An overview of symmetry protected topological phases

Ken Shiozaki　YITP

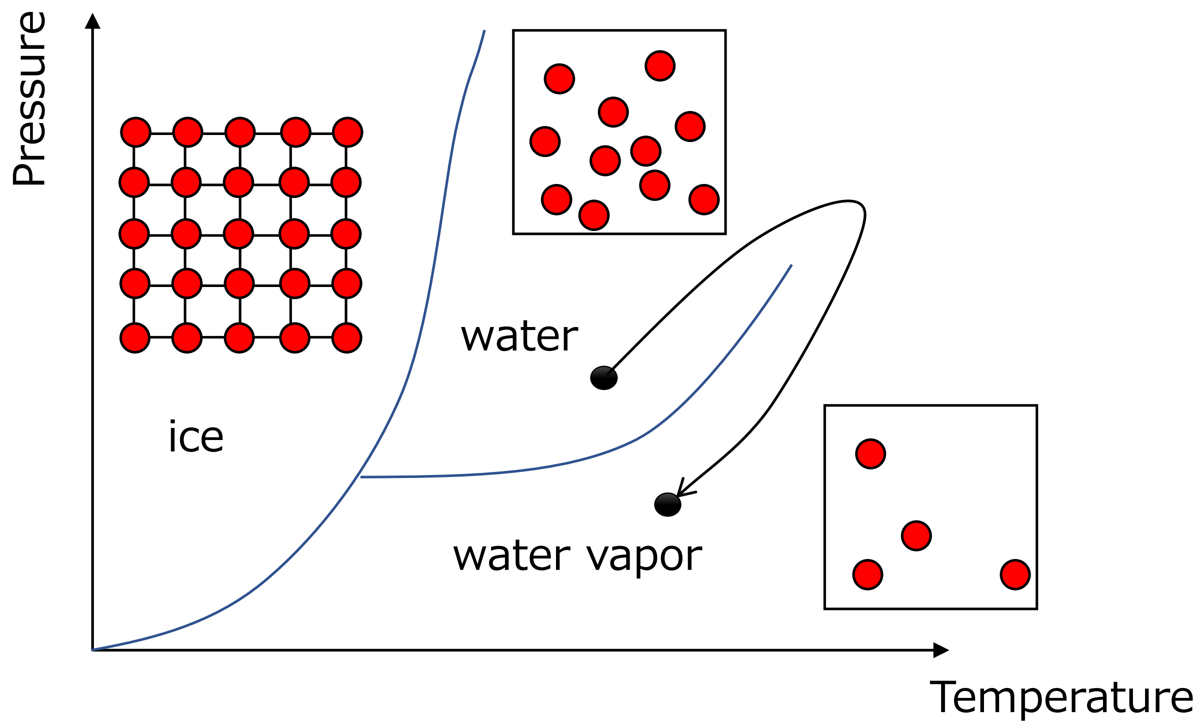
5/24/2019

@Nagoya Univ.

Contents

- Partition functions and classification of SPT phases
- Bulk-boundary correspondence
- Model construction

Topological equivalence



- “Topological” equivalence: If there exists a pass connecting two phases A and B without a phase transition, A and B are considered to be in a same phase.
- Ice ≠ water
- Water = water vapor
- SSB of translation symmetry between {ice} and {water, water vapor}

Topological phases of matter

- Logically, there may exists phase distinctions without SSB in a certain class of phases of matter.

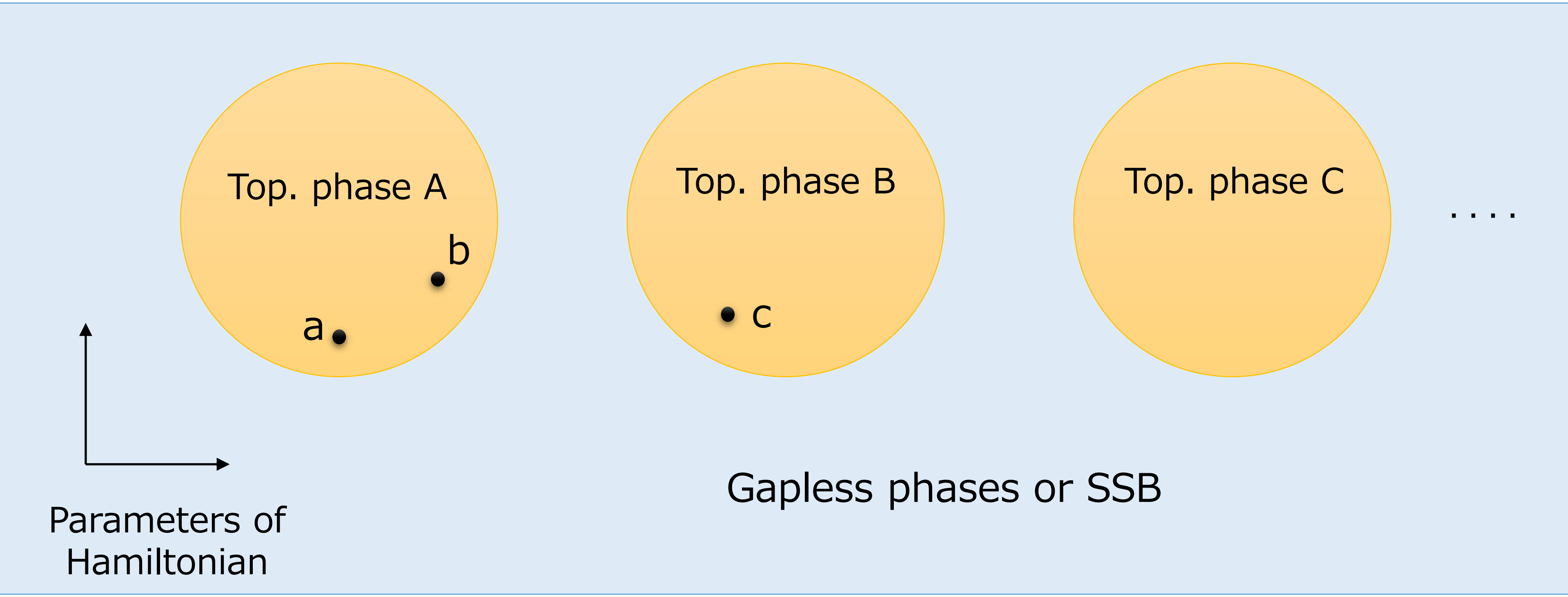
- In topological phases, we consider the following setup:

  ✓ Zero temperature

  ✓ Gapped phases (there exists a finite energy gap in between the ground and the first excited state.)

  ✓ With symmetry (Z2 Ising, U(1) particle conservation, time-reversal, …)

- Let’s imagine a phase diagram for a given dof.



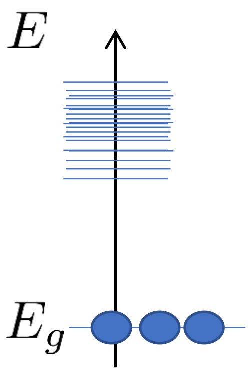
- A topological phase := an equivalence class under the equivalence relation.

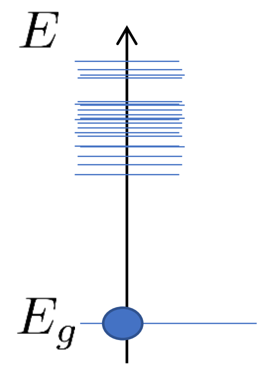
Symmetry Protected Topological phases

- In general, there exists a ground state degeneracy that depends on the global topology of the closed space manifold.

- SPT phases := topological phases that have a unique ground state for any closed space manifolds.

- Long-Range Entangled (LRE) topological phases = topological phases that have a ground state degeneracy for a closed space manifold.





- Exs of SPT phases: Haldane chain, topological insulators/superconductors, …
- Exs of LRE topological phases: Toric code, fluctional quantum Hall effect, …

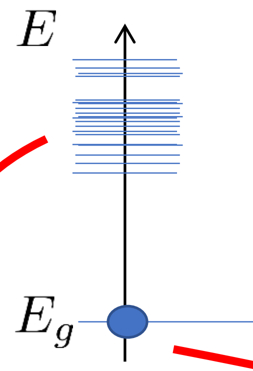
Classification of SPT phases

- How to classify SPT phases?
- Recall that in QFTs,

A theory = a set of correlation functions

- In SPT phases, all information should be encoded in the ground state.
- Excited states do not affect which topological phases a given gapped phases belongs to.
- No excited states
  -> no scale
    -> Topological quantum field theory (TQFT)
- Hilbert space is one-dimensional
  -> no operators
    -> Only partition functions are correlation functions.
- A set of correlation functions = a set of partition functions.

- The classification of SPT phases
  = the classification of U(1)-valued partition functions without a continues parameter [Chen-Gu-Liu-Wen, Kapustin, Freed-Hopkins, Yonekura, ⋯]



$$Z(M) = \int \prod_{\text{matter fields}} D\phi e^{-S_M[\phi]}$$

$$= \left|Z(M)\right| \times e^{iS_{\text{top}}(M)}$$

Euclidian spacetime manifold

Excited states

Characterizes SPT phases

Spacetime manifolds and external fields

- The partition function over a closed space manifold X and time circle S^1 is always unity:

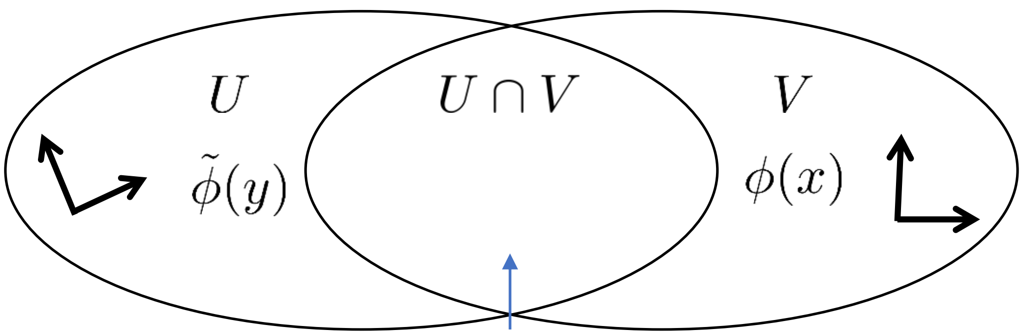
$$Z(X \times S^1) = \mathrm{Tr}(1) = \langle GS|GS \rangle = 1.$$

- Therefore, to distinguish different SPT phases, we should employ generic closed spacetime manifolds.

- We also have a external field A (a G-bundle) introduced by (pre)gauging global G symmetry.

