Entanglement entropy of fermions

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I. RÉNYI ENTANGLEMENT ENTROPY FOR FERMIONS

Rényi entanglement entropy is defined by

$$S_N := \frac{1}{1-N} \log \mathrm{tr}\,[\rho_I^N]. \tag{1.1}$$

Let us consider a fermionic reduced density matrix

$$\rho_I = \int \prod_{i\in I} d\bar\alpha_i d\alpha_i d\bar\beta_i d\beta_i e^{-\sum_{i\in I}(\bar\alpha_i\alpha_i + \bar\beta_i\beta_i)} \rho_I(\{\bar\alpha_i\},\{\beta_i\}) \left|\{\alpha_i\}\right\rangle \left\langle\{\bar\beta_i\}\right| . \tag{1.2}$$

From a straightforward calculation we have

$$\mathrm{tr}\,[\rho_I^N] = \int \prod_{i\in I, n=1,\dots,N} d\bar{\alpha}_{n,i} d\alpha_{n,i} \rho_I(\{-\bar{\alpha}_{1,i}\},\{\alpha_{1,i}\})\cdots\rho_I(\{-\bar{\alpha}_{N,i}\},\{\alpha_{N,i}\}) e^{\sum_{i\in I}(-\bar{\alpha}_{1,i}\alpha_{N,i}+\bar{\alpha}_{2,i}\alpha_{1,i}+\cdots\bar{\alpha}_{N,i}\alpha_{N-1,i})}. \tag{1.3}$$

Here, $n = 1\dots,N$ are the replica indices. For a pure state $|\psi\rangle$, the reduced density matrix becomes

$$\rho_I(\{-\bar{\alpha}_i\}_{i\in I},\{\alpha_i\}_{i\in I}) = \int \prod_{i\notin I} d\bar{\alpha}_i d\alpha_i \psi(\{-\bar{\alpha}_i\}_{i\in I},\{-\bar{\alpha}_i\}_{i\notin I})\psi^*(\{\alpha_i\}_{i\in I},\{\alpha_i\}_{i\notin I}) e^{-\sum_{i\notin I}\bar{\alpha}_i\alpha_i}. \tag{1.4}$$

Then,

$$\mathrm{tr}\,[\rho_I^N] = \int \prod_{i\in \mathrm{full}, n=1,\dots,N} d\bar{\alpha}_{n,i} d\alpha_{n,i} \psi(\{-\bar{\alpha}_{1,i}\})\psi^*(\{\alpha_{1,i}\})\cdots\psi(\{-\bar{\alpha}_{N,i}\})\psi^*(\{\alpha_{N,i}\}) \tag{1.5}$$

$$e^{-\sum_{i\notin I,n}\bar{\alpha}_{n,i}\alpha_{n,i}} e^{\sum_{i\in I}(-\bar{\alpha}_{1,i}\alpha_{N,i}+\bar{\alpha}_{2,i}\alpha_{1,i}+\cdots\bar{\alpha}_{N,i}\alpha_{N-1,i})} \tag{1.6}$$

$$= \int \prod_{i\in \mathrm{full}, n=1,\dots,N} d\bar{\alpha}_{n,i} d\alpha_{n,i} \Psi(\{-\bar{\alpha}_{n,i}\})\Psi^*(\{\alpha_{n,i}\}) e^{-\sum_{i\notin I,n}\bar{\alpha}_{n,i}\alpha_{n,i}} e^{\sum_{i\in I}(-\bar{\alpha}_{1,i}\alpha_{N,i}+\bar{\alpha}_{2,i}\alpha_{1,i}+\cdots\bar{\alpha}_{N,i}\alpha_{N-1,i})}, \tag{1.7}$$

where we introduced the *replica ground state*

$$\Psi(\{-\bar{\alpha}_{n,i}\}) := \psi(\{-\bar{\alpha}_{1,i}\})\psi(\{-\bar{\alpha}_{2,i}\})\cdots\psi(\{-\bar{\alpha}_{N,i}\}), \tag{1.8}$$

$$\Psi^*(\{\alpha_{n,i}\}) := \psi^*(\{\alpha_{N,i}\})\psi^*(\{\alpha_{N-1,i}\})\cdots\psi^*(\{\alpha_{1,i}\}). \tag{1.9}$$

$\mathrm{tr}\,[\rho_I^N]$ can be expressed as the expectation value of the *partial replica permutation operator* T_I on the replica ground state $|\Psi\rangle$. For a while, we ignore site indices. Let $\{f_n\}$ be complex fermions associated with the Grassmann variables $\{\alpha_n\}$. We define the replica permutation operator T so that T satisfies

$$T f_1 T^{-1} = f_N, \qquad T f_n T^{-1} = -f_{n-1}\ (n = 2,\dots N). \tag{1.10}$$

Note that

$$T^N \sim \begin{cases} 1 & (N:\mathrm{odd}) \\ (-1)^F & (N:\mathrm{even}) \end{cases} \tag{1.11}$$

up to a $U(1)$ phase factor. By introducing real fermions $\{c_n^R, c_n^L\}$ by

$$f_n^\dagger = \frac{c_n^R + ic_n^L}{2}, \qquad f_n = \frac{c_n^R - ic_n^L}{2}, \tag{1.12}$$

the permutation operator T is given by

$$T = e^{\frac{\pi}{4}c_N^R c_{N-1}^R} e^{\frac{\pi}{4}c_{N-1}^R c_{N-2}^R}\cdots e^{\frac{\pi}{4}c_2^R c_1^R}\cdot e^{\frac{\pi}{4}c_N^L c_{N-1}^L} e^{\frac{\pi}{4}c_{N-1}^L c_{N-2}^L}\cdots e^{\frac{\pi}{4}c_2^L c_1^L} \tag{1.13}$$

$$= (1 - f_{N-1}^\dagger f_{N-1} - f_N^\dagger f_N - f_{N-1}^\dagger f_N + f_N^\dagger f_{N-1} + 2 f_{N-1}^\dagger f_N^\dagger f_N f_{N-1}) \tag{1.14}$$

$$\cdots(1 - f_1^\dagger f_1 - f_2^\dagger f_2 - f_1^\dagger f_2 + f_2^\dagger f_1 + 2 f_1^\dagger f_2^\dagger f_2 f_1). \tag{1.15}$$

T is normalized as

$$T^N = \begin{cases} 1 & (N:\mathrm{odd}) \\ (-1)^F & (N:\mathrm{even}) \end{cases} \tag{1.16}$$

(KS: I checked this for $N = 2, 3, 4, 5$ from a direct calculation.) Its matrix element is given by

$$\langle\alpha| T |\beta\rangle \tag{1.17}$$
$$= \int \prod d\gamma_n d\delta_n e^{-\sum_n \gamma_n\delta_n} \langle\alpha| (1 - f_{N-1}^\dagger f_{N-1} - f_N^\dagger f_N - f_{N-1}^\dagger f_N + f_N^\dagger f_{N-1} + 2f_{N-1}^\dagger f_N^\dagger f_N f_{N-1}) \tag{1.18}$$
$$\cdots |\delta\rangle \langle\gamma| (1 - f_1^\dagger f_1 - f_2^\dagger f_2 - f_1^\dagger f_2 + f_2^\dagger f_1 + 2f_1^\dagger f_2^\dagger f_2 f_1) |\beta\rangle \tag{1.19}$$
$$= \int \prod d\gamma_n d\delta_n e^{-\sum_n \gamma_n\delta_n} \langle\alpha| (1 - f_{N-1}^\dagger f_{N-1} - f_N^\dagger f_N - f_{N-1}^\dagger f_N + f_N^\dagger f_{N-1} + 2f_{N-1}^\dagger f_N^\dagger f_N f_{N-1}) \tag{1.20}$$
$$\cdots |\delta\rangle e^{-\gamma_1\beta_2+\gamma_2\beta_1+\sum_{n=3}^N \gamma_n\beta_n} \tag{1.21}$$
$$= \langle\alpha| (1 - f_{N-1}^\dagger f_{N-1} - f_N^\dagger f_N - f_{N-1}^\dagger f_N + f_N^\dagger f_{N-1} + 2f_{N-1}^\dagger f_N^\dagger f_N f_{N-1}) \tag{1.22}$$
$$\cdots (1 - f_2^\dagger f_2 - f_3^\dagger f_3 - f_2^\dagger f_3 + f_3^\dagger f_2 + 2f_2^\dagger f_3^\dagger f_3 f_2) |-\beta_2, \beta_1, \beta_3, \dots, \beta_N\rangle \tag{1.23}$$
$$= \cdots \tag{1.24}$$
$$= \langle\alpha| - \beta_2, -\beta_3, \dots, -\beta_N, \beta_1\rangle \tag{1.25}$$
$$= e^{-\sum_{n=1}^{N-1} \alpha_n\beta_{n+1}+\alpha_N\beta_1}. \tag{1.26}$$

