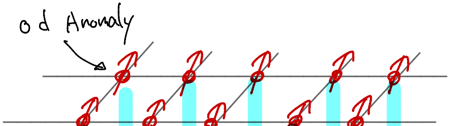
G	$4'mm'$	$4'/m$	$4'/m'$	$4'/mmm'$	$4/mm'm'$	$4/m'mm$	$4'/m'm'm$	$32'$	$\overline{3}m'$	$\overline{3}'m$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$E_{0,0}^{\infty}$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}^3$	$\mathbb{Z}_2$
$E_{2,-2}^{\infty}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

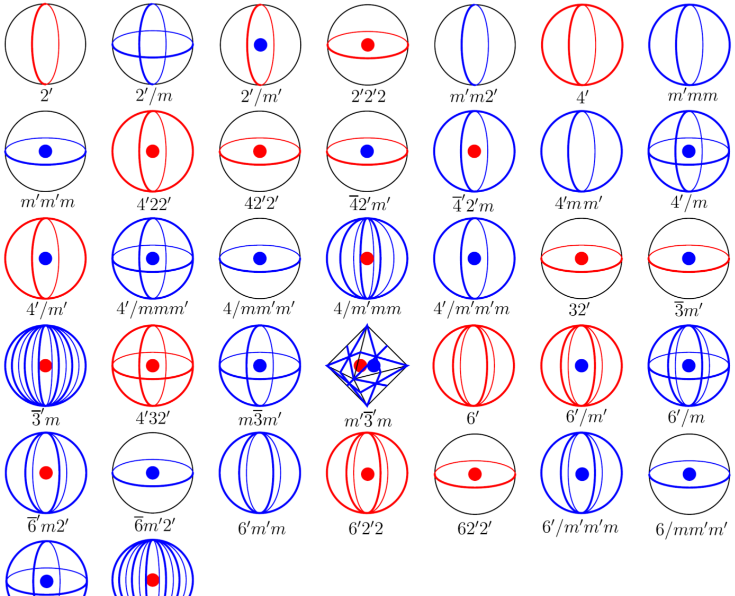
G	$4'32'$	$m\overline{3}m'$	$m'\overline{3}'m$	$6'$	$6'/m'$	$6'/m$	$\overline{6}'m2'$	$\overline{6}m'2'$	$6'm'm$	$6'2'2$	$62'2'$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}_2^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^6$
$E_{0,0}^{\infty}$	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$
$E_{2,-2}^{\infty}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

G	$6'/m'm'm$	$6/mm'm'$	$6'/mmm'$	$6/m'mm$	Others†
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	0

LSM type theorems

- Im $d^r_{r,-r}$ = SPT-LSM theorm [Lu 1705.04691, Yang=Jiang=Vishwanath=Ran 1705.05421]:
  In DOF consisting only of Im $d^r_{r,-r} \subset E^1_{0,-1}$, if we get an invertible state, this state must be a nontrivial invertible state.
- $E^\infty_{0,-1}$ = LSM theorem:
  DOF consisting only of $E^\infty_{0,-1} \subset E^1_{0,-1}$ can not have an invertible state.
  LSM-type theorems forbid the system with a sort of dof having a unique symmetric gapped ground state in the presence of translation symmetry and others. [Chen-Gu-Wen 11, Watanabe-Po-Vishwanath-Zaletel 15]
- The group $E^\infty_{0,-1}$ is the classification of the LSM theorem with crystalline G symmetry. (cf. Po-Watanabe-Jian-Zaletel 17,)
- See [KS-Xiong-Gomi 18, Else-Thorngren 19, Jiang-Cheng-Qi 19] for the detail.
- “ A LSM theorem as a boundary of an SPT phase” [Metlitski-Thorngren, ...]
- Using the AHSS, one can systematically classify the LSM-type theorems for a given space group and onsite symmetry. Many symmetry classes are remain unclassified.




$2'$
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$4/mm'm'$
$4/m'mm$
$4'/m'm'm$
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$\overline{3}'m$
$4'32'$
$m\overline{3}m'$
$m'\overline{3}'m$
$6'$
$6'/m'$
$6'/m$
$\overline{6}'m2'$
$\overline{6}m'2'$
$6'm'm$
$6'2'2$
$62'2'$
$6'/m'm'm$
$6/mm'm'$

Summary for part 3

- The AHSS gives us a useful tool to study the SPT phases and LSM theorems with crystalline symmetry with respect to high-symmetry regions in the real space.
- The differentials of the AHSS can be physically understood, thus they are computable from physical arguments. See KS-Xiong-Gomi for various worked examples of higher-differentials.
- The E^{∞}-page itself has a physical meaning. It represents the classification of higher-order SPT phases, anomalies, and adiabatic pumps. In particular, $E^{\infty}_{-1,0}$ is the classification of LSM theorems.