1) generic spacetime manifolds

- We would like to define a theory over an arbitrary spacetime manifold.
- Recall that a manifold M is a set of patches and patch transformations.
- The low-energy dof would be described by a topological field theory.
- There, spacetime rotation symmetry is effectively emergent.
- The spacetime rotation symmetry can be used to define the patch transformation for the field.



$$\tilde{\phi}(y) = U\left(\frac{\partial x_\mu}{\partial y_\alpha}(y)\right)\phi(x(y))$$

- We get partition functions over closed spacetime manifolds.

$$Z(M) \in U(1)$$

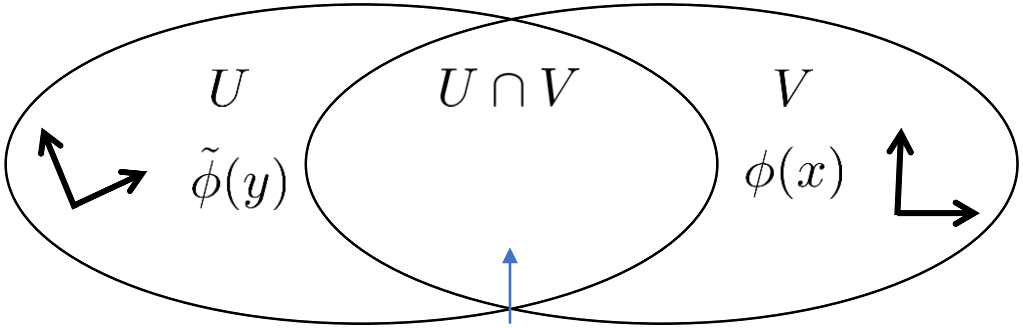
2) external fields

- We have global G symmetry.

- The field $\phi(x)$ is equipped with a G-action, i.e. $\phi(x)$ is a representation of G.

$$\phi(x) \mapsto g\phi(x), \quad g \in G.$$

- This G-action can be used to define the background G-field.



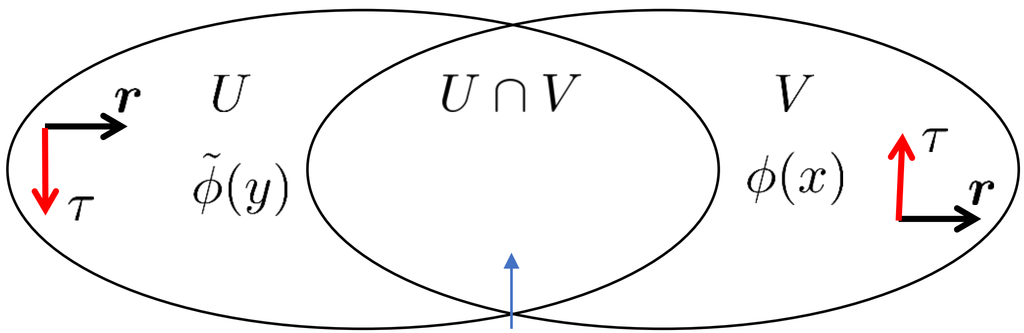
$$\tilde{\phi}(y) = g(x) U \left(\frac{\partial x_\mu}{\partial y_\alpha}(y) \right) \phi(x(y))$$

- We get partition functions over closed spacetime manifolds with G-field (G-bundles, says).

$$Z(M, A) \in U(1)$$

3) Non-orientable spacetime manifolds

- If an orientation-reversing symmetry (e.g. time-reversal, reflection) is present, one can define partition functions over non-orientable manifolds.



$$\tilde{\phi}(\boldsymbol{r}, -\tau) = T\phi(\boldsymbol{r}, \tau)$$

- We get partition functions over non-orientable manifold with G-field, if an orientation-reversing symmetry is present.

$$Z(M, A) \in U(1)$$

- In sum,

$$Z(M, A) = \int \prod_{\text{matter fields}} D\phi e^{-S_M(\phi, A)} \in U(1)$$

Background G-field

Closed spacetime manifold

- A Comment: Manifolds for fermions are more involved. The fermion field is rotated by the Spin(d) group, a double cover of SO(d), meaning that the spacetime manifold has (variants of) spin structure.

Some terminologies

- Gapped phases

  $\sim$ Invertible phases

  $\sim$Theories with U(1)-valued partition functions

- SPT phases (in my definition)

  $\sim$ deformation invariant invertible phases

  $\sim$ Theories with U(1)-valued partition functions without a continuous parameter

- For example, the 4d topological theta term of the background U(1)-field

$$\frac{\theta}{4\pi^2}\int_M F \wedge F$$

has continuous parameter $\theta \in [0,2\pi]$. This is not the partition function of SPT phases, but the partition functions of invertible phases.

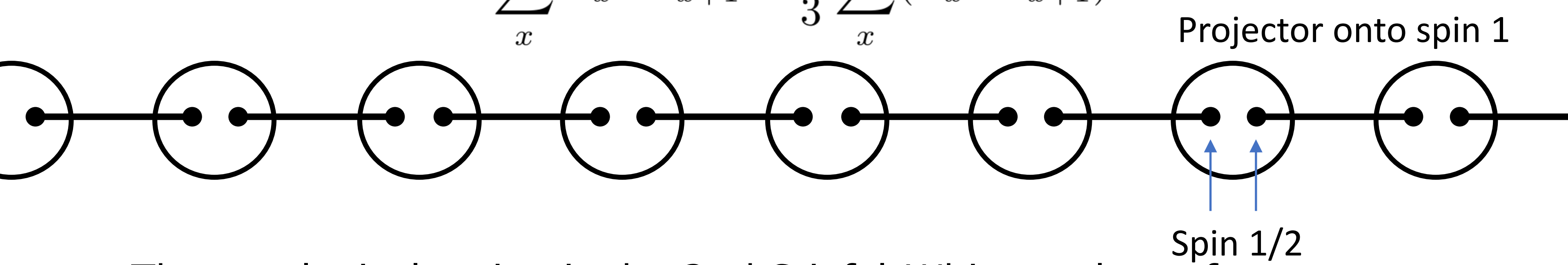
Ex: Haldane chain w/ TRS

- (1+1)d bosonic SPT phases w/ TRS
- Classification : Z_2
- Model: 1d antiferromagnetic spin chain with S=1 [Haldane].

$$H = \sum_x \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1}.$$

- An exactly solvable model: AKLT chain [Affleck-Lieb-Kennedy-Tasaki]

$$H = \sum_x \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \frac{1}{3} \sum_x (\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2.$$



- The topological action is the 2nd Stiefel-Whitney class of tangent bundle of spacetime manifold.

$$e^{iS_{\mathrm{top}}[M]} = e^{i\theta \int_M w_2(TM)}, \quad \theta \in \{0, \pi\}.$$

- Semiclassical description of the AF spin chain:

  Fluctuation field $\vec{n}(x) \in S^2$ from the AF ground state

$$\vec{S}_x \sim (-1)^x \, \vec{n}(x)$$

- TR-symmetry:

$$\hat{T}\, \vec{S}_x \, \hat{T}^{-1} = -\vec{S}_x \quad \Rightarrow \quad T.\vec{n}(x,\tau) = -\vec{n}(x,-\tau)$$

- Action O(3) NLσ model with topological theta term [Haldane]:

$$S[\vec{n}] = \frac{1}{2g} \int d\tau dx \, \{ (\partial_\tau \vec{n})^2 + (\partial_x \vec{n})^2 \} + 2\pi i S \; \times Q,$$

$$Q = \frac{1}{4\pi} \int d\tau \, dx \, \vec{n} \cdot \partial_\tau \, \vec{n} \times \partial_x \, \vec{n} \;\in\; \mathrm{Z},$$

where, S is the spin quantum number.

- For simplicity, let's consider the easy plane limit by setting

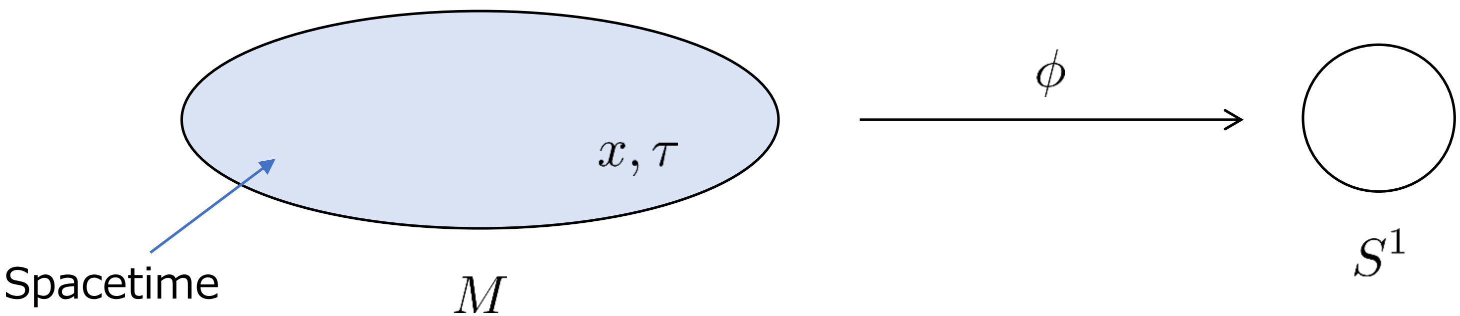
$$\vec{n} = (\cos\phi, \sin\phi, 0) \quad .$$

- The meron configuration becomes a single vortex.

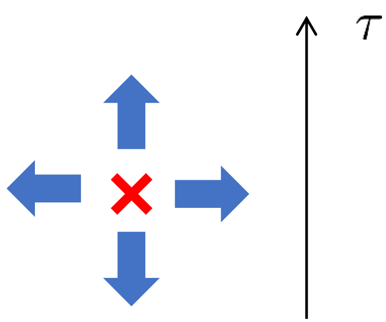
2d abelian sigma model

- A toy model of Haldane chain phase protected by TR symmetry. (For example, see [Takayoshi-Pujol-Tanaka, arXiv:1609.01316])
- Target space is S^1.
  "the easy plane limit of semiclassical description of the AF chain"

$$\phi : M \to \mathbb{R}/2\pi\mathbb{Z} \cong S^1$$



- With vortex events.
  (The field ϕ can be singular.)