We define the partial N-fold rotation T_I on the interval I by

$$T_I := \prod_{i\in I} e^{\frac{\pi}{4}c_{N,i}^R c_{N-1,i}^R} e^{\frac{\pi}{4}c_{N-1,i}^R c_{N-2,i}^R} \cdots e^{\frac{\pi}{4}c_{2,i}^R c_{1,i}^R} \cdot e^{\frac{\pi}{4}c_{N,i}^L c_{N-1,i}^L} e^{\frac{\pi}{4}c_{N-1,i}^L c_{N-2,i}^L} \cdots e^{\frac{\pi}{4}c_{2,i}^L c_{1,i}^L} \tag{1.27}$$

Then,

$$\langle\Psi|T_I|\Psi\rangle = \mathrm{tr}\left[T_I |\Psi\rangle \langle\Psi|\right] \tag{1.28}$$
$$= \int \prod d\bar{\alpha}_{n,i} d\alpha_{n,i} e^{-\sum_{n,i} \bar{\alpha}_{n,i}\alpha_{n,i}} \langle-\bar{\alpha}|T_I|\Psi\rangle \langle\Psi|\alpha\rangle \tag{1.29}$$
$$= \int \prod d\bar{\alpha}_{n,i} d\alpha_{n,i} d\bar{\beta}_{n,i} d\beta_{n,i} e^{-\sum_{n,i} \bar{\alpha}_{n,i}\alpha_{n,i}} e^{-\sum_{n,i} \bar{\beta}_{n,i}\beta_{n,i}} \langle-\bar{\alpha}|T_I|\beta\rangle \langle\bar{\beta}|\Psi\rangle \langle\Psi|\alpha\rangle \tag{1.30}$$
$$= \int \prod d\bar{\alpha}_{n,i} d\alpha_{n,i} d\bar{\beta}_{n,i} d\beta_{n,i} e^{-\sum_{n,i} \bar{\alpha}_{n,i}\alpha_{n,i}} e^{-\sum_{n,i} \bar{\beta}_{n,i}\beta_{n,i}} \tag{1.31}$$
$$e^{-\sum_{n,i\notin I} \bar{\alpha}_{n,i}\beta_{n,i}} e^{\sum_{i\in I}[\sum_{n=1}^{N-1} \bar{\alpha}_{n,i}\beta_{n+1,i} - \bar{\alpha}_{N,i}\beta_{1,i}]} \langle\bar{\beta}|\Psi\rangle \langle\Psi|\alpha\rangle \tag{1.32}$$
$$= \int \prod d\alpha_{n,i} d\bar{\beta}_{n,i} e^{\sum_{n,i\notin I} \bar{\beta}_{n,i}\alpha_{n,i}} e^{\sum_{i\in I}[\bar{\beta}_{1,i}\alpha_{N,i} - \sum_{n=2}^N \bar{\beta}_{n,i}\alpha_{n-1,i}]} \langle\bar{\beta}|\Psi\rangle \langle\Psi|\alpha\rangle \tag{1.33}$$
$$= \int \prod d\bar{\alpha}_{n,i} d\alpha_{n,i} e^{-\sum_{n,i\notin I} \bar{\alpha}_{n,i}\alpha_{n,i}} e^{\sum_{i\in I}[-\bar{\alpha}_{1,i}\alpha_{N,i} + \sum_{n=2}^N \bar{\alpha}_{n,i}\alpha_{n-1,i}]} \langle-\bar{\alpha}|\Psi\rangle \langle\Psi|\alpha\rangle \tag{1.34}$$
$$= \mathrm{tr}\left[\rho_I^N\right]. \tag{1.35}$$

Here we used

$$\int \prod_n d\bar{\alpha}_n d\beta_n e^{-\sum_n \bar{\alpha}_n\alpha_n} e^{-\sum_n \bar{\beta}_n\beta_n} e^{\sum_{n=1}^{N-1} \bar{\alpha}_n\beta_{n+1} - \bar{\alpha}_N\beta_1} \tag{1.36}$$
$$= \int (\beta_2 - \alpha_1) d\beta_1 (\beta_3 - \alpha_2) d\beta_2 \cdots (\beta_N - \alpha_{N-1}) d\beta_{N-1} (-\beta_1 - \alpha_N) d\beta_N e^{-\sum_n \bar{\beta}_n\beta_n} \tag{1.37}$$
$$= \int d\beta_1 (\beta_1 + \alpha_N) d\beta_2 (\beta_2 - \alpha_1) \cdots d\beta_N (\beta_N - \alpha_{N-1}) e^{-\sum_n \bar{\beta}_n\beta_n} \tag{1.38}$$
$$= e^{\bar{\beta}_1\alpha_N - \sum_{n=2}^N \bar{\beta}_n\alpha_{n-1}}. \tag{1.39}$$

It is useful to introduce a basis of fermions which diagonalizes the permutation operator T. T is written as

$$T f_n^\dagger T^{-1} = f_m^\dagger [U_T]_{mn}, \qquad U_T = \begin{pmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & \cdots & & \\ & & & 0 & -1 \\ 1 & & & & 0 \end{pmatrix}. \tag{1.40}$$

The matrix U_T is diagonalized by

$$U_T v_k = -\omega_k v_k, \qquad v_k = \frac{1}{\sqrt{N}}(1, \omega_k, \omega_k^2, \cdots, \omega_k^{N-1})^T, \qquad \omega_k = e^{\frac{\pi i}{N}(2k-1)} \qquad (k = 1, \dots, N). \tag{1.41}$$

We introduce new fermions $\psi_k^\dagger (k = 1, \dots, N)$ by

$$\psi_k^\dagger := \frac{1}{\sqrt{N}}(f_1^\dagger + \omega_k f_2^\dagger + \omega_k^2 f_3^\dagger + \cdots + \omega_k^{N-1} f_N^\dagger), \tag{1.42}$$

$$T\psi_k^\dagger T^{-1} = -\omega_k \psi_k^\dagger. \tag{1.43}$$

If the pure state $|\psi\rangle$ has the particle number symmetry (Spin^c structure), the Rényi entanglement entropy for the subregion I factorizes into the N number of the partial $U(1)$ rotations

$$\mathrm{tr}\,[\rho_I^N] = \langle \Psi | T | \Psi \rangle = \prod_{k=-\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}} \langle \psi | U_k|_I | \psi \rangle \,, \tag{1.44}$$

where $|\psi\rangle$ is the ground state of original single system and

$$U_k|_I = \exp\Big[2\pi i \frac{k}{N} \sum_{i \in I} f_i^\dagger f_i\Big]. \tag{1.45}$$

II. PARTIAL TIME-REVERSAL TRANSFORMATION (CLASS AI)

There are three types of partial anti-unitary transformations: class AI, AII, and AIII. Here we consider class AI time-reversal twisting. For a class AI TRS defined by

$$\Theta f_i^\dagger \Theta^{-1} = f_j^\dagger [\mathcal{U}_\Theta]_{ji}, \qquad \mathcal{U}_\Theta^T = \mathcal{U}_\Theta, \tag{2.1}$$

the partial time-reversal twisting on a subregion I_1 is defined by

$$U_\Theta^{I_1} \rho_I^{\Theta_1} [U_\Theta^{I_1}]^\dagger = \int \prod_{i \in I} d\bar{\gamma}_i d\gamma_i d\bar{\delta}_i d\delta_i e^{-\sum_i (\bar{\gamma}_i \gamma_i + \bar{\delta}_i \delta_i)} \rho_I(\{\bar{\gamma}_i\}, \{\delta_i\}) \left| \{i\bar{\delta}_i [\mathcal{U}_\Theta]_{ij}\}_{j \in I_1}, \{\gamma_i\}_{i \in I_2} \right\rangle \left\langle \{i[\mathcal{U}_\Theta^*]_{ij} \gamma_j\}_{i \in I_1}, \{\bar{\delta}_i\}_{i \in I_2} \right| , \tag{2.2}$$

where $U_T^{I_1}$ is the unitary part of the time-reversal transformation on I_1. Here, we consider the quantity

$$S_N^{\Theta_1} := \mathrm{tr}\,[(\rho_I^{\Theta_1})^N] = \mathrm{tr}\,[(U_\Theta^{I_1} \rho_I^{\Theta_1} [U_\Theta^{I_1}]^\dagger)^N]. \tag{2.3}$$

Since $S_N^{\Theta_1}$ does not depend on the choice of $U_\Theta^{I_1}$, we can simply set $\mathcal{U}_\Theta = 1$:

$$U_\Theta^{I_1} \rho_I^{\Theta_1} [U_\Theta^{I_1}]^\dagger = \int \prod_{i \in I} d\bar{\gamma}_i d\gamma_i d\bar{\delta}_i d\delta_i e^{-\sum_i (\bar{\gamma}_i \gamma_i + \bar{\delta}_i \delta_i)} \rho_I(\{\bar{\gamma}_i\}, \{\delta_i\}) \left| \{i\bar{\delta}_i\}_{i \in I_1}, \{\gamma_i\}_{i \in I_2} \right\rangle \left\langle \{i\gamma_i\}_{i \in I_1}, \{\bar{\delta}_i\}_{i \in I_2} \right| . \tag{2.4}$$