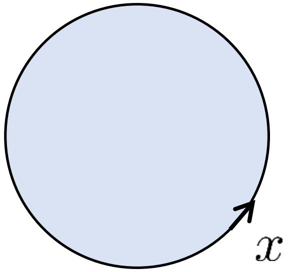
- Theta term

$$Z[M] = \int D\phi \exp\Big[- S_{\rm kin}[\phi] + i\theta(\#\ \text{of vortices})\Big], \qquad \theta \in [0, 2\pi]$$

Unimportant for topological phases

✓ Ex: The ground state functional on S[1] (Disc state):

$$GS[\phi(x)] = e^{i\theta \oint_{S^1} d\phi} = e^{i\theta(\text{winding number})}.$$



✓ Ex: Partition function over a closed oriented manifold:

$$Z[M] = 1.$$

- TR transformation

$$T\hat{\boldsymbol{S}}T^{-1} = -\hat{\boldsymbol{S}} \qquad \Rightarrow \qquad \phi(x,\tau) \mapsto \phi(x,-\tau) + \pi$$

- TR symmetry = the theory is invariant under the relabeling of path-integral variables by

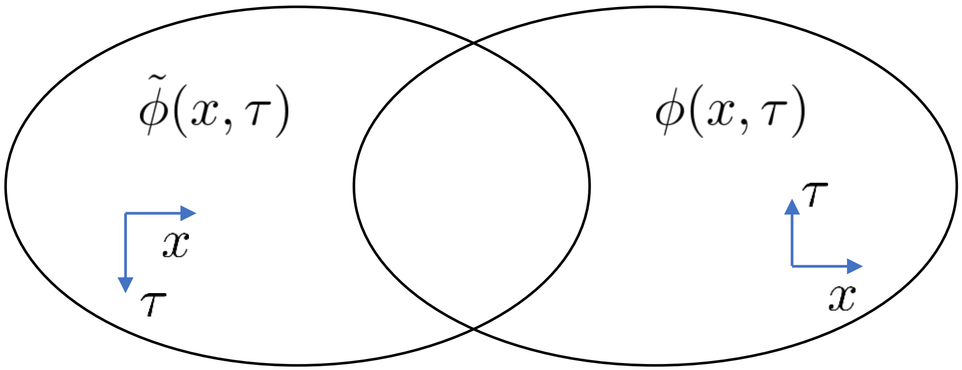
$$\phi(x,\tau) \mapsto \phi(x,-\tau) + \pi.$$

- In the presence of TR symmetry, θ is quantized.

$$\theta \in \{0, \pi\}$$

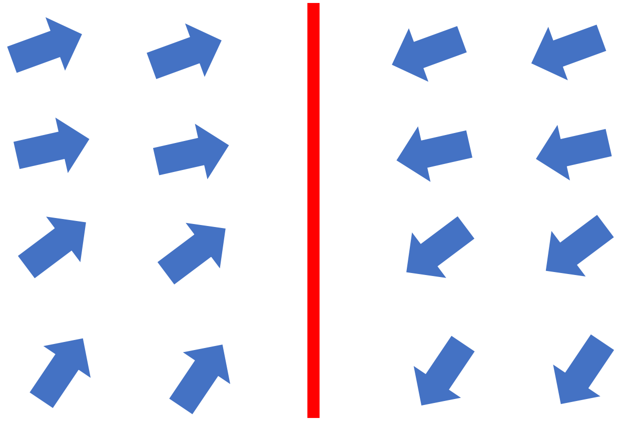
- $\theta = \pi$ is known to be a nontrivial SPT phase.
- How to detect θ?

- “Gauging” the TR symmetry = to define the theory on unoriented manifolds by the use of TR transformation.

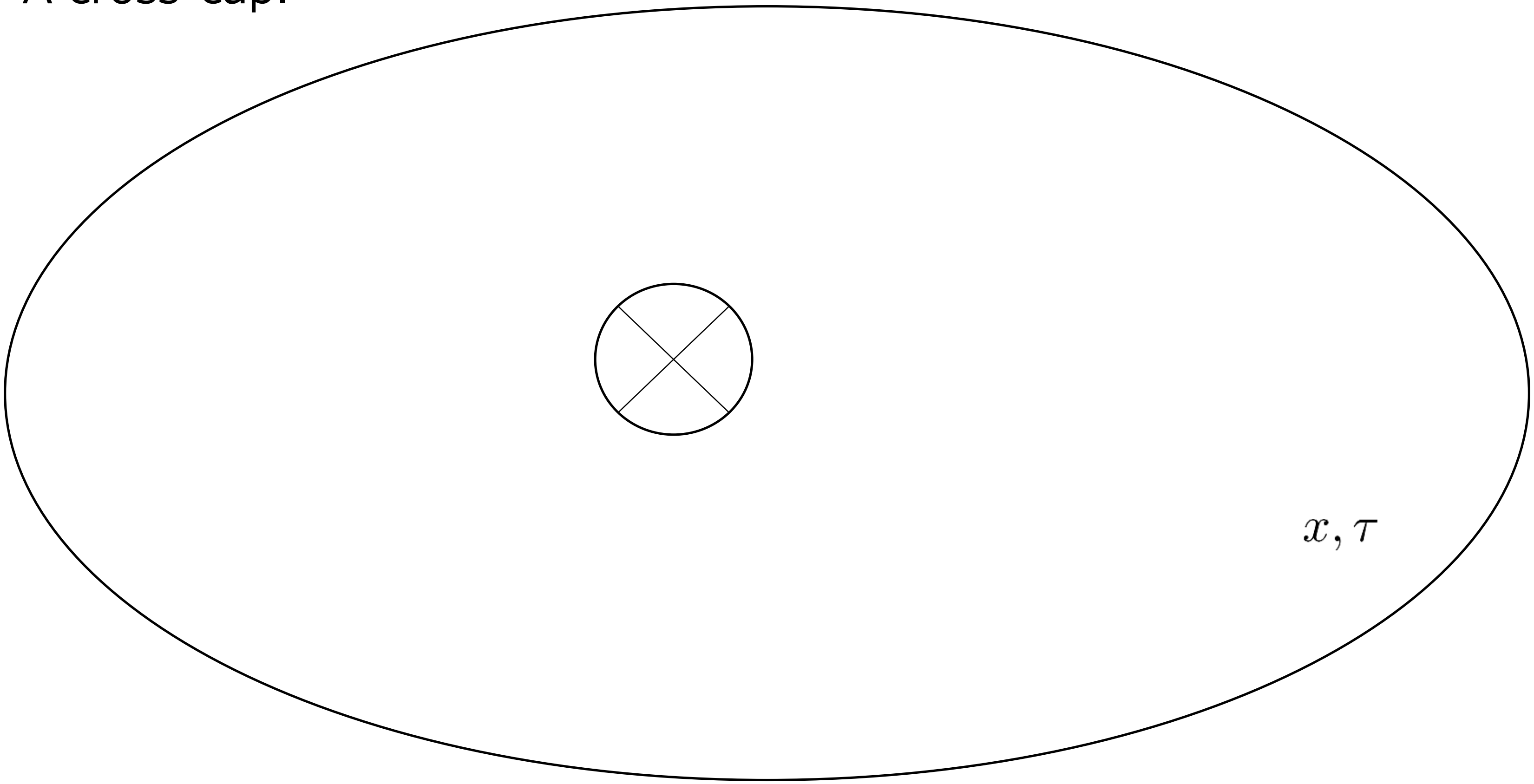


$$\tilde{\phi}(x,-\tau)=\phi(x,\tau)+\pi$$

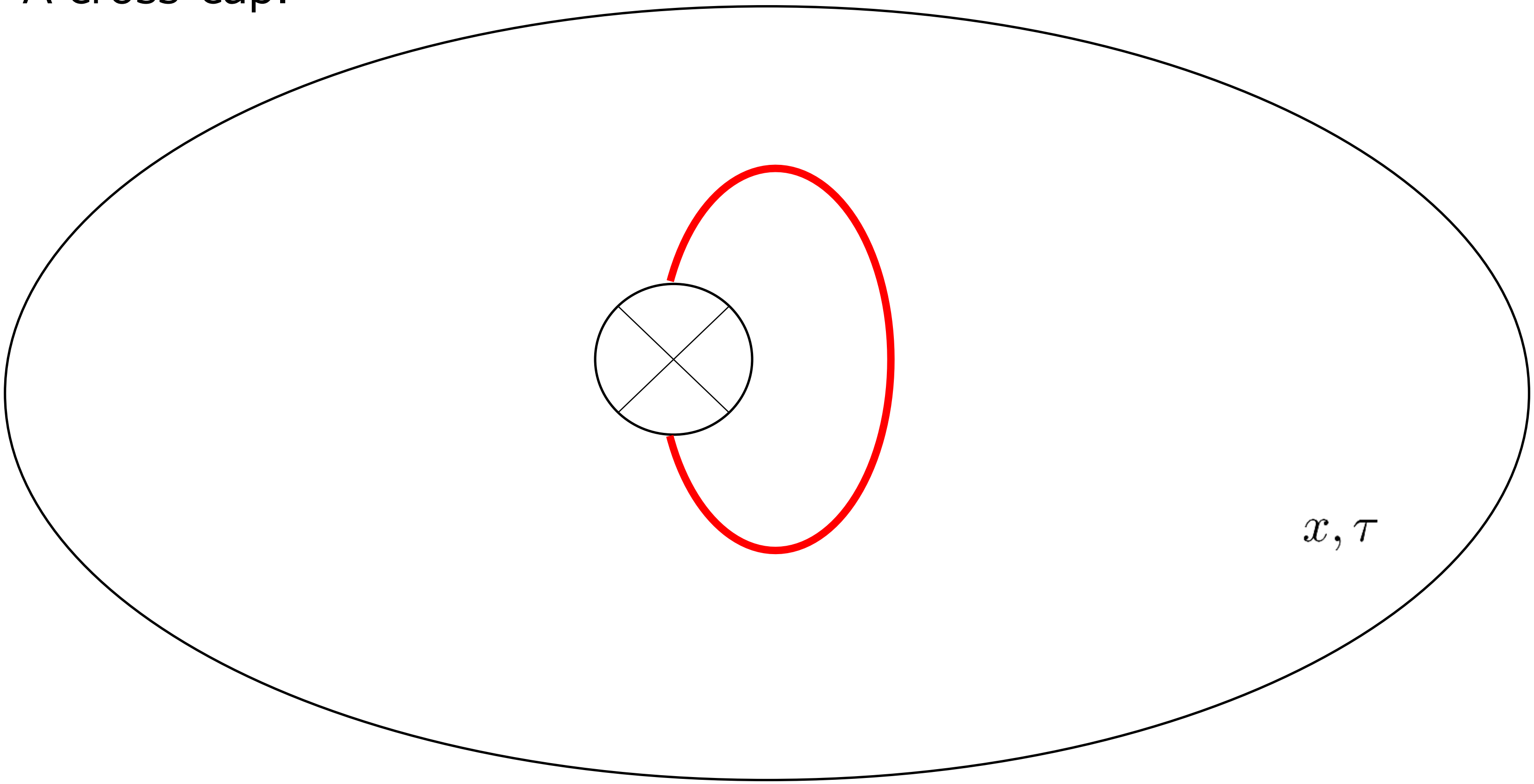
- At orientation reversing patches, the filed is shifted by π.