One can show that

$$S_N^{\Theta_1} = \int \prod_{i \in I_1 \cup I_2, n=1,\dots,N} d\bar{\alpha}_{n,i} d\alpha_{n,i} e^{\sum_{i \in I_1} (\sum_{n=1}^{N-1} \bar{\alpha}_{n,i} \alpha_{n+1,i} - \bar{\alpha}_{N,i} \alpha_{1,i})} e^{\sum_{i \in I_2} (\sum_{n=1}^{N-1} \bar{\alpha}_{n+1,i} \alpha_{n,i} - \bar{\alpha}_{1,i} \alpha_{N,i})} \prod_n \rho(\{-\bar{\alpha}_{n,i}\}, \{\alpha_{n,i}\}). \tag{2.5}$$

For a pure state $|\psi\rangle$,

$$S_N^{\Theta_1} = \int \prod_{i,n=1,\dots,N} d\bar{\alpha}_{n,i} d\alpha_{n,i} e^{\sum_{i \in I_1} (\sum_{n=1}^{N-1} \bar{\alpha}_{n,i} \alpha_{n+1,i} - \bar{\alpha}_{N,i} \alpha_{1,i})} e^{\sum_{i \in I_2} (\sum_{n=1}^{N-1} \bar{\alpha}_{n+1,i} \alpha_{n,i} - \bar{\alpha}_{1,i} \alpha_{N,i})} e^{-\sum_{i \notin I_1 \cup I_2} \sum_{n=1}^{N} \bar{\alpha}_{n,i} \alpha_{n,i}} \tag{2.6}$$

$$\psi(\{-\bar{\alpha}_{1,i}\}) \psi(\{-\bar{\alpha}_{2,i}\}) \cdots \psi(\{-\bar{\alpha}_{N,i}\}) \psi^*(\{\alpha_{N,i}\}) \cdots \psi^*(\{\alpha_{2,i}\}) \psi^*(\{\alpha_{1,i}\}). \tag{2.7}$$

This can be written as an expectation value of the partial N-fold permutation operator on the replicated ground state $|\Psi\rangle$. Noticing that

$$\langle\alpha|T^{-1}|\beta\rangle = \langle -\alpha_2, -\alpha_3, \dots, -\alpha_N, \alpha_1|\beta\rangle = e^{\alpha_1\beta_N - \sum_{n=2}^N \alpha_n\beta_{n-1}}, \tag{2.8}$$

we get

$$S_N^{\Theta_1} = \langle\Psi|T_{I_1}^{-1}T_{I_2}|\Psi\rangle \tag{2.9}$$

with

$$T_{I_1}^{-1} = \prod_{i\in I_1} e^{\frac{\pi}{4}c^R_{1,i}c^R_{2,i}}e^{\frac{\pi}{4}c^R_{2,i}c^R_{3,i}}\cdots e^{\frac{\pi}{4}c^R_{N-1,i}c^R_{N,i}}\cdot e^{\frac{\pi}{4}c^L_{1,i}c^L_{2,i}}e^{\frac{\pi}{4}c^L_{2,i}c^L_{3,i}}\cdots e^{\frac{\pi}{4}c^L_{N-1,i}c^L_{N,i}}, \tag{2.10}$$

$$T_{I_2} = \prod_{i\in I_2} e^{\frac{\pi}{4}c^R_{N,i}c^R_{N-1,i}}e^{\frac{\pi}{4}c^R_{N-1,i}c^R_{N-2,i}}\cdots e^{\frac{\pi}{4}c^R_{2,i}c^R_{1,i}}\cdot e^{\frac{\pi}{4}c^L_{N,i}c^L_{N-1,i}}e^{\frac{\pi}{4}c^L_{N-1,i}c^L_{N-2,i}}\cdots e^{\frac{\pi}{4}c^L_{2,i}c^L_{1,i}}. \tag{2.11}$$

If the pure state $|\psi\rangle$ has the particle number conservation symmetry (Spin^c structure), $\langle\Psi|T_{I_1}^{-1}T_{I_2}|\Psi\rangle$ factorizes into the N number of the partial $U(1)$ rotations

$$S_N^{\Theta_1} = \prod_{k=-\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}} \langle\psi|U_k|_I|\psi\rangle\,, \tag{2.12}$$

$$U_k|_I = \exp\Big[-\frac{2\pi i k}{N}\sum_{i\in I_1} f_i^\dagger f_i + \frac{2\pi i k}{N}\sum_{i\in I_2} f_i^\dagger f_i\Big]. \tag{2.13}$$

III. MAJORANA CHAIN

A. Jordan-Wigner transformation in (1+1)d

Introducing real fermions

$$f^\dagger_{n,i} = \frac{c^R_{n,i} + ic^L_{n,i}}{2}, \qquad f_{n,i} = \frac{c^R_{n,i} - ic^L_{n,i}}{2}, \tag{3.1}$$

we introduce spin 1/2 variables $\{\sigma^\mu_{n,i}\}$ by

$$c^R_{n,i} = -\sigma^x_{n,1}\sigma^x_{n,2}\cdots\sigma^x_{n,i-1}\sigma^y_{n,i}, \tag{3.2}$$

$$c^L_{n,i} = \sigma^x_{n,1}\sigma^x_{n,2}\cdots\sigma^x_{n,i-1}\sigma^z_{n,i}. \tag{3.3}$$

The following relations hold:

$$(-1)^F = \prod_{n,i}(-ic^L_{n,i}c^R_{n,i}) = \prod_{n,i}\sigma^x_{n,i}, \tag{3.4}$$

$$f^\dagger_{n,i}f_{n,i} = \frac{1-\sigma^x_{n,i}}{2}, \tag{3.5}$$

$$P_n = \prod_i \sigma^x_{n,i}, \tag{3.6}$$

$$f^\dagger_{n,i} = -\sigma^x_{n,1}\cdots\sigma^x_{n,i-1}\sigma^-_{n,i}, \quad \sigma^-_{n,i} = \frac{\sigma^y_{n,i} - i\sigma^z_{n,i}}{2}, \tag{3.7}$$

$$f_{n,i} = -\sigma^x_{n,1}\cdots\sigma^x_{n,i-1}\sigma^+_{n,i}, \quad \sigma^+_{n,i} = \frac{\sigma^y_{n,i} + i\sigma^z_{n,i}}{2}, \tag{3.8}$$

$$ic^L_{n,i}c^R_{n,i} = -\sigma^x_{n,i}, \tag{3.9}$$

$$ic^R_{n,i}c^L_{n,i+1} = -\sigma^z_{n,i}\sigma^z_{n,i+1}, \quad ic^R_{n,L}c^L_{n,1} = P_n\sigma^z_{n,L}\sigma^z_{n,1}, \tag{3.10}$$

$$ic^L_{n,i}c^R_{n,i+1} = \sigma^y_{n,i}\sigma^y_{n,i+1}, \quad ic^L_{n,L}c^R_{n,1} = -P_n\sigma^y_{n,L}\sigma^y_{n,1}. \tag{3.11}$$

1. Majorana chain

$$H_n = \sum_{i=1}^{L}(f^\dagger_{n,i}f_{n,i} + h.c.) + J\sum_{i=1}^{L-1}(-f^\dagger_{n,i}f_{n,i+1} - f_{n,i}f_{n,i+1} + h.c.) \pm J(-f^\dagger_{n,L}f_{n,1} - f_{n,L}f_{n,1} + h.c.) \tag{3.12}$$

$$= \sum_{i=1}^{L} ic^L_{n,i}c^R_{n,i} + J\sum_{i=1}^{L-1} ic^R_{n,i}c^L_{n,i+1} \pm Jic^R_{n,L}c^L_{1,1} \tag{3.13}$$

$$= -\sum_{i=1}^{L}\sigma^x_{n,i} - J\sum_{i=1}^{L-1}\sigma^z_{n,i}\sigma^z_{n,i+1} \pm JiP_n\sigma^z_{n,L}\sigma^z_{n,1}. \tag{3.14}$$

$J = 1$ is the critical point. We introduce new fermions $\psi^\dagger_k (k = 1, \ldots, N)$ by

$$\psi^\dagger_k := \frac{1}{\sqrt{N}}(f^\dagger_1 + \omega_k f^\dagger_2 + \omega_k^2 f^\dagger_3 + \cdots + \omega_k^{N-1} f^\dagger_N), \qquad \omega_k = e^{\frac{\pi i}{N}(2k-1)}, \tag{3.15}$$

$$T\psi^\dagger_k T^{-1} = -\omega_k\psi^\dagger_k. \tag{3.16}$$

This inverse transformation is

$$f^\dagger_n = \frac{1}{\sqrt{N}}(\bar{\omega}_1^{n-1}\psi^\dagger_1 + \bar{\omega}_2^{n-1}\psi^\dagger_2 + \cdots + \bar{\omega}_N^{n-1}\psi^\dagger_N), \tag{3.17}$$

$$f_n = \frac{1}{\sqrt{N}}(\omega_1^{n-1}\psi_1 + \omega_2^{n-1}\psi_2 + \cdots + \omega_N^{n-1}\psi_N). \tag{3.18}$$

$$\sum_n f_{n,i}f_{n,i+1} = \frac{1}{N}\sum_{n,k,l}\omega_k^{n-1}\omega_l^{n-1}\psi_{k,i}\psi_{l,i+1} = \sum_k \psi_{k,i}\psi_{1-k,i+1}. \tag{3.19}$$

IV. DIRAC FERMION (Spin^c)

Model Hamiltonian

$$H(\theta) = -\sum_{i=1}^{L-1}[f^\dagger_i f_{i+1} + h.c.] - e^{i\theta}[f^\dagger_L f_1 + h.c.] \tag{4.1}$$

$$\tag{4.2}$$

A. Bosonization

- Spin^c $\leftrightarrow$ $U(1)$?
- Dirac fermion $\leftrightarrow$ compactified boson ?