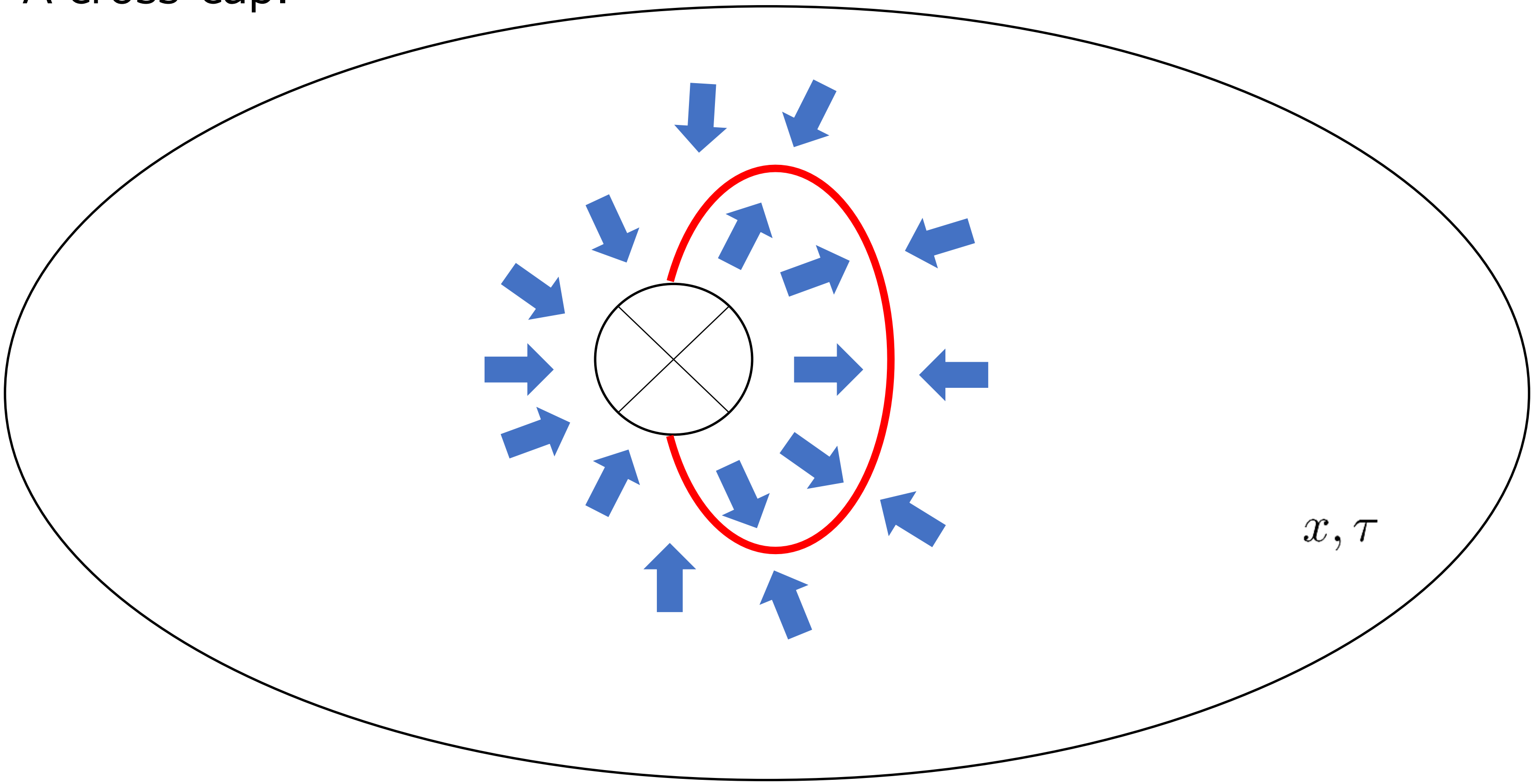
- A cross-cap.



- A cross-cap.



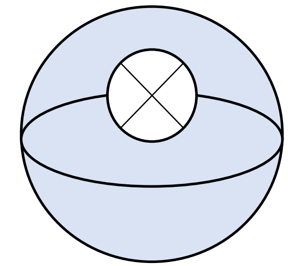
- A cross-cap.



- Around a cross cap, the vortex number should be odd.

- The partition function over the real projective plane:

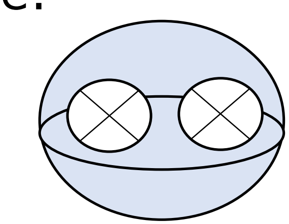
$$Z[RP^2] = e^{i\theta} = \begin{cases} 1 & (\theta = 0, \text{trivial}), \\ -1 & (\theta = \pi, \text{Haldane}). \end{cases}$$



Sphere with a cross-cap
= Real projective plane

- Cf. The partition function over the Klein bottle:

$$Z[KB] = e^{2i\theta} = 1$$



Klein bottle

- The partition function over the real projective plane RP^2 is the SPT invariant of Haldane chain phase.

- Consistent with the classification of topological action

$$e^{iS_{\text{top}}[M]} = e^{i\theta \int_M w_2(TM)}, \quad \theta \in \{0, \pi\}.$$

Classification of SPT phases

$$Z(M, A) = e^{2\pi i S_{\mathrm{top}}(M,A)} \in U(1)$$

- is the classification of U(1)-valued "topological" partition functions.
- Ex: Dijkgraaf-Witten topological actions classified by the group cohomology

$$H^d\big(BG, U(1)\big) \cong H^d\big(G, U(1)\big).$$

  -> bosonic nonchiral SPT phases.

- In the end, SPT phases are known to be classified by the Anderson dual of the cobordism group

$$(D\Omega)^d_{str}(BG) \cong Free\ \Omega^{str}_{d+1}(BG) \times Tor\ \Omega^{str}_d(BG) \quad \text{(non-canonical).}$$

- Global symmetry can be higher-form symmetries.
  -> Eilenberg–MacLane space $K(G, n)$ for n-form G symmetry.

Dijkgraaf-Witten theory [Dijkgraaf-Witten, 90]

- Motivation: What is a suitable generalization of the Chern-Simons action of U(1) gauge theory to that of finite groups?

- Given d-cocycle $\omega \in Z^d(BG, R/Z)$, the topological action is given by

$$S_M^\omega[A] = \int_M A^*(\omega), \qquad A: M \to BG \,,$$

  where BG is the classifying space of the group G.

  -> Twisted G gauge theory

$$\int DA\, W_1(A) W_2(A) \cdots \exp 2\pi i\, S_M^\omega\, [A]\,.$$

- By design, the action $S_M^\omega[A]$ is invariant under continuous deformation of G-bundle $A \to M$.

- The classification of DW G gauge theory is given by the group cohomology

$$H^d(BG, R/Z\,) \cong H^d\big(G, U(1)\big).$$

Relation to SPT phases

- The DW topological action

$$S_M^{\omega}[A] = \int_M A^*(\omega), \qquad A: M \to BG$$

gives topological partition functions classified by $H^d\big(G, U(1)\big)$.

$$Z_{SPT}(M,A) = \int \prod_{matter} D\,\phi\, e^{-S_M[\phi,A]} = |Z(M,A)| \times \exp 2\pi i\; S_M^{\omega}[A]$$

- Gauging G symmetry gives the DW twisted gauge theory:

$$Z_{DW}(M) = \int DA\, Z_{SPT}(M,A).$$

Exs of group cohomology

(for example, see [Chen-Gu-Liu-Wen, arXiv:1106.4772])

G	$d=1$	$d=2$	$d=3$	$d=4$
Z_2^T	0	Z_2	0	Z_2
Z_n	Z_n	0	Z_n	0
$Z_n \times Z_m$	$Z_n \times Z_m$	$Z_{\mathrm{gcd}(n,m)}$	$Z_n \times Z_m \times Z_{\mathrm{gcd}(n,m)}$	$Z_{\mathrm{gcd}(n,m)}^{\times 2}$
$SO(3)$	0	Z_2	Z	0
$U(1)$	Z	0	Z	0

Inequivalent 1d irreps

Inequivalent factor systems of projective reps

More involved anomalies

Ex: 1d bosonic SPT phases with $Z_2 \times Z_2$ symmery

- Group cohomology: $H^2\left(Z_2 \times Z_2, U(1)\right) = Z_2$.
- Topological action:
  - ✓ Let A_1, A_2 be $(Z_2 \times 1)$- and $(1 \times Z_2)$- fields, respectively. ($Z_2 = \{0,,1\}$).
  - ✓ $S_M^{\omega}(A_1, A_2) = \int_M A_1 \cup A_2 \in \{0,1\}$
- Trivial SPT phase:

$$Z_{triv}(M, A_1, A_2) = 1.$$

- Nontrivial SPT phase:

$$Z_{nontriv}(M, A_1, A_2) = \exp \pi i \int_M A_1 \cup A_2 .$$

- From this explicit form of topological action, we find thaet the SPT phase can be detected by the torus partition function with background holonomies.

$$Z_{nontriv}\left(T^2, \oint_x A_1 = 1, \oint_y A_2 = 1\right) = Tr_{twised\ bdy\ cond.for\ (Z_2 \times 1)}\left[\hat{O}_{1 \times Z_2\ charge}\right] = -1.$$

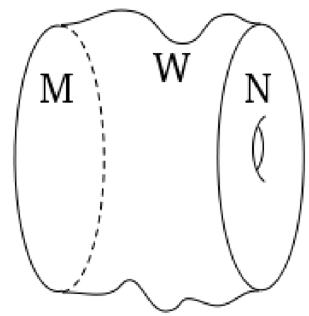
Cobordism classification

- Group cohomology classification does not explain SPT phases described by topological actions defined only by the spacetime manifold itself.
- From the same reason, the group cohomology can not describe SPT phases of fermionic dof.
- The cobordism was proposed to classify all SPT phase. [Kapustin]
- Precisely speaking, SPT phases are classified by the Anderson dual of the cobordism group

$$(D\Omega)^{d}_{str}(BG) \cong Free\ \Omega^{str}_{d+1}(BG) \times Tor\ \Omega^{str}_{d}(BG) \quad \text{(non-canonical).}$$

Cobordism

- Cobordism invariance ≠ topological invariance
- Two d-manifolds M, N are said to be cobordant iff there exists a $(d+1)$-manifold W whose boundary is $M \sqcup (-N)$.



(from wikipedia)

- This gives an equivalence relation.
- Disjoint union yields the structure of Abelian group.

$$[M] + [N] := [M \sqcup N]$$

-> Cobordism group Ω_d^{str}

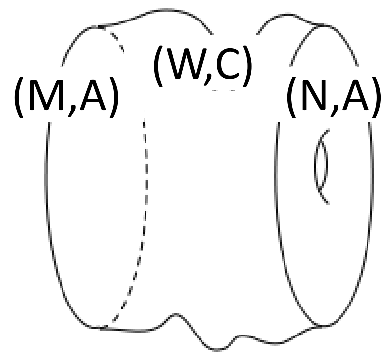
✓ Unoriented manifold

✓ Fermionic dof -> (variants of) spin structure

- With background G-fields -> $\Omega_d^{str}(BG)$, cobordism of manifolds with G-fields

Cobordism invariant theories

$$(M, A) \sim (N, B) \quad \Rightarrow \quad Z(M, A) = Z(N, B).$$



- A homomorphism from the cobordism group to U(1) group is a cobordism invariant invertible theory.

$$Hom(\Omega_d^{str}(BG), U(1))$$

Deformation invariant invertible theories

- SPT phases (~ deformation invariant invertible theories) are said to be classified by the Anderson dual of the cobordism group. [Kapustin, Freed-Hopkins]

$$(D\Omega)^{d}_{str}(BG) \cong Free\ \Omega^{str}_{d+1}(BG) \times Tor\ \Omega^{str}_{d}(BG)$$

- Torsion part: no continuous parameter

- Free part : theta term -> continuous parameter θ -> not an SPT phase.

- However, the Z classification in d-dim. implies the existence of the Chern-Simons action in (d-1)-dim. whose coefficient is in Z.

  -> SPT phases.

  -> describes thermal and quantum Hall effects.

From theta term to the Chern-Simons action

- Ex: theta term of the 4d background U(1)-field

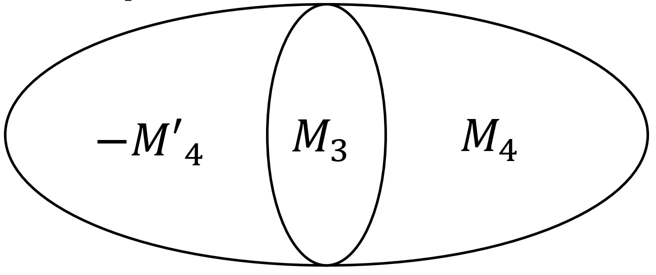
$$\frac{1}{4\pi^2}\int_{N_4} F^2 \in Z \Rightarrow \exp\left[2\pi i \times \frac{1}{4\pi^2}\int_{M_4} F^2\right] = 1.$$

- Here, N_4 is a closed 4-manifold.

- For a U(1)-bundle $A \to M_3$ over a 3-manifold M_3, set an extension of A to a 4-manifold M_4 with the boundary $\partial M_4 = M_3$. The 3d Chern-Simons action is defined by

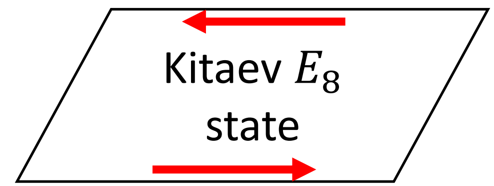
$$\exp\frac{ik}{2\pi i}\int_{M_3} "AdA" := \exp\left[k \times 2\pi i \times \frac{1}{4\pi^2}\int_{M_4} F^2\right], \qquad k \in Z$$

- This is well-defined because the U(1) value does not depend on extensions, thanks to $\frac{1}{4\pi^2}\int_{N_4} F^2 \in Z$.



✓ Ex1: $\Omega_4^{SO}(pt) = Z$. (bosonic systems with no symmetry)

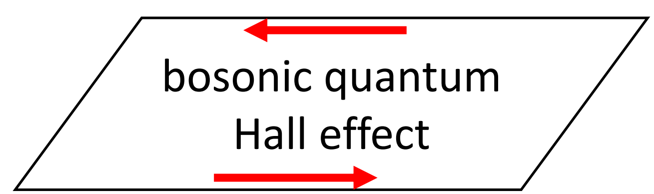
- 4d topological action is the signature of manifold
- $(const) \times \int_M R^2 \in Z$
- 3d SPT action is the gravitation CS form (Kitaev E_8 state)
- $\exp 2\pi i\, k \times (const) \times \int_M Tr\left[\omega d\omega + \frac{2}{3}\omega^3\right],\ k \in Z$



$c_- = 8k$ chiral central charge

✓ Ex2: $\Omega_4^{SO}(BU(1)) = Z \times Z$. (bosonic systems with U(1) symmetry)

- Another 4d topological action is the theta term of U(1)-field.
- $\frac{1}{4\pi^2}\int_M F^2 \in Z$
- 3d SPT action is the U(1) CS form (bosonic quantum Hall effect)
- $\exp \frac{ik}{2\pi}\int_M A\, dA,\ \ k \in Z.$



$\sigma_{xy}=2k$: quantum Hall conductivity

$c_- = 0$: no thermal Hall effect

Short summary

- SPT phases are gapped quantum phases with symmetry.
- SPT phases are characterized by U(1)-valued partition functions.
- SPT phases are classified by the Anderson dual of the cobordism group.

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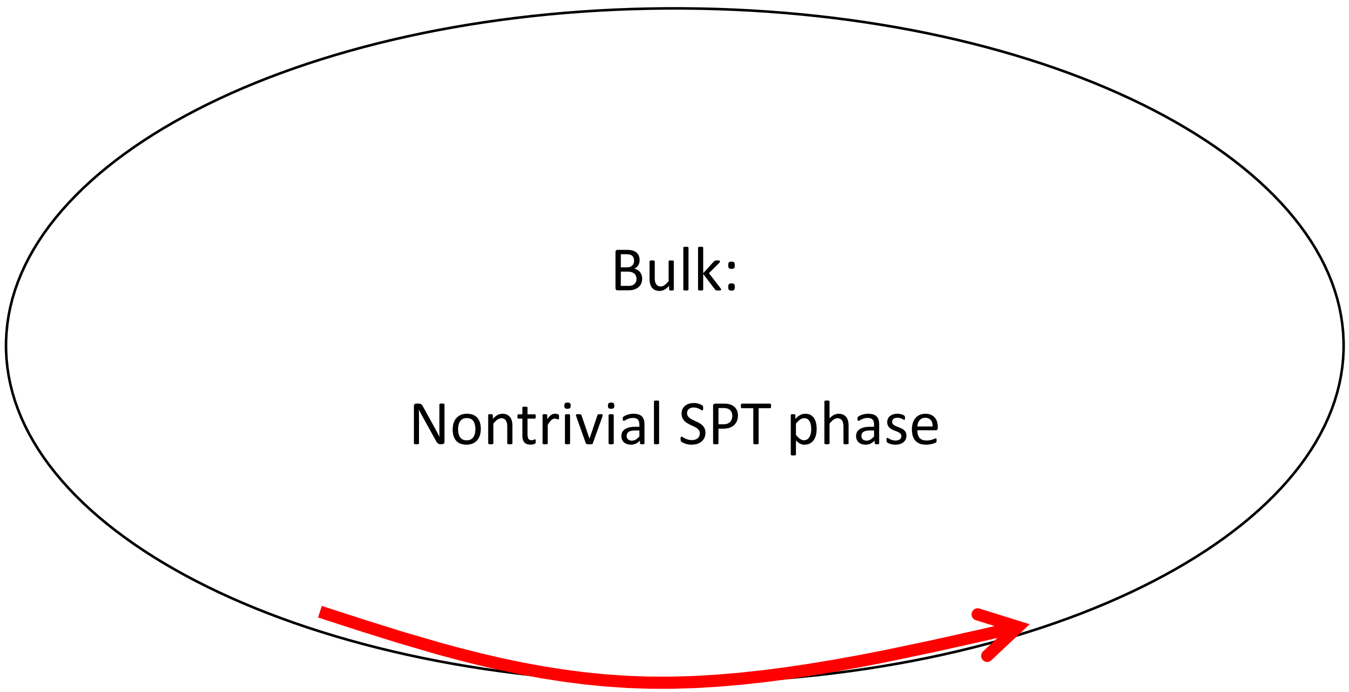
Bulk-boundary correspondence

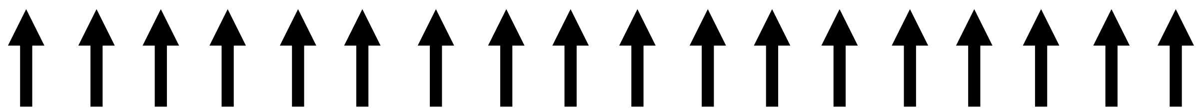


Boundary: dof with a quantum anomaly

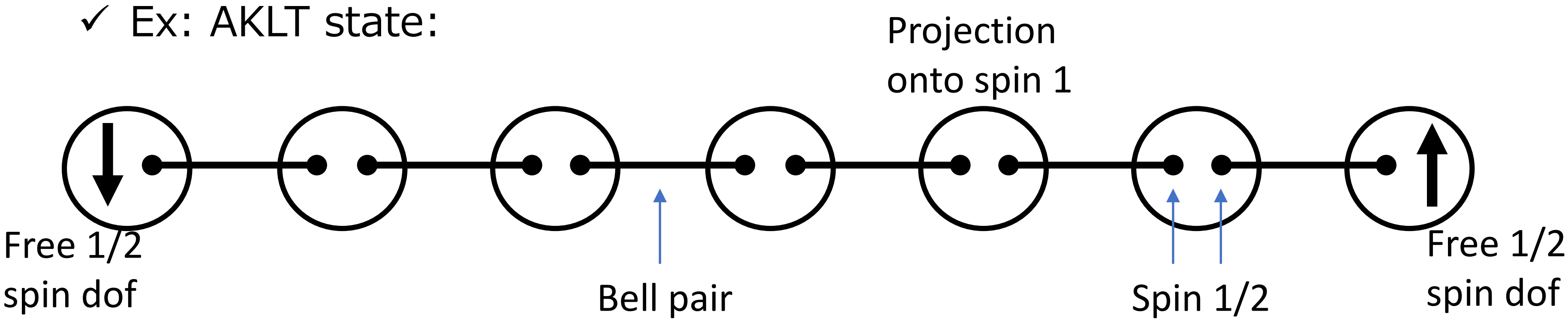
Bulk-boundary correspondence

- The emergent low-energy dof is the signature of nontrivial SPT phases.
- The trivial SPT phase: tensor product state
  - ✓ On the open bdy condition, the system is still gapped as bulk.

$$|\Psi\rangle = \otimes_x \ |\uparrow\rangle_x, \qquad |\uparrow\rangle_x \in \mathcal{H}_x$$



- Nontrivial SPT phases: nontrivial short-range entanglement
  - ✓ A "nontrivial" low-energy dof appears.
  - ✓ Ex: AKLT state:



Ex: decorated domain wall state

- 1d bosonic systems with $Z_2 \times Z_2$ symmetry

- Dof: two flavors of Ising spins $\sigma_j^\mu, \tau_{j+\frac{1}{2}}^\mu, j \in Z$.

- Hamiltonian (cluster Hamiltonian)

$$H = -\sum_j A_{j+\frac{1}{2}} - \sum_j B_j := -\sum_{j \in Z} \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_{j \in Z} \tau_{j-\frac{1}{2}}^z \sigma_j^x \tau_{j+\frac{1}{2}}^z$$

- $Z_2 \times Z_2$ symmetry operators:

$$U_\sigma = \prod_j \sigma_j^x, \qquad U_\tau = \prod_j \tau_{j+\frac{1}{2}}^x$$

- All terms are commuted with each other.