Critical Dirac fermion (Spin^c structure):

$$H(\theta) = -\sum_{i=1}^{L-1}[f^\dagger_i f_{i+1} + h.c.] - e^{i\theta}[f^\dagger_L f_1 + h.c.] \tag{4.3}$$

$$= -\frac{1}{2}\sum_{i=1}^{L-1}[ic^L_i c^R_{i+1} - ic^R_i c^L_{i+1}] - \frac{e^{i\theta}}{2}[ic^L_L c^R_1 - ic^R_L c^L_1] \tag{4.4}$$

$$= -\frac{1}{2}\sum_{i=1}^{L-1}[\sigma^y_i\sigma^y_{i+1} + \sigma^z_i\sigma^z_{i+1}] + \frac{e^{i\theta}}{2}P[\sigma^y_L\sigma^y_1 + \sigma^z_L\sigma^z_1] \tag{4.5}$$

$$= -\sum_{i=1}^{L-1}[\sigma^+_i\sigma^-_{i+1} + \sigma^-_i\sigma^+_{i+1}] + e^{i\theta}P[\sigma^+_L\sigma^-_1 + \sigma^-_L\sigma^+_1]. \tag{4.6}$$

$$(-1)^F = \prod_i (1-2f_i^\dagger f_i) = P = \prod_i \sigma_i^x, \tag{4.7}$$

$$T_{I,k} = \exp\Big[\frac{2\pi ik}{N}\sum_{i\in I} f_i^\dagger f_i\Big] = \exp\Big[\frac{\pi ik}{N}|I| - \frac{\pi ik}{N}\sum_{i\in I}\sigma_i^x\Big], \tag{4.8}$$

$$T_{I,k}^{-1} = \exp\Big[-\frac{2\pi ik}{N}\sum_{i\in I} f_i^\dagger f_i\Big] = \exp\Big[-\frac{\pi ik}{N}|I| + \frac{\pi ik}{N}\sum_{i\in I}\sigma_i^x\Big], \tag{4.9}$$

$$k = -\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}. \tag{4.10}$$

$$\left\{\begin{array}{l} e^{-2\pi ibf^\dagger f} f^\dagger e^{2\pi ibf^\dagger f} = e^{-2\pi ib} f^\dagger \\ e^{-2\pi ibf^\dagger f} f e^{2\pi ibf^\dagger f} = e^{2\pi ib} f \end{array}\right. \leftrightarrow \left\{\begin{array}{ll} e^{-\pi ib+\pi ib\sigma^x}\sigma^y e^{\pi ib-\pi ib\sigma^x} & = \sigma^y\cos(2\pi b) + \sigma^z\sin(2\pi b) \\ e^{-\pi ib+\pi ib\sigma^x}\sigma^z e^{\pi ib-\pi ib\sigma^x} & = \sigma^z\cos(2\pi b) - \sigma^y\sin(2\pi b) \end{array}\right. \tag{4.11}$$

$$\leftrightarrow \left\{\begin{array}{ll} e^{-\pi ib+\pi ib\sigma^x}\sigma^+ e^{\pi ib-\pi ib\sigma^x} & = e^{-2\pi ib}\sigma^+ \\ e^{-\pi ib+\pi ib\sigma^x}\sigma^- e^{\pi ib-\pi ib\sigma^x} & = e^{2\pi ib}\sigma^- \end{array}\right. \tag{4.12}$$

B. Toeplitz determinant

Let $|\psi\rangle$ be a occupied state

$$|\psi\rangle = \prod_{|k|<k_F} f_k^\dagger |0\rangle \tag{4.13}$$

of complex fermions f_k with momentum $k \in \frac{2\pi n}{L}, n \in \mathbb{Z}$, $-a/\pi < k < \pi/a$ with a the lattice constant. In the present section, we do not assume any forms of the dispersion ϵ_k of the complex fermions. What we want to compute is the following expectation value of the sequence of the partial $U(1)$ transformation

$$\Big\langle \psi \Big| \exp\Big[2\pi ib_1 \sum_{x\in I_1} f_x^\dagger f_x + 2\pi ib_2 \sum_{x\in I_2} f_x^\dagger f_x + \cdots\Big] \Big| \psi \Big\rangle. \tag{4.14}$$

The above expectation value can be estimated by use of the Fisher-Hartwig theorem.[1] The matrix element of the sequence for momentum basis

$$|n\rangle = \sqrt{\frac{a}{L}} \sum_x |x\rangle\, e^{2\pi inx/L} \tag{4.15}$$

is given approximated as

$$\phi_{n-m} = \Big\langle n \Big| \exp\Big[2\pi ib_1 \sum_{x\in I_1} f_x^\dagger f_x + 2\pi ib_2 \sum_{x\in I_2} f_x^\dagger f_x + \cdots\Big] \Big| m \Big\rangle \tag{4.16}$$

$$= \frac{a}{L}\sum_x \left\{\begin{array}{ll} e^{2\pi ib_j} & (x\in I_j) \\ 1 & \text{otherwise} \end{array}\right\} e^{-i(n-m)\frac{2\pi x}{L}} \tag{4.17}$$

$$\sim \frac{1}{2\pi}\oint_0^{2\pi} d\theta \left\{\begin{array}{ll} e^{2\pi ib_j} & (\theta\in I_j) \\ 1 & \text{otherwise} \end{array}\right\} e^{-i(n-m)\theta} \qquad \text{for } \frac{a}{L} \ll 1. \tag{4.18}$$

Here, we used the same notations I_j for the intervals in $S^1 = [0, 2\pi]$. The generating function of the Toeplitz matrix

$$T_{k_F}[\phi] = (\phi_{n-m}), \quad -\frac{Lk_F}{2\pi} < n, m < \frac{Lk_F}{2\pi}. \tag{4.19}$$

is given by

$$\phi(\theta) = \left\{\begin{array}{ll} e^{2\pi ib_j} & (\theta\in I_j) \\ 1 & \text{otherwise} \end{array}\right. \tag{4.20}$$

The expectation value (4.14) is given by the determinant of the Toeplitz matrix

$$(4.14) \sim \det T_{k_F}[\phi] \tag{4.21}$$

Notice that the Toeplitz matrix (4.19) is equivalent to

$$T_{k_F}[\phi] = (\phi_{n-m}), \quad 0 < n, m < \frac{Lk_F}{\pi}. \tag{4.22}$$

We can immediately apply the Fisher-Hartwig theorem, which is reviewed in Appendix A, to (4.14).

C. Single interval

Here we consider the partial $e^{2\pi ib}$ transformation and the Rényi entanglement entropy on an interval $[0,\ell]$. The data of generating function of the Toeplitz matrix is

$$e^{V(e^{i\theta})} = e^{i\theta_1 b}, \quad 0 = \theta_0 < \theta_1 = \frac{2\pi\ell}{L}, \quad \alpha_0 = \alpha_1 = 0, \quad \beta_0 = -b, \beta_1 = b. \tag{4.23}$$

There is ambiguity in β_0 and β_1 which arises from

$$\beta_i \mapsto \beta_i + n_i, \quad \sum_i n_i = 0. \tag{4.24}$$

β_i should be chosen to minimize

$$\sum_i (\text{Re } \beta_i)^2. \tag{4.25}$$

If there are multiple minimized points $\{\beta_i\}$, all the minimized points $\{\beta_i\}$ contribute the determinant of the Toeplitz matrix in an equal footing.

1. *Partial $e^{2\pi ib}$ transformation, $b \neq 1/2$.*

For $b \neq 1/2$, β is fixed to be $|\beta| < 1/2$, thus we assume $-1/2 < b < 1/2$. The partial $e^{2\pi ib}$ transformation is given by

$$\begin{aligned} Z(b) &:= \langle\psi| e^{2\pi ib \sum_{0\le x\le \ell} f_x^\dagger f_x} |\psi\rangle & (4.26)\\ &\cong \left[2\sin\left(\frac{\pi\ell}{L}\right)\right]^{-2b^2} \cdot \left(\frac{Lk_F}{\pi}\right)^{-2b^2} \cdot e^{2ik_F\ell b} \cdot [G(1+b)G(1-b)]^2, & (4.27)\end{aligned}$$

where $G(z)$ is the Barnes G-function

$$G(1+z) = (2\pi)^{z/2} \exp\Big(-\frac{z+(1+\gamma)z^2}{2}\Big) \prod_{r=1}^{\infty} \Big[\big(1+\frac{z}{r}\big)^r e^{\frac{z^2}{2r}-z}\Big]. \tag{4.28}$$

The logarithm of the partial $e^{2\pi ib}$ transformation is

$$\begin{aligned} \log Z(b) &= \log \langle\psi| e^{2\pi ib \sum_{0\le x\le \ell} f_x^\dagger f_x} |\psi\rangle & (4.29)\\ &\cong -2b^2 \log\left[\frac{Lk_F}{\pi} \cdot 2\sin\left(\frac{\pi\ell}{L}\right)\right] + 2ik_F\ell b + 2\log[G(1+b)G(1-b)]. & (4.30)\end{aligned}$$

The pure imaginary part is a trivial contribution from the fermi sea. In fact, this can be removed by the redefinition the $U(1)$ charge operator as

$$Z'(b) := \langle\psi| e^{2\pi ib \sum_{0\le x\le \ell}(f_x^\dagger f_x - \nu)} |\psi\rangle\,, \tag{4.31}$$

with ν the filling number par a site. In the present case, ν is given by

$$\nu = \frac{N_F}{L/a} = \frac{2k_F/(2\pi/L)}{L/a} = \frac{k_F a}{\pi}, \tag{4.32}$$

where N_F is the total fermion number of the fermi sea. Then,

$$-2\pi i b\nu \sum_{0\leq x\leq \ell} = -2\pi i b\frac{k_F a}{\pi}\cdot\frac{\ell}{a} = -2ibk_F\ell, \tag{4.33}$$

which cancels the pure imaginary part of $\log Z(b)$.

2. Partial $(-1)^F$ transformation

Let us consider the partial fermion parity flip on a single interval $I = [0,\ell]$

$$Z((-1)^F_I) = \langle\psi|e^{\pi i\sum_{0\leq x\leq\ell} f^\dagger_x f_x}|\psi\rangle\,. \tag{4.34}$$

We have two minima $b = \pm 1/2$, both of which contribute $Z((-1)^F_I)$. This is something like zero modes in the partition function. We have that

$$Z((-1)^F_I) = \sum_{b=\pm 1/2} Z(b) \tag{4.35}$$

$$= 2\cos(k_F\ell)\cdot\left[2\sin\left(\frac{\pi\ell}{L}\right)\right]^{-1/2}\cdot\left(\frac{Lk_F}{\pi}\right)^{-1/2}\cdot[G(1/2)G(3/2)]^2. \tag{4.36}$$

3. Rényi entanglement entropy

The Rényi entanglement entropy is

$$S_n = \frac{1}{1-n}\sum_{k=-\frac{n-1}{2},\dots,\frac{n-1}{2}}\ln Z(b=k/n) \tag{4.37}$$

$$= \frac{n+1}{6n}\log\left[\frac{Lk_F}{\pi}\cdot 2\sin\left(\frac{\pi\ell}{L}\right)\right] + \frac{2}{1-n}\sum_{k=-(n-1)/2}^{(n-1)/2}\log[G(1+k/n)G(1-k/n)] \tag{4.38}$$

$$= \frac{n+1}{6n}\log\left[\frac{Lk_F}{\pi}\cdot 2\sin\left(\frac{\pi\ell}{L}\right)\right] + \frac{4}{1-n}f(n), \tag{4.39}$$

where $f(n)$ is given as

$$f(n_o) = \frac{n_o^2-1}{n_o}\left(\frac{1}{12}-\log A\right) + \frac{\log n_o}{12n_o} + \sum_{k=1}^{(n_o-1)/2}\Big[(k/n_o)\log\Gamma(k/n_o) - (k/n_o)\log\Gamma(1-k/n_o)\Big] \tag{4.40}$$

$$f(n_e) = \frac{n_e^2+1/2}{n_e}\left(\frac{1}{12}-\log A\right) - \frac{\log(n_e/2)}{24n_e} + \sum_{k=1/2}^{(n_e-1)/2}\Big[(k/n_e)\log\Gamma(k/n_e) - (k/n_e)\log\Gamma(1-k/n_e)\Big] \tag{4.41}$$

for odd (even) integers n_o (n_e). A is the GlaisherKinkelin constant. Note that the Rényi entanglement entropy S_n satisfies

$$S_n(\ell) = S_n(L-\ell). \tag{4.42}$$

D. Smooth kink adding operator

Let us consider the expectation value of the operator adding a one kink by

$$O = \langle\psi|e^{2\pi i\sum_x x/L}|\psi\rangle\,. \tag{4.43}$$

The corresponding generating function of the Toeplitz matrix is

$$\psi(\theta) = e^{i\theta}. \tag{4.44}$$

The Toeplitz determinant is zero since O is nothing but the momentum shift

$$Of_k^\dagger O^{-1} = f_{k+2\pi/L}. \tag{4.45}$$

Thus, there is no overlap

$$\langle\psi|O|\psi\rangle = 0. \tag{4.46}$$

E. Smooth function without winding

next, let us consider the expectation value of the operator parametrized by a smooth function $V : S^1 \to \mathbb{C}^\times$ without winding number.

$$O[V] = \langle\psi|e^{\sum_x V(2\pi x/L)f_x^\dagger f_x}|\psi\rangle\,. \tag{4.47}$$

The generating function of the Toeplitz matrix is

$$\psi(\theta) = e^{V(\theta)}. \tag{4.48}$$

This has no singularity. The Toeplitz determinant is given as

$$\langle\psi|O[V]|\psi\rangle = \exp\Big[\sum_k kV_kV_{-k}\Big] = \exp\Big[\frac{1}{2\pi}\oint_0^{2\pi} d\theta V(\theta)(-i\partial_\theta)V(\theta)\Big] \tag{4.49}$$

with V_k the Fourier component of $V(\theta)$,

$$V_k = \frac{1}{2\pi}\oint_0^{2\pi} d\theta V(\theta)e^{-ik\theta}. \tag{4.50}$$

V. (2+1)D CHERN INSULATOR

As an application of the formula, we estimate the entanglement entropy associated with a disc region for the Chern insulator with a single right-mover chiral mode. We employ the bulk-boundary correspondence: the reduced density matrix ρ_D of a disc region is given by the thermal state of the physical edge excitations

$$\rho_D = \frac{e^{-\frac{\xi}{v}H}}{\operatorname{tr} e^{-\frac{\xi}{v}H}}, \tag{5.1}$$

$$H = \frac{2\pi v}{L}\sum_{m\in\mathbb{Z}+\frac{1}{2}} m :\gamma_m^\dagger\gamma: -\frac{1}{24}, \tag{5.2}$$

where the edge excitations are written by the bulk fermions as

$$\gamma^\dagger(\frac{L\phi}{2\pi}) \sim \big(e^{-\phi/2-\pi i/4}\psi_1^\dagger(r,\phi) + e^{\phi/2+\pi i/4}\psi_2^\dagger(r,\phi)\big)e^{-\int^r dr' m(r')}, \tag{5.3}$$

and $L = |\partial D|$. The partial $U_{b,D} = e^{-2\pi ib\sum_{x\in D}\psi_x^\dagger\psi_x}$ transformation induces the U_b action on the edge CFT. We get

$$\langle\psi|U_{b,D}|\psi\rangle \sim \frac{\operatorname{tr}\big[e^{-2\pi ib\tilde{Q}}e^{-\frac{\xi}{v}H}\big]}{\operatorname{tr} e^{-\frac{\xi}{v}H}}, \qquad \tilde{Q} = \sum_{m\in\mathbb{Z}+\frac{1}{2}} :\gamma_m^\dagger\gamma_m:\,. \tag{5.4}$$

This is the partition function of the free Dirac fermions. We have

$$\langle\psi|U_{b,D}|\psi\rangle \sim \frac{Z_{\frac{1}{2},b+\frac{1}{2}}(\frac{i\xi}{L})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{i\xi}{L})} = \frac{Z_{\frac{1}{2}-b,\frac{1}{2}}(\frac{iL}{\xi})}{Z_{\frac{1}{2},\frac{1}{2}}(\frac{iL}{\xi})}. \tag{5.5}$$

For $b \neq 1/2$, the numerator is approximated by the unique vacuum state as

$$Z_{\frac{1}{2}-b,\frac{1}{2}}(\frac{iL}{\xi}) \sim \exp\Big[-\frac{2\pi L}{\xi}(\frac{1}{2}b^2-\frac{1}{24})\Big], \qquad (-\frac{1}{2}<b<\frac{1}{2}). \tag{5.6}$$

Thus,

$$\langle\psi|U_{b,D}|\psi\rangle \sim \exp\Big[-\frac{2\pi L}{\xi}\cdot\frac{1}{2}b^2\Big], \qquad (-\frac{1}{2}<b<\frac{1}{2}). \tag{5.7}$$