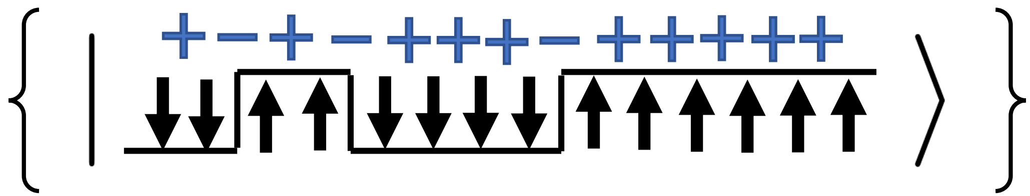
- The ground state is the state with $A_{j+\frac{1}{2}}|\Psi\rangle = B_j|\Psi\rangle = |\Psi\rangle$.

- On the closed ring S^1, all terms are independent, meaning that the ground state is unique. (cf. toric code)

- Moreover, an excited state has at least the energy $E = 2$. -> gapped

- Let us write the ground state with the bases of $\sigma_j^z = \{\uparrow, \downarrow\}, \tau_{j+\frac{1}{2}}^x = \{+. -\}$.

- 1st terms -> $\sigma_j^z \sigma_{j+1}^z = \tau_{j+\frac{1}{2}}^x$ -> decorated domain walls (DDWs)



- 2nd terms fluctuate the decorated domain walls

  -> The ground state is the equal-weight superposition of the decorated domain walls.

$$|\Psi\rangle = \sum_{DDWs} |\, DDW \,\rangle$$

- Hamiltonian on the open chain

$$H = -\sum_{j=1}^{N-1} \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_{j=1}^{N} \tau_{j-\frac{1}{2}}^z \sigma_j^x \tau_{j+\frac{1}{2}}^z$$

$$\tau_{\frac{1}{2}} \quad \sigma_1 \quad \tau_{\frac{3}{2}} \qquad \bullet \quad \bullet \quad \bullet \qquad \tau_{N-\frac{1}{2}} \ \sigma_N \ \tau_{N+\frac{1}{2}}$$

- # (dof) = 2N+1, #($A_{j+\frac{1}{2}}, B_j$) = 2N-1

  -> The ground state is 4-fold degenerate.

$$|\Psi(a,b)\rangle = \sum_{DDWs} \left|\left(\tau_{\frac{1}{2}}^x = a\right) - DDW - \left(\tau_{N+\frac{1}{2}}^x = b\right)\right\rangle, \qquad a, b \in \{+,-\}.$$

- This 4-fold degeneracy is not accidental, but protected by the $Z_2 \times Z_2$ symmetry!

- To see this, let's consider how $Z_2 \times Z_2$ symmetry operators act on the ground states.

$$U_\sigma \Big|_\Psi = \prod_j \sigma_j^x \Big|_\Psi = \prod_j (\tau^z_{j-\frac{1}{2}} \tau^z_{j+\frac{1}{2}}) = \tau^z_{\frac{1}{2}} \otimes \tau^z_{N+\frac{1}{2}} =: U_\sigma^L \otimes U_\sigma^R,$$

$$U_\tau \Big|_\Psi = \prod_j \tau_j^x \Big|_\Psi = \tau^x_{\frac{1}{2}} \left(\prod_j (\sigma_j^z \sigma_{j+1}^z) \right) \tau^x_{N+\frac{1}{2}} = \tau^x_{\frac{1}{2}} \sigma_1^z \otimes \sigma_N^z \tau^x_{N+\frac{1}{2}} =: U_\tau^L \otimes U_\tau^R.$$

- $Z_2 \times Z_2$ symmetry operations split into ones for the left and right dof.

- The most important point is that on the one side of edges $Z_2 \times Z_2$ acts on dof as a nontrivial projective representation that is classified by $H^2(Z_2 \times Z_2, U(1)) = Z_2$, which can be seen in the algebra

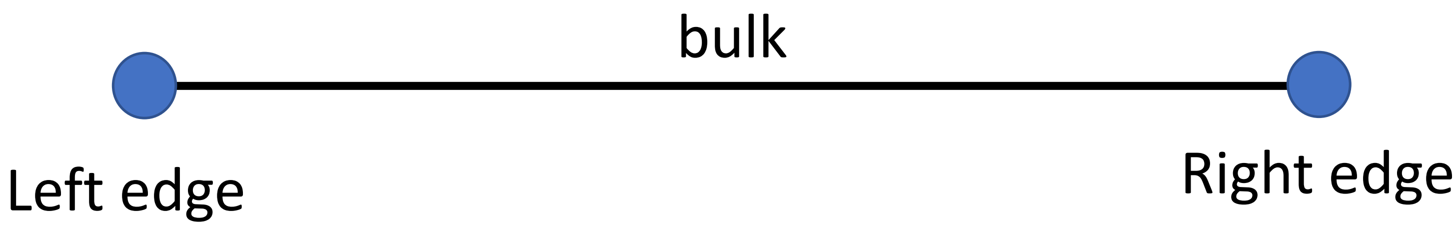
$$U_\sigma^R \, U_\tau^R = -U_\tau^R U_\sigma^R.$$

- There is no 1-dimensional rep of nontrivial projective representations, meaning that the one side of edge should be degenerate, unless the bulk gap is closed.

- This is an example of the quantum anomaly.

Symmetry fractionalization

- Assumption: the bulk dof obey a linear representation of G.
- The classification of 1d bosonic SPT phases with global G symmetry is the classification of how G symmetry can act on the edge dof protectively. "symmetry fractionalization"



$$U_g \Big|_{low-energy} = U_g^L \otimes U_g^R$$

- The total symmetry action $U_g|_{low-energy}$ is linear representation of G.

Projective representations

Let G be a group. A set of matrices $\{D_g\}_{g\in G}$ is called a projective representation iff

$$D_g D_h = \omega_{g,h}\, D_{gh}, \qquad \omega_{g,h} \in U(1).$$

The associativity $(D_g D_h)D_k = D_g(D_h D_k)$ yields the 2-cocyle condition

$$\omega_{g,h}\omega_{gh,k} = \omega_{g,hk}\, \omega_{h,k}.$$

The redefinition $D_g \mapsto \alpha_g\, D_g, \alpha_g \in U(1)$ yields the equivalence relation

$$\omega_{g,h} \sim \omega_{g,h}\, \alpha_h \alpha_{gh}^{-1} \alpha_g$$

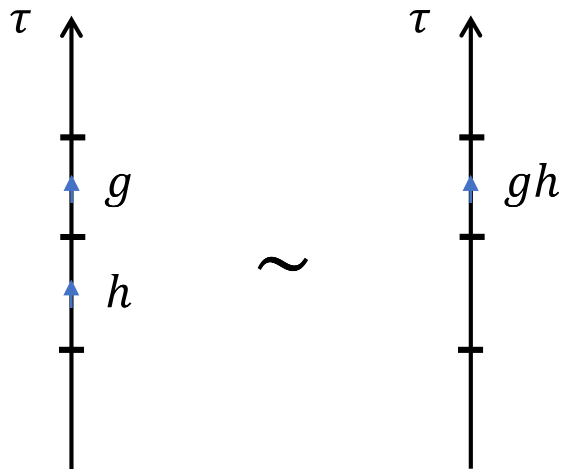
The factor system $\omega_{g,h}$ is classified by the group cohomology

$$H^2\big(G, U(1)\big) = Z^2(G, U(1))/B^2(G, U(1)).$$

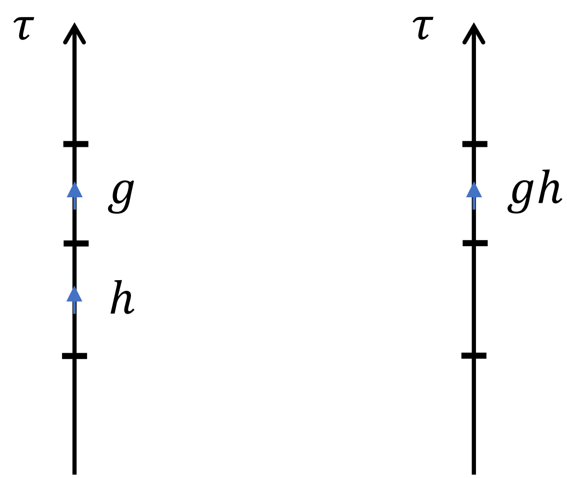
Why quantum anomaly?

- Let's consider a (0+1)d system obeying a linear representation of G.
- The linearity of G-action is nothing but the gauge equivalence on the background G-field.

$$U_g U_h = U_{gh}$$



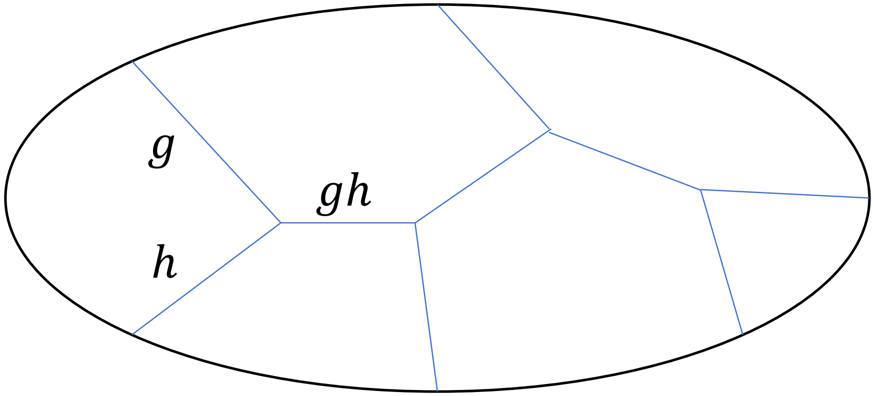
- However, if G-action obeys an nontrivial projective representation, the partition function can not be gauge independent by a counter term (1-coboundary).



$$U_g U_h = \omega_{g,h} U_{gh} \quad \neq \quad U_{gh}$$

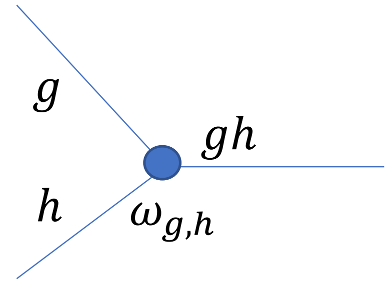
2d DW theory

- Let's consider the relationship between edge anomaly and the bulk DW action.
- A G-field A over M is a symmetry defect network on M.
- When a matter field passes a defect line labeled by $g \in G$, the matter field is charged by g.