The Rényi entanglement entropy is

$$S_N \sim \frac{1}{1-N}\sum_{k=-\frac{N-1}{2},\dots,\frac{N-1}{2}}\Big[-\frac{2\pi L}{\xi}\cdot\frac{1}{2}\Big(\frac{k}{N}\Big)^2\Big] \tag{5.8}$$

$$= \frac{N+1}{24N}\cdot\frac{2\pi L}{\xi}. \tag{5.9}$$

The Von Neumann Entanglement entropy is

$$S = \lim_{N\to1} S_N = \frac{1}{12}\cdot\frac{2\pi L}{\xi}. \tag{5.10}$$

VI. (3+1)D TOPOLOGICAL INSULATOR

Let us consider the three-ball entanglement entropy of the $(3+1)d$ insulator

$$H = \sum_k \psi_k^\dagger\Big[(\frac{k^2}{2m}-\mu)\tau_z + v\tau_x\boldsymbol{\sigma}\cdot\boldsymbol{k}\Big]\psi_k. \tag{6.1}$$

The partial $U(1)$ transformation is approximated as

$$\langle\psi|U_{a,D}|\psi\rangle \sim \frac{\mathrm{tr}\,\big[e^{-2\pi i a Q}e^{-\frac{\xi}{v}H}\big]}{\mathrm{tr}\,e^{-\frac{\xi}{v}H}}, \tag{6.2}$$

$$H = \frac{v}{R}\sum_{n\in\mathbb{Z},n>0}\ \sum_{m=-(n-\frac{1}{2}),\dots,n-\frac{1}{2}}\big[2n\chi_{n,m}^\dagger\chi_{n,m} + 2n\chi_{-n,m}\chi_{-n,m}^\dagger\big], \tag{6.3}$$

$$Q = \sum_{n\in\mathbb{Z},n>0}\ \sum_{m=-(n-\frac{1}{2}),\dots,n-\frac{1}{2}}\big[\chi_{n,m}^\dagger\chi_{n,m} - \chi_{-n,m}\chi_{-n,m}^\dagger\big], \tag{6.4}$$

where R is the radius of the three-ball D. A direct calculation shows

$$\langle\psi|U_{a,D}|\psi\rangle \sim \frac{\prod_{n=1}^\infty(1+e^{-2\pi i a}q^n)^{2n}(1+e^{2\pi i a}q^n)^{2n}}{\prod_{n=1}^\infty(1+q^n)^{2n}(1+q^n)^{2n}} \qquad (q=e^{-2\xi/R}) \tag{6.5}$$

$$\sim \exp\Big[-2(2\xi/R)^{-2}\big\{\mathrm{Li}_3(-e^{-2\pi i a})+\mathrm{Li}_3(-e^{2\pi i a})\big\} - 3(2\xi/R)^{-2}\zeta(3) - \frac{1}{3}\log|\cos(\pi a)|\Big]. \tag{6.6}$$

The Rényi entanglement entropy is

$$S_N \sim \frac{1}{1-N}\sum_{k=-\frac{N-1}{2},\dots,\frac{N-1}{2}}\Big[-2(2\xi/R)^{-2}\big\{\mathrm{Li}_3(-e^{-2\pi i k/N})+\mathrm{Li}_3(-e^{2\pi i k/N})\big\} - 3(2\xi/R)^{-2}\zeta(3) - \frac{1}{3}\log|\cos(\pi k/N)|\Big] \tag{6.7}$$

$$= \frac{1}{1-N}\sum_{k=0}^{N-1}\Big[-\frac{R^2}{\xi^2}\mathrm{Li}_3(e^{\pi i/N}e^{2\pi i k/N}) - \frac{3\zeta(3)}{4}\frac{R^2}{\xi^2} - \frac{1}{3}\log\sin\Big(\frac{\pi}{N}\big(k+\frac{1}{2}\big)\Big)\Big] \tag{6.8}$$

$$= \frac{3}{4}\zeta(3)\frac{1+N+N^2}{N^2}\frac{R^2}{\xi^2} - \frac{1}{3}\log 2. \tag{6.9}$$

The Von Neumann Entanglement entropy is

$$S = \lim_{N\to 1} S_N = \frac{9}{4}\zeta(3)\frac{R^2}{\xi^2} - \frac{1}{3}\log 2. \tag{6.10}$$

Here we used

$$\sum_{k=0}^{N-1} \mathrm{Li}_s(ze^{2\pi i k/N}) = N^{1-s}\mathrm{Li}_s(z^N), \qquad \mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3), \tag{6.11}$$

$$\sum_{k=0}^{N-1} \log\sin\Big(\frac{\pi}{N}\big(k+\frac{1}{2}\big)\Big) = -(N-1)\log 2. \tag{6.12}$$

Appendix A: Toeplitz determinant and Fisher-Hartwig theorem

The symbols of Fisher-Hartwig (FH) class have the following form[1]

$$f(z) = e^{V(z)} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^{m} |z-z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad 0 \le \theta \le 2\pi, \tag{A1}$$

for some $m = 0, 1, 2, \ldots$, where

$$z_j = e^{i\theta_j}, \quad j = 0, 1, \ldots, m, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi, \tag{A2}$$

$$g_{z_j,\beta_j}(z) = g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j}, & 0 \le \arg z < \theta_j, \\ e^{-i\pi\beta_j}, & \theta_j \le \arg z < 2\pi, \end{cases} \tag{A3}$$

$$\mathrm{Re}\ \alpha_j > -\frac{1}{2}, \quad \beta_j \in \mathbb{C}, \quad j = 0, 1, \ldots, m, \tag{A4}$$

and $V(e^{i\theta})$ is a sufficiently smooth function on S^1. Note that $e^{V(e^{i\theta})}$ has no winding number.

Let $\{n_j\}_{j=0,\ldots,m}$ be a set of integers with $\sum_j n_j = 0$, and let $f(z;\hat{\beta})$ denote the Fisher-Hartwig (FH) symbol obtained by replacing β_j with $\hat{\beta}_j = \beta_j + n_j$ and $e^{V(z)}$ by $(\prod_j z_j^{n_j})e^V = e^{V + i\sum_j n_j\theta_j}$. Then, $f(z;\hat{\beta})$ gives another FH representation for $f(z)$,

$$f(z) = f(z;\hat{\beta}), \quad \hat{\beta}_j = \beta_j + n_j, \quad \sum_j n_j = 0. \tag{A5}$$

Given β, we call

$$O_\beta = \{\hat{\beta}_j = \beta_j + n_j, \sum_j n_j = 0\} \tag{A6}$$

the *orbit* of β. We consider the discrete minimization problem

$$F_\beta = \min_{\hat{\beta}\in O_\beta} \sum_j (\mathrm{Re}\ \hat{\beta}_j)^2. \tag{A7}$$

Let

$$\mathcal{M}_\beta = \{\hat{\beta} \in O_\beta | \sum_j (\mathrm{Re}\ \hat{\beta}_j)^2 = F_\beta\}. \tag{A8}$$

We say $\mathcal{M}_\beta$ is *non-degenerate* if $\alpha_j \pm \hat{\beta}_j \neq -1, -2, \ldots$ for all j and all $\hat{\beta} \in \mathcal{M}_\beta$.

Theorem Suppose $\mathcal{M}_\beta$ is non-degenerate. Then, as $n \to \infty$,

$$D_n(f) = \sum_{\hat{\beta}\in\mathcal{M}_\beta} \Big[R_n\big(f(\hat{\beta})\big)(1+o(1))\Big], \tag{A9}$$

Note that this is the **sum of minima**, not the product.

$$R_n(f) = E(e^V, \alpha_0, \dots, \alpha_m, \beta_0, \dots, \beta_m, \theta_0, \dots, \theta_m) \cdot n^{\sum_j (\alpha_j^2 - \beta_j^2)} e^{nV_0}(1 + o(1)), \quad V_0 = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\theta})d\theta, \tag{A10}$$

$$E(e^V, \alpha_0, \dots, \alpha_m, \beta_0, \dots, \beta_m, \theta_0, \dots, \theta_m) \tag{A11}$$

$$= E(e^V) \prod_{j=0}^{m} \left[b_+(z_j)^{-\alpha_j+\beta_j} b_-(z_j)^{-\alpha_j-\beta_j} \right] \tag{A12}$$

$$\cdot \prod_{0 \le j < k \le m} \left[|z_j - z_k|^{2(\beta_j\beta_k - \alpha_j\alpha_k)} \left(\frac{z_k}{z_j e^{i\pi}} \right)^{\alpha_j\beta_k - \alpha_k\beta_j} \right] \cdot \prod_{j=0}^{m} \frac{G(1+\alpha_j+\beta_j)G(1+\alpha_j-\beta_j)}{G(1+2\alpha_j)}. \tag{A13}$$

$$b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \quad b_-(z) = e^{\sum_{k=-1}^{-\infty} V_k z^k}, \tag{A14}$$

$$E(e^V) = e^{\sum_{k=1}^{\infty} k V_k V_{-k}}, \quad V_k = \text{Fourier coefficient of } V(e^{i\theta}), \tag{A15}$$

$$G(z) = \text{Barnes G-function}, \tag{A16}$$

$$G(1+z) = (2\pi)^{z/2} \exp\Big(-\frac{z + (1+\gamma)z^2}{2} \Big) \prod_{k=1}^{\infty} \Big[\big(1 + \frac{z}{k}\big)^k e^{\frac{z^2}{2k} - z} \Big]. \tag{A17}$$

$$\ln G(1+z) = \frac{z}{2}\log(2\pi) - \frac{z + (1+\gamma)z^2}{2} + \sum_{r=1}^{\infty} \Big[r\log(1 + z/r) + \frac{z^2}{2r} - z \Big] \tag{A18}$$

$$= \frac{z}{2} - \frac{z + (1+\gamma)z^2}{2} + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} z^{k+1}, \tag{A19}$$