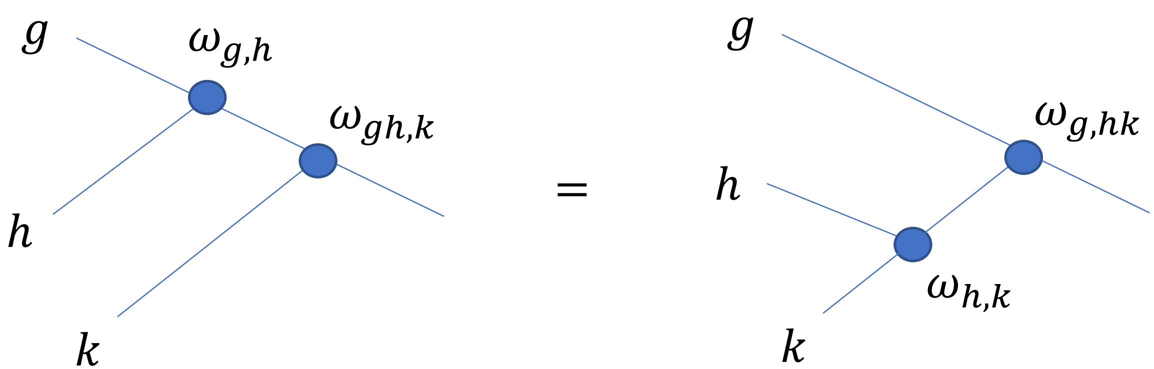


- The ansatz for U(1)-valued topological actions:

$$Z(M, A) = \prod_{junctions} \omega_{g,h}\,, \qquad \omega_{g,h} \in U(1).$$



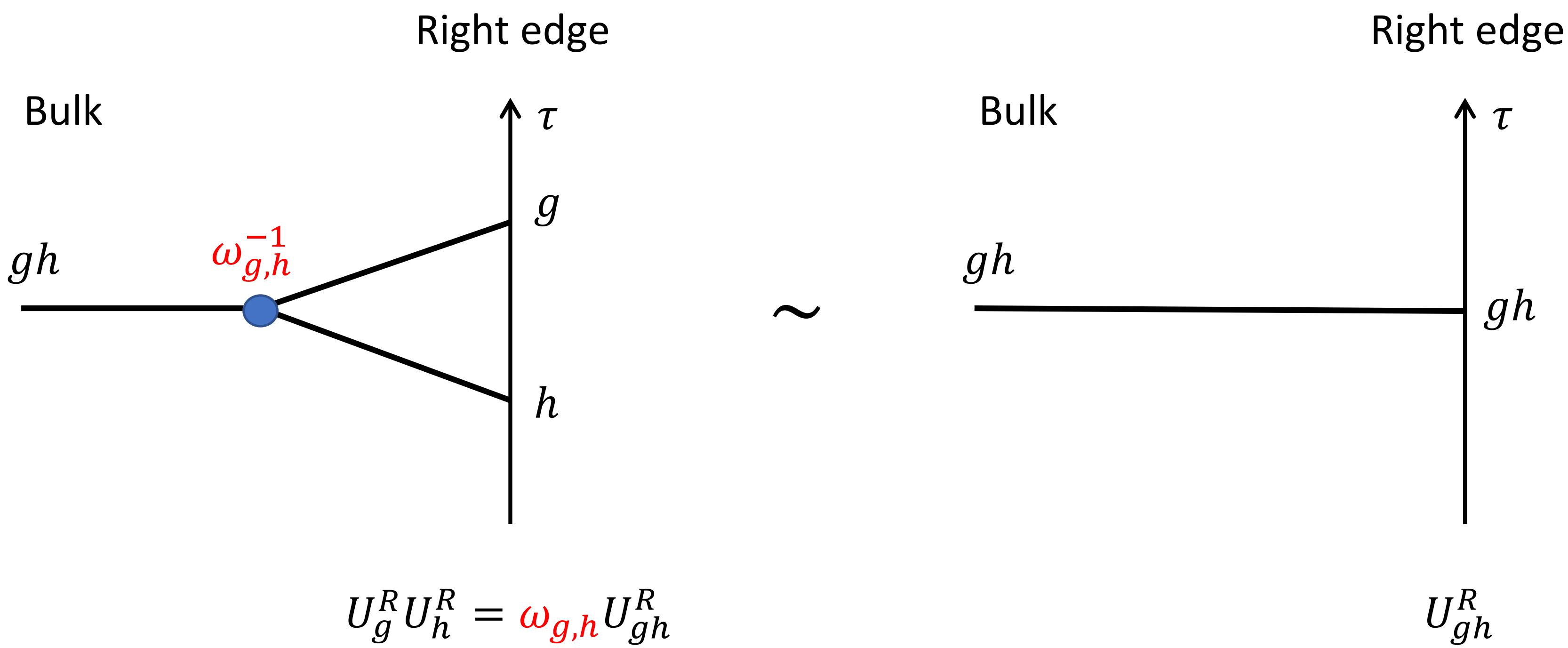
- Topological invariance requires the 2-cocycle condition on $\omega_{g,h}$.



- We got the 2d DW topological action labeled by a 2-group cocycle $\omega_{g,h}$.

Anomaly cancellation

- The total system composed of the bulk and the boundary is anomaly free.



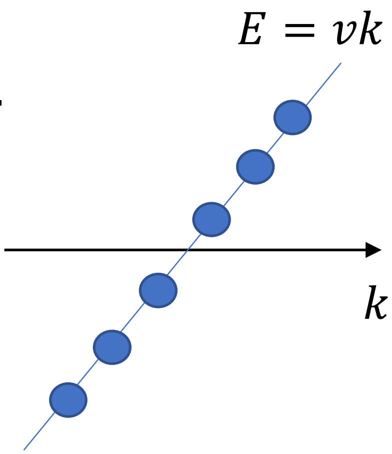
(2+1)d example: the integer quantum Hall state

- Bulk dof: Dirac fermions.
- Classification of SPT phases: $D\Omega^3_{\mathrm{spin}^c}(\mathrm{pt}) \cong Free\ \Omega_4^{spin^c}(pt) = Z \times Z$
- One of Z is generated by the integer quantum Hall state.
- Bulk model (free fermion):

$$H_{bulk} = -i\,\sigma_x \partial_x \; - i\,\sigma_y\,\partial_y + (m - \epsilon\partial^2)\sigma_z, \qquad m, \epsilon > 0.$$

- Boundary: chiral Dirac fermion

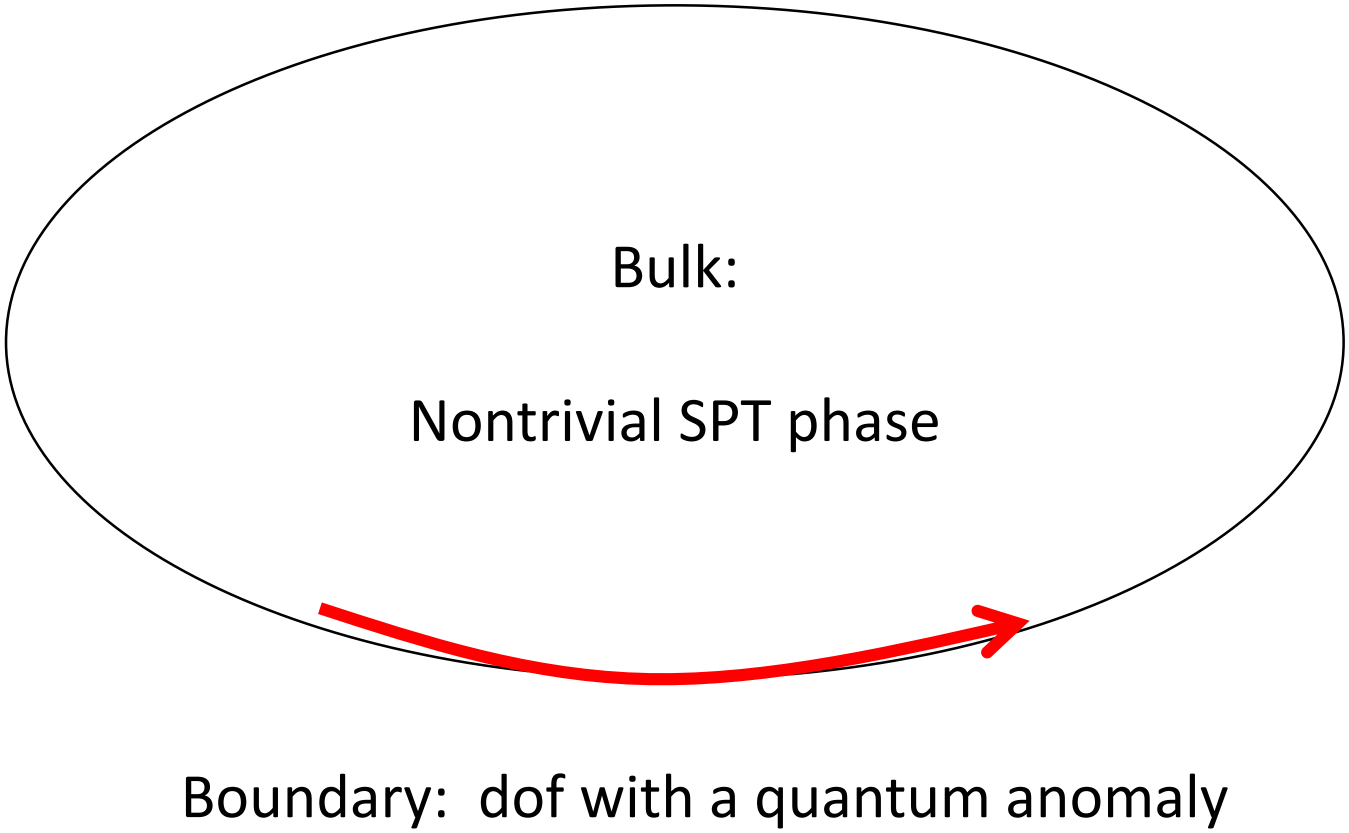
$$\hat{H}_{bdy} = \sum_{k \in Z + \frac{\theta}{2\pi}} vk\,\psi_k^\dagger \psi_k$$



- Anomaly under the global gauge transformation $\theta \mapsto \theta + 2\pi$.
- The bulk action is the CS 3-form $\exp \frac{i}{4\pi}\int_M AdA$, which is gauge dependent on open manifolds.
- The anomaly on the boundary is cancelled by the bulk CS action.

Bulk-boundary correspondence

- On the boundaries of nontrivial SPT phases, there appears low-energy dof.
- As a standalone system, the boundary dof has an anomaly.
- The anomaly on the boundary is cancelled by the bulk.



Contents

- Partition functions and classification of SPT phases
- Bulk-boundary correspondence
- Model construction

Model construction [Chen-Gu-Liu-Wen]

- So far, we discussed a lot about DW topological action, which is the response theory obtained by integrating out the matter dof.

- Is there a canonical way to construct SPT phases?