For $\alpha_j = 0$,

$$E(e^V, \{\alpha_j = 0\}, \{\beta_j\}, \{\theta_j\}) \tag{A20}$$

$$= E(e^V) \prod_{j=0}^{m} \left[b_+(z_j)^{\beta_j} b_-(z_j)^{-\beta_j} \right] \cdot \prod_{0 \le j < k \le m} |z_j - z_k|^{2\beta_j\beta_k} \cdot \prod_{j=0}^{m} [G(1+\beta_j)G(1-\beta_j)]. \tag{A21}$$

Here we used $G(1) = 1$. Moreover, if $e^{V(z)}$ is constant, we have that

$$E(e^V = \text{const.}, \{\alpha_j = 0\}, \{\beta_j\}, \{\theta_j\}) = \prod_{0 \le j < k \le m} |z_j - z_k|^{2\beta_j\beta_k} \cdot \prod_{j=0}^{m} [G(1+\beta_j)G(1-\beta_j)], \tag{A22}$$

That is to say,

$$R_n(f) = \prod_{0 \le j < k \le m} |z_j - z_k|^{2\beta_j\beta_k} \cdot \prod_{j=0}^{m} [G(1+\beta_j)G(1-\beta_j)] \cdot n^{\sum_j (\alpha_j^2 - \beta_j^2)} e^{nV_0}(1 + o(1)) \tag{A23}$$

$$= \prod_{0 \le j < k \le m} \left[2 - 2\cos(\theta_j - \theta_k) \right]^{\beta_j\beta_k} \cdot \prod_{j=0}^{m} [G(1+\beta_j)G(1-\beta_j)] \cdot n^{\sum_j (\alpha_j^2 - \beta_j^2)} e^{nV_0}(1 + o(1)). \tag{A24}$$

1. $\sum_{k=-\frac{n-1}{2},\dots,\frac{n-1}{2}} \ln G(1 + k/n)$

Let

$$f(n) := \sum_{k=-\frac{n-1}{2},\dots,\frac{n-1}{2}} \ln G(1 + k/n) \tag{A25}$$

$$= -\frac{(1+\gamma)(n^2-1)}{24n} + \sum_{r=1}^{\infty} \Big[\frac{n^2-1}{24nr} + r\log\Big[\Big(\frac{1}{nr}\Big)^n \Big(n\Big(r - \frac{1}{2}\Big) + \frac{1}{2} \Big)_n \Big] \Big], \tag{A26}$$

where

$$(a)_n = a(a+1)\cdots(a+n) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{A27}$$

Some examples (by Mathematica):

$$f(1) = 0, \tag{A28}$$

$$f(2) = \frac{1}{16}\left(-36\log(A) + 3 + 4\log\left(\Gamma\left(\frac{1}{4}\right)\right) - 4\log\left(\Gamma\left(\frac{3}{4}\right)\right)\right), \tag{A29}$$

$$f(3) = \frac{1}{36}\left(-96\log(A) + 8 + \log(3) + 12\log\left(\Gamma\left(\frac{1}{3}\right)\right) - 12\log\left(\Gamma\left(\frac{2}{3}\right)\right)\right), \tag{A30}$$

$$f(4) = \frac{1}{96}\left(-396\log(A) + 33 - \log(2) + 12\log\left(\Gamma\left(\frac{1}{8}\right)\right) + 36\log\left(\Gamma\left(\frac{3}{8}\right)\right) - 36\log\left(\Gamma\left(\frac{5}{8}\right)\right) - 12\log\left(\Gamma\left(\frac{7}{8}\right)\right)\right), \tag{A31}$$

$$f(5) = \frac{1}{60}\left(\log(5) - 12\left(24\log(A) - 2 + \log\left(\frac{\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{1}{5}\right)}\right) - 2\log\left(\Gamma\left(\frac{2}{5}\right)\right) + 2\log\left(\Gamma\left(\frac{3}{5}\right)\right)\right)\right), \tag{A32}$$

$$f(6) = \frac{1}{144}\Big[-876\log(A) + 73 - \log(3) + 36\log\left(\Gamma\left(\frac{1}{4}\right)\right) - 36\log\left(\Gamma\left(\frac{3}{4}\right)\right) + 12\log\left(\Gamma\left(\frac{1}{12}\right)\right) \tag{A33}$$

$$+ 60\log\left(\Gamma\left(\frac{5}{12}\right)\right) - 60\log\left(\Gamma\left(\frac{7}{12}\right)\right) - 12\log\left(\Gamma\left(\frac{11}{12}\right)\right)\Big], \tag{A34}$$

$$f(7) = \frac{1}{84}\left(\log(7) - 12\left(48\log(A) - 4 + \log\left(\frac{\Gamma\left(\frac{6}{7}\right)}{\Gamma\left(\frac{1}{7}\right)}\right) - 2\log\left(\Gamma\left(\frac{2}{7}\right)\right) - 3\log\left(\Gamma\left(\frac{3}{7}\right)\right) + 3\log\left(\Gamma\left(\frac{4}{7}\right)\right) + 2\log\left(\Gamma\left(\frac{5}{7}\right)\right)\right)\right). \tag{A35}$$

These suggests the following general expressions

$$f(n_o) = \frac{n_o^2 - 1}{n_o}\Big(\frac{1}{12} - \log A\Big) + \frac{\log n_o}{12 n_o} + \sum_{k=1}^{(n_o-1)/2}\Big[(k/n_o)\log\Gamma(k/n_o) - (k/n_o)\log\Gamma(1 - k/n_o)\Big] \tag{A36}$$

$$f(n_e) = \frac{n_e^2 + 1/2}{n_e}\Big(\frac{1}{12} - \log A\Big) - \frac{\log(n_e/2)}{24 n_e} + \sum_{k=1}^{n_e/2}\Big[\frac{k-1/2}{n_e}\log\Gamma\Big(\frac{k-1/2}{n_e}\Big) - \frac{k-1/2}{n_e}\log\Gamma\Big(1 - \frac{k-1/2}{n_e}\Big)\Big] \tag{A37}$$

$$= \frac{n_e^2 + 1/2}{n_e}\Big(\frac{1}{12} - \log A\Big) - \frac{\log(n_e/2)}{24 n_e} + \sum_{k=1/2}^{(n_e-1)/2}\Big[(k/n_e)\log\Gamma(k/n_e) - (k/n_e)\log\Gamma(1 - k/n_e)\Big]. \tag{A38}$$

where $n_o(n_e)$ represents an odd (even) integer, and A is the GlaisherKinkelin constant $A \cong 1.28243$. Recall that $\zeta'(-1,0) = 1/12 - \ln A$.

Appendix B: Backup

$$\ln[G(1+z)G(1-z)] = (1+\gamma)z^2 - \sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}z^{2k+2}. \tag{B1}$$

$$\ln G(1+z) = \frac{z(1-z)}{2} + \frac{z}{2}\ln 2\pi + z\ln\Gamma(z) - \int_0^z dx\ln\Gamma(x), \tag{B2}$$

$$G(1+z) = \Gamma(z)G(z), \quad G(1) = 1. \tag{B3}$$

Some useful formulae:[2]

$$\ln G(1+z) = z\ln\Gamma(z) + \zeta'(-1) - \zeta'(-1,z) \tag{B4}$$

$$= z(\zeta'(0,z) - \zeta'(0)) + \zeta'(-1) - \zeta'(-1,z), \tag{B5}$$

Here, $\zeta(s,a)$ is the Hurwitz zeta function which is defined by the analytic continuation of

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\quad \mathrm{Re}\ s>1, \mathrm{Re}\ a>0. \tag{B6}$$

$$\zeta'(t,z)=\frac{d}{dt}\zeta(t,z), \tag{B7}$$

$$\zeta'(-1)=\zeta'(-1,0)=2\int_0^{\infty}dx\frac{x\ln x}{e^{2\pi x}-1}=\frac{1}{12}-\ln A,\quad \zeta'(0)=-\frac{1}{2}\ln(2\pi), \tag{B8}$$

$$\zeta'(-1,z)=\frac{z^2}{2}\ln z-\frac{z^2}{4}-\frac{z}{2}\ln z+2z\int_0^{\infty}dx\frac{\tan^{-1}(x/z)}{e^{2\pi x}-1}+\int_0^{\infty}dx\frac{x\ln(x^2+z^2)}{e^{2\pi x}-1},\quad \mathrm{Re}\ z>0. \tag{B9}$$

Here, A is the GlaisherKinkelin constant $A\cong 1.28243$. We will use

$$\sum_{p=0}^{q-1}\zeta(s,a+p/q)=q^s\zeta(s,qa),\qquad \frac{\partial}{\partial a}\zeta(s,a)=-s\zeta(s+1,a). \tag{B10}$$