- For SPT phases described by the group cohomology, yes. [Chen-Gu-Liu-Wen]

- Mathematically, we use the dual expression of the group cohomology by the “homogenous cochain”.

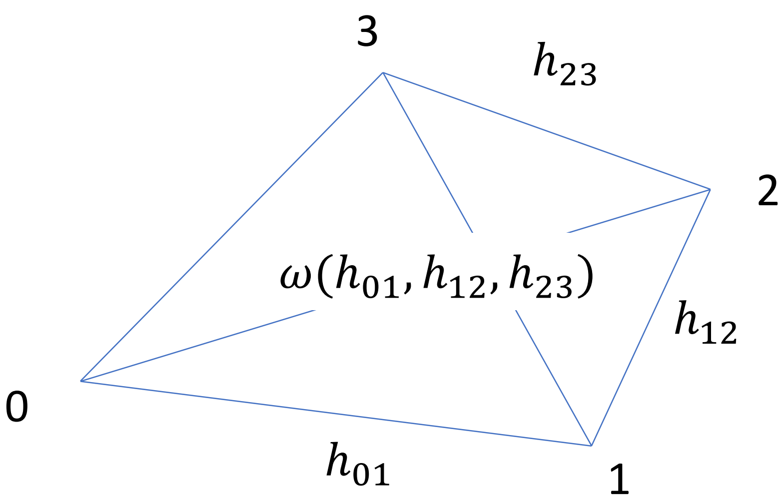
A heuristic derivation

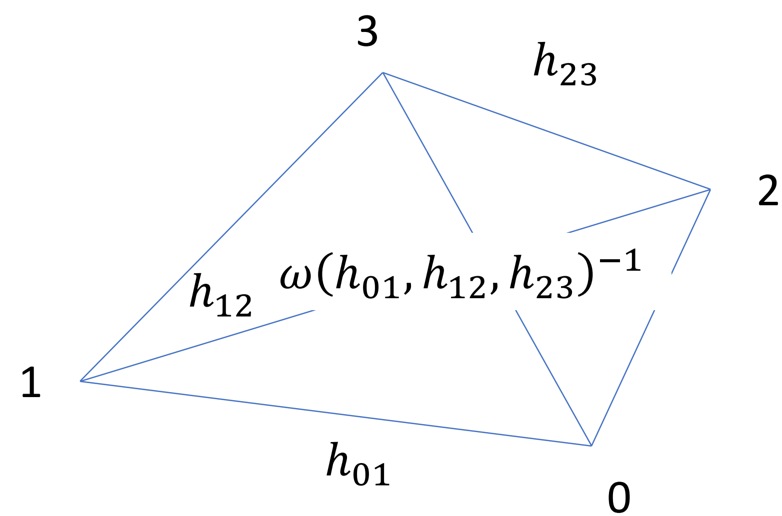
- For concreteness, we consider (2+1)d SPT phases. (The derivation is parallel for general dimensions.)

- For a given 3-cocycle $\omega(h_{01}, h_{12}, h_{23}) \in Z^3(G, U(1))$, the DW action is given by

$$Z_{DW}^{\omega}\left(M, \{h_{ij}\}\right) = \prod_{\Delta^3} \omega(h_{01}, h_{12}, h_{23})^{\sigma(\Delta^3)},$$

where Δ^3 runs all 3-simplexes, and $\sigma(\Delta^3) \in \{\pm 1\}$ is the sign of the simplex Δ^3.





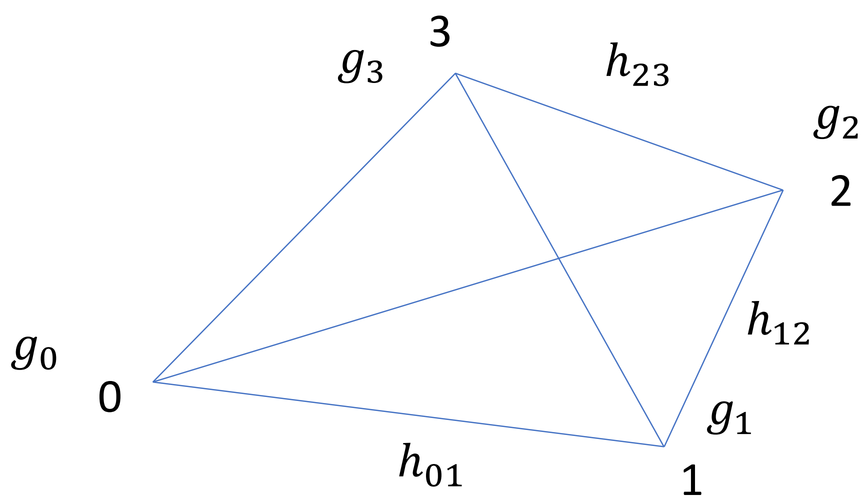
- Take the gauge transformation $g_v\colon \{vertices\} \to G$, we have

$$Z_{DW}^{\omega}\big(M,\{h_{ij}\}\big) = \prod_{\Delta^3} \omega(g_0^{-1}h_{01}g_1, g_1^{-1}h_{12}g_2, g_2^{-1}h_{23}g_3)^{\sigma(\Delta^3)}.$$

- Since this does not depend on the gauge transformation, summing up the all gauge transformations, we get

$$Z_{DW}^{\omega}\big(M,\{h_{ij}\}\big) = \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \prod_{\Delta^3} \omega(g_0^{-1}h_{01}g_1, g_1^{-1}h_{12}g_2, g_2^{-1}h_{23}g_3)^{\sigma(\Delta^3)},$$

Where N_v is the total number of vertices.



$$Z_{DW}^{\omega}\left(M,\{h_{ij}\}\right) = \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \prod_{\Delta^3} \omega(g_0^{-1}h_{01}g_1, g_1^{-1}h_{12}g_2, g_2^{-1}h_{23}g_3)^{\sigma(\Delta^3)},$$

- The field $g_v: \{\text{vertices}\} \to G$ can be regarded as a matter field living on the vertices.

- In particular, without a background G field, we have the partition function of the matter field

$$Z_{SPT}^{\omega}(M) = \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \prod_{\Delta^3} \omega(g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3)^{\sigma(\Delta^3)}.$$

- $\nu(g_0, g_1, g_2, g_3) := \omega(g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3)$ is called the homogenious cochain, because it holds that

$$\nu(gg_0, gg_1, gg_2, gg_3) = \nu(g_0, g_1, g_2, g_3).$$

- The 3-cocycle condition of ω becomes

$$\nu(g_1, g_2, g_3, g_4)\nu(g_0, g_2, g_3, g_4)^{-1}\,\nu(g_0, g_1, g_3, g_4)\,\nu(g_0, g_1, g_2, g_4)^{-1}\,\nu(g_0, g_1, g_2, g_3) = 1.$$

- In sum, for a given homogenous 3-cocycle $\nu(g_0, g_1, g_2, g_3) \in Z^3(G, U(1))$, the Lagrangian of the matter theory is given by

$$e^{i \int_{\Delta^3} \mathcal{L}} = \nu(g_0, g_1, g_2, g_3)^{\sigma(\Delta^3)},$$

and the partition function is

$$Z_{SPT}^{\omega}(M) = \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \prod_{\Delta^3} \nu(g_0, g_1, g_2, g_3)^{\sigma(\Delta^3)}.$$

- This is the generalization of the theta term of the NLσ model to finite groups.

- The wave function $\Psi_\nu(\{g_v\}_{v\in\partial M})$ over the 2-manifold ∂M is given by the path-integral of internal dof of M

$$\Psi_\nu(\{g_v\}_{v\in\partial M}) = \frac{1}{|G|^{N_v^{internal}}} \sum_{\{g_v\}, v\in internal} \nu(g_0, g_1, g_2, g_3)^{\sigma(\Delta^3)} \in \mathcal{H}_{\partial M}.$$

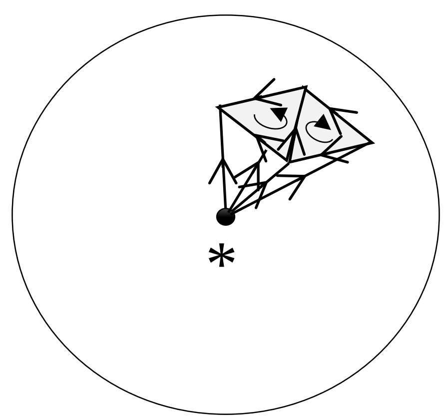
- In particular, the ground state wave function over the 2-sphere is given by

$$\Psi_\nu(\{g_v\}_{v\in S^2}) = \frac{1}{|G|} \sum_{g_*} \prod_{\Delta^3} \nu(g_*, g_1, g_2, g_3)^{\sigma(\Delta^3)}.$$

- Using the 3-cocyle condition, we arrive at the simple expression

$$\Psi_\nu(\{g_v\}_{v\in S^2}) = \prod_{\Delta^2} \nu(1, g_1, g_2, g_3)^{\sigma(\Delta^2)},$$

where Δ^2 runs over the 2-simpleces on the 2-sphere.



- The ground state is given by

$$|\Psi_\nu\rangle = \frac{1}{\sqrt{|G|^{N_v}}} \sum_{\{g_v\}} \prod_{\Delta^2} \nu(1, g_1, g_2, g_3)^{\sigma(\Delta^2)} \, |\{g_v\}\rangle \, .$$

- Here, $\{\,|g_v\rangle\,\}_{g\in G}$ is the basis of local Hilbert space at the vertex v equipped with the G-action $\hat{g}\,|h\rangle = |gh\rangle$.

- The exactly solvable commuting projector Hamiltonian is given as follows.

- We first introduce the trivial Hamiltonian H_0 as a “disordered Hamiltonian“

$$H_0 = -\sum_v P_v, \qquad P_v = \cdots \, Id \otimes |\phi_v\rangle\langle\phi_v| \otimes Id \cdots,$$

$$|\phi_v\rangle = \frac{1}{\sqrt{|G|}} \sum_{g\in G} |g_v\rangle \, .$$

- The ground state of H_0 is the trivial tensor product state

$$|\Psi_0\rangle = \otimes_v \; |\phi_v\rangle.$$

- The SPT Hamiltonian H_ν is defined so that the ground state of H_ν is $|\Psi_\nu\rangle$.
- Using the nonlocal unitary transformation

$$U_\nu := \sum_{\{g_v\}} \prod_{\Delta^2} \nu(1, g_1, g_2, g_3)^{\sigma(\Delta^2)} \, |\{g_v\}\rangle \langle\{g_v\}|,$$

we have the SPT Hamiltonian

$$H_\nu = U_\nu \, H_0 \, U_\nu^{-1} = -\sum_v U_\nu \, P_v \, U_\nu^{-1}.$$

- It is evident that this Hamiltonian is short-ranged and exactly solvable.
- See [Chen-Gu-Liu-Wen] for various examples.