The former one is shown as

$$\zeta(s,qa)=\sum_{n=0}^{\infty}\frac{1}{(qa+n)^s}=q^{-s}\sum_{n=0}^{\infty}\frac{1}{(a+n/q)^s}=q^{-s}\sum_{p=0}^{q-1}\sum_{n'=0}^{\infty}\frac{1}{(a+p/q+n')^s}=q^{-s}\sum_{p=0}^{q-1}\zeta(s,a+p/q). \tag{B11}$$

At integer $s=n$,

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1}. \tag{B12}$$

The following may be useful

$$\ln G(1+z)=z\ln\Gamma(1+z)-z\ln z+\zeta'(-1)-\zeta'(-1,z), \tag{B13}$$

a. $\zeta'(-1,z)$

$$\frac{d}{ds}\Big|_{s\to-1}\sum_{k=-\frac{n-1}{2},\ldots,\frac{n-1}{2}}\zeta(s,k/n)=\frac{d}{ds}\Big|_{s\to-1}\sum_{k=0}^{n-1}\zeta\Big(s,-\frac{n-1}{2n}+\frac{k}{n}\Big) \tag{B14}$$

$$=\frac{d}{ds}\Big|_{s\to-1}n^s\zeta\Big(s,-\frac{n-1}{2}\Big) \tag{B15}$$

$$=\frac{1}{n}\Big[\ln(n)\zeta\Big(-1,-\frac{n-1}{2}\Big)+\zeta'\Big(-1,-\frac{n-1}{2}\Big)\Big]. \tag{B16}$$

Here,

$$\zeta\Big(-1,-\frac{n-1}{2}\Big)=\begin{cases}-(n^2-1)/8+\zeta(-1) & (n:\ \text{odd})\\ -(n^2+2n-4)/8+\zeta(-1,1/2) & (n:\ \text{even})\end{cases} \tag{B17}$$

One can use

$$\zeta'(1-2k,1/2)=-\frac{B_{2k}\ln 2}{4^k k}-\frac{(2^{2k-1}-1)\zeta'(-2k+1)}{2^{2k-1}}, \tag{B18}$$

$$\zeta'(-1,1/2)=-\frac{B_2\ln 2}{4}-\frac{\zeta'(-1)}{2}=\frac{1}{2}\ln A-\frac{1}{24}(1+\ln 2), \tag{B19}$$

$$\zeta(-1)=-\frac{1}{12},\quad \zeta(-1,1/2)=\frac{1}{24}. \tag{B20}$$

b. $z\zeta'(0,z)$

$$\frac{d}{ds}\Big|_{s\to 0}\sum_{k=-\frac{n-1}{2},\ldots,\frac{n-1}{2}}(k/n)\zeta(s,k/n)=\frac{d}{ds}\Big|_{s\to 0}\sum_{k=0}^{n-1}\Big(-\frac{n-1}{2n}+\frac{k}{n}\Big)\zeta\Big(s,-\frac{n-1}{2n}+\frac{k}{n}\Big). \tag{B21}$$

The first part is given by

$$-\frac{n-1}{2n}\frac{d}{ds}\Big|_{s\to 0}\sum_{k=0}^{n-1}\zeta\Big(s,-\frac{n-1}{2n}+\frac{k}{n}\Big)=-\frac{n-1}{2n}\zeta\Big(0,\frac{n-1}{2}\Big)=\frac{1}{4n}(n-1)(n-2). \tag{B22}$$

The second part is

$$\frac{d}{ds}\Big|_{s\to 0}\sum_{k=1}^{n-1}\frac{k}{n}\zeta\Big(s,-\frac{n-1}{2n}+\frac{k}{n}\Big) \tag{B23}$$

c. *Approach from* z^{2k+2}

$$f(n,k):=\sum_{r=-\frac{n-1}{2},\ldots,\frac{n-1}{2}}(r/n)^{2k+2}=n^{-2k-2}\left(\zeta\left(-2k-2,\frac{1-n}{2}\right)-\zeta\left(-2k-2,\frac{1-n}{2}+n\right)\right). \tag{B24}$$

$$f(1,k)=1,\qquad \frac{d}{dn}f(n,k)\Big|_{n\to 1}=-2(k+1)\zeta(-2k-1)=B_{2k+2}. \tag{B25}$$

A lessen is that z^{2k+2} is replaced by the Bernoulli numbers B_{2k+2}.

$$\Big[\frac{1}{n-1}\cdot\sum_{r=-\frac{n-1}{2},\ldots,\frac{n-1}{2}}\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}(r/n)^{2k+2}\Big]\Big|_{n\to 1}=\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}B_{2k+2}. \tag{B26}$$

This does not converge...

d. $z\ln\Gamma(1+z)$

$$\ln\Gamma(1+z)=-\gamma z+\int_0^\infty\frac{e^{-zt}-1+zt}{t(e^t-1)}dt \tag{B27}$$

$$=\int_0^\infty\frac{e^{-zt}-ze^{-t}-1+z}{t(e^t-1)}dt. \tag{B28}$$

Then,

$$\sum_{k=-\frac{n-1}{2},\ldots,\frac{n-1}{2}}(k/n)\Big[\frac{e^{-zt}-ze^{-t}-1+z}{t(e^t-1)}\Big]_{z\to k/n} \tag{B29}$$

$$=\frac{e^{-t}}{12n\left(e^t-1\right)t\left(e^{t/n}-1\right)^2}\cdot\Big[-2\left(n^2-1\right)e^{\frac{t}{n}+t}+\left(n^2-1\right)e^{\frac{2t}{n}+t}+2\left(n^2-1\right)e^{t/n}-\left(n^2-1\right)e^{\frac{2t}{n}} \tag{B30}$$

$$+\left(n^2-1\right)e^t-n^2+6(n-1)e^{\frac{(n+1)t}{2n}}-6(n-1)e^{\frac{3(n+1)t}{2n}}-6(n+1)e^{\frac{(n+3)t}{2n}}+6(n+1)e^{\frac{3nt+t}{2n}}+1\Big] \tag{B31}$$

$$=:f(t,n). \tag{B32}$$

We have

$$f(t,1)=0, \quad \frac{d}{dn}f(t,n)\Big|_{n\to 1}=\frac{e^{-t}\left(e^{2t}(6t-8\sinh(t)+4\cosh(t)-3)-1\right)}{6\left(e^t-1\right)^3 t}, \tag{B33}$$

$$\int_0^\infty \left[\frac{d}{dn}f(t,n)\Big|_{n\to 1}\right]dt \sim -0.0857148. \tag{B34}$$

e. $-z\ln z$

f. $\zeta'(-1)-\zeta'(-1,z)$

$$\sum_{k=-\frac{n-1}{2},\dots,\frac{n-1}{2}} \ln G(1+k/n)=\begin{cases}\sum_{k=0}^{n-1}\ln G(1+k/n)-\sum_{k=1}^{(n-1)/2}\ln\Gamma(1-k/n) & (n:\ \text{odd})\\ \sum_{k=0}^{n-1}\ln G(1+1/(2n)+k/n)-\sum_{k=0}^{n/2-1}\ln\Gamma(1-1/(2n)-k/n) & (n:\ \text{even})\end{cases} \tag{B35}$$

Notice that

$$\sum_{k=0}^{n-1}\zeta(s,z+k/n)=n^s\zeta(s,nz), \tag{B36}$$

$$\zeta(s,-q)=(-q)^{-s}+\zeta(s,1-q), \tag{B37}$$

$$\zeta'(s,q)=-sq^{-s-1} \tag{B38}$$

we have for odd n

$$\sum_{k=-\frac{n-1}{2},\dots,\frac{n-1}{2}} \ln G(1+k/n)=\sum_{k=0}^{n-1}\ln G(1+k/n)-\sum_{k=1}^{(n-1)/2}\ln\Gamma(1-k/n) \tag{B39}$$

$$=\sum_{k=0}^{n-1}(k/n)\ln\Gamma(k/n)+n\zeta'(-1)-\sum_{k=0}^{n-1}\zeta'(-1,k/n)-\sum_{k=1}^{(n-1)/2}\ln\Gamma(1-k/n). \tag{B40}$$

$$\frac{d}{ds}\left[\sum_{k=0}^{n-1}\zeta(s,k/n)\right]\Big|_{s\to -1}=\frac{d}{ds}\left[n^s\zeta(s)\right]\Big|_{s\to -1} \tag{B41}$$

$$=\left[n^s(\zeta(s)\ln n+\zeta'(s))\right]\Big|_{s\to -1} \tag{B42}$$

$$=\frac{1-\ln n-12\ln A}{12n} \tag{B43}$$

$$=\frac{1/12-\ln A}{n}-\frac{\ln n}{12n}. \tag{B44}$$

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[1] P. Deift, A. Its, and I. Krasovsky, Communications on Pure and Applied Mathematics **66**, 1360 (2012), arXiv:1207.4990.
[2] V. S. Adamchik, arXiv preprint math/0308086 , 18 (2003), arXiv:0308086 [math].